Journal of Computational Mathematics, Vol.19, No.3, 2001, 259–268.

D-CONVERGENCE OF RUNGE-KUTTA METHODS FOR STIFF DELAY DIFFERENTIAL EQUATIONS*1)

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Abstract

This paper is concerned with the numerical solution of delay differential equations(DDEs). We focus on the error behaviour of Runge-Kutta methods for stiff DDEs. We investigate D-convergence properties of algebraically stable Runge-Kutta methods with three kinds of interpolation procedures.

Key words: Delay Differential equations, Runge-Kutta methods, D-convergence.

1. Introduction

When considering the applicability of numerical methods for the solution of the delay differential equation (DDE) $y'(t) = f(t, y(t), y(t-\tau))$, it is necessary to analyze the error behaviour of the methods. In fact, many papers have investigated the local and global error behaviour of DDE solvers (cf.[1,2,14]). These error analyses are based on the assumption that the function $f(t, y, z)$ satisfies Lipschitz conditions in both the last two variables. They are suitable for nonstiff DDEs because the Lipschitz constants are moderate-sized. However, they can not be applied to stiff DDEs. For example, consider Hutchinson's equation (cf.[9])

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = a\frac{\partial^2}{\partial x^2}u(x,t) + u(x,t)[1-u(x,t-\tau)], \quad t>0, x\in(0,1), \\ u(x,t)=\phi(x,t), \quad t\in[-\tau,0], x\in(0,1), \\ u(0,t)=u(1,t)=0, t\geq -\tau, \end{cases} \tag{1.1}$$

where $a > 0$ is the diffusion coefficient, $\phi(x,t)$ is continuous. We transform the partial DDE (1.1) into a system of ordinary DDE by discretising the space variable x into $(N+2)$ discrete values $(N > 0)$, with a constant stepsize in space, $\Delta x = 1/(N+1)$, so that $x_j = j\Delta x, j = 0, 1, \cdots, N+1$. Using the standard central difference operator to approximate the Laplacian we obtain a system with

$$f(t,y(t),y(t-\tau)) = \frac{a}{\Delta x^2}\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ \vdots \\ y_N(t) \end{bmatrix} + \begin{bmatrix} y_1(t)(1-y_1(t-\tau)) \\ y_2(t)(1-y_2(t-\tau)) \\ \vdots \\ \vdots \\ y_N(t)(1-y_N(t-\tau)) \end{bmatrix}, \tag{1.2}$$

* Received April 24, 1998; Final revised October 21, 1999.
1) Project supported by NSF of China (No.19871070) and China Postdoctoral Science Foundation.

where $y_j(t)$ denotes the approximation to $u(x_j,t), j=1,2,\cdots,N$. In this case, the Lipschitz constant L of the function $f(t,y,z)$ with respect to y will contain negative powers of the meshwidth Δx in space. As a consequence, L will be very large for fine space grids, and the error estimates based on L are not realistic. On the other hand, the one-sided Lipschitz constant α is only moderate. Hence estimates based on α are often considerably more realistic than that based on L. In fact, Frank et al. introduced the concept of B-convergence for Runge-Kutta methods applied to stiff ODEs, and established the following basic criteria (cf.[6,7,8])

algebraic stability + diagonal stability + stage order p ⇒ B-convergence with order p.

Burrage and Hundsdorfer [4] further discussed the conditions which guarantee that a Runge-Kutta method has order one higher than the stage order. Li [13] further extended these studies to general linear methods and to initial value problems in Hilbert spaces and established a more efficient theory. Recently, the concept of D-convergence [16] for DDEs, which is a generalization of the concept of B-convergence, was introduced. Zhang and Zhou [16] discussed D-convergence of a class of Runge-Kutta methods, and some first and second order D-convergent methods were found. We proved in [10] that the order of D-convergence equals the consistent order in classical sense for A-stable one-leg methods with linear interpolation. In this paper, we further discuss D-convergence of algebraically stable Runge-Kutta methods. We will discuss D-convergence of general linear methods in other paper.

2. Runge-Kutta Methods for DDEs

Let $\langle\cdot,\cdot\rangle$ be an inner product on C^N and $\|\cdot\|$ the corresponding norm. Consider the following nonlinear equation

$$\begin{cases} y'(t)=f(t,y(t),y(t-\tau)), & t\ge 0,\\ y(t)=\phi_1(t), & t\le 0,\end{cases} \tag{2.1}$$

where τ is a positive delay term , ϕ_1 is a continuous function, and $f:[0,+\infty)\times C^N\times C^N\to C^N$, is a given mapping which satisfies the following conditions:

$$\mathrm{Re}\langle u_1-u_2, f(t,u_1,v)-f(t,u_2,v)\rangle \le \alpha\|u_1-u_2\|^2, \quad t\ge 0, u_1,u_2,v\in C^N, \tag{2.2}$$

$$\|f(t,u,v_1)-f(t,u,v_2)\|\le\beta\|v_1-v_2\|, \quad t\ge 0, u,v_1,v_2\in C^N, \tag{2.3}$$

where α and β are real constants. In order to make the error analysis feasible, we always assume that the problem (2.1) has a unique solution $y(t)$ which is sufficiently differentiable and satisfies

$$\|\frac{d^iy(t)}{dt^i}\|\le M_i.$$

Remark 2.1. When $\beta=0$, the above problem class has been used widely in stiff ODEs field (cf.[5,12]).

Now we consider the adaptation of Runge-Kutta methods to (2.1). Let (A,b,c) denote a given Runge-Kutta method with $s\times s$ matrix $A=(a_{ij})$ and vectors $b=(b_1,\cdots,b_s)^T, c=(c_1,\cdots,c_s)^T$. In this paper we always assume that $0\le c_i\le 1 (i=1,\cdots,s)$. Let $h>0$ be a given stepsize and $y_0=\phi_1(0)$. Define gridpoints $t_n (n=0,1,2,\cdots)$ by $t_n=nh$. Then approximation y_{n+1} to $y(t_{n+1}) (n=0,1,2,\cdots)$ are defined by

$$Y_i^{(n)}=y_n+h\sum_{j=1}^s a_{ij}f(t_n+c_jh, Y_j^{(n)}, \bar{Y}_j^{(n)}), \quad i=1,\cdots,s, \tag{2.4a}$$

$$y_{n+1}=y_n+h\sum_{j=1}^s b_jf(t_n+c_jh, Y_j^{(n)}, \bar{Y}_j^{(n)}). \tag{2.4b}$$

The argument $\bar{Y}_j^{(n)}$ is defined by $\bar{Y}_j^{(n)}=\phi_1(t_n+c_jh-\tau)$ (whenever $t_n+c_jh-\tau\le 0$), and denotes an approximation to $y(t_n+c_jh-\tau)$ (whenever $t_n+c_jh-\tau>0$) which is obtained by a specific interpolation procedure at the point $t = t_n + c_j h - \tau$ using values $Y_i^{(k)}$ and y_k with $k \le n$.

Process (2.4) is defined completely by the Runge-Kutta method (A, b, c) and the interpolation procedure for $\bar{Y}_j^{(n)}$.

Definition 2.2. *(cf.[10,16]) A method (A, b, c) with an interpolation procedure is said to be D-convergent of order p if the global error satisfies a bound*

$$\|y(t_n) - y_n\| \le \varrho(t_n)h^p, \quad n \ge 1, h \in (0, h_0], \tag{2.5}$$

where the function $\varrho(t)$ and the maximum stepsize h_0 depend only on the method, some of the bounds M_i, the parameters α, β and τ.

In addition to general D-convergence results, we are also interested in the error behaviour of numerical methods with constrained mesh that the stepsize h satisfies

$$hm = \tau, \tag{2.6}$$

where m is a positive integer, because the results in this special case are also useful for practical applications.

Proposition 2.3. *D-convergence implies B-convergence.*

Remark 2.4. When β is moderate, the error estimate based on D-convergence is significant to practical applications. When β is very large, the error estimate is worthy of further investigation.

It is well known that there exist three kinds of interpolation procedures for $\bar{Y}_j^{(n)}$. Let $\tau = (m - \delta)h$ with integer m and $\delta \in [0, 1)$, $c_j + \delta = l_j + x_j$ with integer l_j and $x_j \in [0, 1)$ for $1 \le j \le s$, then $0 \le l_j \le 1$. Let $\nu, \mu \ge 0$ be integers.

(i) Interpolation by using y_k. Let $y_k = \phi_1(t_k)$ for $k \le 0$. Define(cf.[11])

$$\bar{Y}_j^{(n)} = \sum_{i=-\mu}^{\nu} L_i(x_j) y_{n-m+l_j+i}, \quad t_n + c_j h - \tau > 0, \nu + 1 \le m, \tag{2.7}$$

where

$$L_i(x) = \prod_{\substack{k = -\mu \\ k \ne i}}^{\nu} \left(\frac{x - k}{i - k}\right), \quad x \in [0, 1), \tag{2.8}$$

and we assume $m \ge \nu + 1$ so as to guarantee that, in the interpolation procedure for $\bar{Y}_j^{(n)}$, no unknown values y_k with $k > n$ are used.

(ii) Interpolation by using $Y_j^{(k)}$. Let $Y_j^{(k)} = \phi_1(t_k + c_j h)$ for $k < 0, 1 \le j \le s$. Define(cf.[11])

$$\bar{Y}_j^{(n)} = \sum_{i=-\mu}^{\nu} L_i(\delta) Y_j^{(n-m+i)}, \quad t_n + c_j h - \tau > 0, \nu + 1 \le m. \tag{2.9}$$

where we assume $m \ge \nu + 1$ so as to guarantee that, in the interpolation procedure for $\bar{Y}_j^{(n)}$, no unknown values $Y_j^{(k)}$ with $k \ge n$ are used.

(iii) Interpolation by using continuous extensions of Runge-Kutta methods. Define(cf.[14,15])

$$\bar{Y}_j^{(n)} = y_{n-m+l_j} + h \sum_{i=1}^{s} b_i(x_j) f(t_{n-m+l_j} + c_i h, Y_i^{(n-m+l_j)}, \bar{Y}_i^{(n-m+l_j)}), \quad t_n + c_j h - \tau > 0, m \ge 2. \tag{2.10}$$

where we assume $m \ge 2$ so as to guarantee $n > n - m + l_j$, and functions $b_i(x)$ satisfy $b_i(0) = 0$ and $b_i(1) = b_i$ so as to assure the continuity of the interpolation.

Now we introduce some notations. For any given $k \times l$ real matrix $G = [g_{ij}]$ we define the corresponding linear operator $G : C^{lN} \to C^{kN}$,

$$GU = V = (v_1^T, v_2^T, \cdots, v_k^T)^T \in C^{kN}, U = (u_1^T, u_2^T, \cdots, u_l^T)^T \in C^{lN}, u_i \in C^N,$$

with

$$v_i = \sum_{j=1}^{l} g_{ij} u_j, i = 1, 2, \cdots, k.$$

The inner product and norm on C^{kN} are defined by

$$\langle U, V\rangle = \sum_{i=1}^{k} \langle u_i, v_i\rangle, \|U\| = \left(\sum_{i=1}^{k} \|u_i\|^2\right)^{1/2},$$

where $U = (u_1^T, u_2^T, \cdots, u_k^T)^T \in C^{kN}, V = (v_1^T, v_2^T, \cdots, v_k^T)^T \in C^{kN}, u_i, v_i \in C^N, i = 1, 2, \cdots, k.$ And the norm $\|G\|$ of the linear operator G is defined by the spectral norm of the matrix G. It is easily seen that $\|GU\| \leq \|G\| \cdot \|U\|$.

Thus, the process (2.4) can be written in the more compact form

$$Y^{(n)} = ey_n + hAF(t_n, Y^{(n)}, \bar{Y}^{(n)}), \tag{2.11a}$$

$$y_{n+1} = y_n + hb^T F(t_n, Y^{(n)}, \bar{Y}^{(n)}), \tag{2.11b}$$

with the following notational conventions:

$$Y^{(n)} = \begin{bmatrix} Y_1^{(n)} \\ Y_2^{(n)} \\ \vdots \\ Y_s^{(n)} \end{bmatrix}, \bar{Y}^{(n)} = \begin{bmatrix} \bar{Y}_1^{(n)} \\ \bar{Y}_2^{(n)} \\ \vdots \\ \bar{Y}_s^{(n)} \end{bmatrix}, F(t_n, Y^{(n)}, \bar{Y}^{(n)}) = \begin{bmatrix} f(t_n + c_1 h, Y_1^{(n)}, \bar{Y}_1^{(n)}) \\ f(t_n + c_2 h, Y_2^{(n)}, \bar{Y}_2^{(n)}) \\ \vdots \\ f(t_n + c_s h, Y_s^{(n)}, \bar{Y}_s^{(n)}) \end{bmatrix},$$

and $e = [1, 1, \cdots, 1]^T \in R^s$.

It is well known that a method (A, b, c) is said to be algebraically stable if $B = \mathrm{diag}(b_1, b_2, \cdots, b_s) \geq 0$ and the matrix

$$BA + A^T B - bb^T$$

is nonnegative definite(cf.[3]). A method is said to be diagonally stable if there exists an $s \times s$ diagonal matrix $Q > 0$ such that the matrix $QA + A^T Q$ is positive definite. A method is said to have stage order p if p is the largest integer such that the following simplifying conditions(cf.[5]) hold,

$$B(p): \quad b^T c^{j-1} = 1/j, \quad j = 1, 2, \cdots, p,$$

$$C(p): \quad Ac^{j-1} = c^j/j, \quad j = 1, 2, \cdots, p,$$

with $c^j = (c_1^j, c_2^j, \cdots, c_s^j)^T$. For the definitions of BS-stability and BSI-stability we refer to [8,13].

3. Main Results and Their Proofs

In this section, we focus on the D-convergence analysis of Runge-Kutta methods. For the sake of simplicity, we always assume that all constants h_i, γ_i and d_i used later are dependent only on the method, some of the bounds M_i, the parameters α, β and τ.

Let p be the stage order of the method (A, b, c). Let $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_s)^T \in R^s$ and $\sigma_0 \in R$ be defined by

$$\sigma = (p!)^{-1}((p+1)^{-1} c^{p+1} - Ac^p), \quad \sigma_0 = (p!)^{-1}((p+1)^{-1} - b^T c^p). \tag{3.1}$$

For any $n \geq 0, R^{(n)} = (R_1^{(n)^T}, R_2^{(n)^T}, \cdots, R_s^{(n)^T})^T$ and $R_0^{(n)}$ are defined by

$$Y(t_n) = ez(t_n) + hAF(t_n, Y(t_n), Y(t_n - \tau)) + R^{(n)}, \tag{3.2a}$$

$$z(t_{n+1}) = z(t_n) + hb^T F(t_n, Y(t_n), Y(t_n - \tau)) + R_0^{(n)}, \tag{3.2b}$$

where the functions $Y(t)$ and $z(t)$ are defined by

$$Y(t) = (y(t + c_1 h)^T, y(t + c_2 h)^T, \cdots, y(t + c_s h)^T)^T, \tag{3.3}$$

$$z(t) = y(t) + \sigma_1 h^{p+1} y^{(p+1)}(t). \tag{3.4}$$

Using Taylor expansion and the conditions $B(p)$ and $C(p)$, we can easily obtain the following results.

Theorem 3.1. *Suppose the method (A, b, c) has stage order p. Then*

$$\|R_i^{(n)} - (\sigma_i - \sigma_1)h^{p+1}y^{(p+1)}(t_n)\| \le \frac{h^{p+2}}{(p+2)!}M_{p+2}(|c_i|^{p+2} + (p+2)\sum_{j=1}^{s}|a_{ij}c_j^{p+1}|), i = 1, 2, \cdots, s, \tag{3.5}$$

$$\|R_0^{(n)} - \sigma_0 h^{p+1}y^{(p+1)}(t_n)\| \le \frac{h^{p+2}}{(p+2)!}M_{p+2}(1 + |\sigma_1|(p+2)! + (p+2)\sum_{j=1}^{s}|b_j c_j^{p+1}|). \tag{3.6}$$

Lemma 3.2. *(cf. [8,13]) The method is BS-stable and BSI-stable if it is diagonally stable.*

Now for any $n \ge 0$ we define $\hat{Y}^{(n)}$ and $\hat{y}_{n+1}$ by

$$\hat{Y}^{(n)} = ez(t_n) + hAF(t_n, \hat{Y}^{(n)}, Y(t_n - \tau)), \tag{3.7a}$$

$$\hat{y}_{n+1} = z(t_n) + hb^T F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau)). \tag{3.7b}$$

It follows from (3.2) and (3.7) that

$$Y(t_n) - \hat{Y}^{(n)} = hA[F(t_n, Y(t_n), Y(t_n - \tau)) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))] + R^{(n)}, \tag{3.8a}$$

$$z(t_{n+1}) - \hat{y}_{n+1} = hb^T[F(t_n, Y(t_n), Y(t_n - \tau)) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))] + R_0^{(n)}. \tag{3.8b}$$

From the definition of BS-stability, we have

Lemma 3.3. *Suppose the method (A, b, c) is BS-stable, then there exist positive constants h_1 and d_1 such that*

$$\|z(t_{n+1}) - \hat{y}_{n+1}\| \le d_1(\|R^{(n)}\| + \|R_0^{(n)}\|), \quad h \in (0, h_1], n = 0, 1, 2, \cdots. \tag{3.9}$$

From the definition of BSI-stability, we have

Lemma 3.4. *Suppose the method (A, b, c) is BSI-stable, then there exist positive constants h_2 and d_2 such that for any $Z^{(n)} = (Z_1^{(n)^T}, Z_2^{(n)^T}, \cdots, Z_s^{(n)^T})^T \in C^{sN}$, we have*

$$\|Y^{(n)} - Z^{(n)}\|^2 \le d_2\|\Delta^{(n)}\|^2, \quad h \in (0, h_2], n = 0, 1, 2, \cdots, \tag{3.10}$$

where

$$\Delta^{(n)} = Y^{(n)} - Z^{(n)} - hA[F(t_n, Y^{(n)}, \bar{Y}^{(n)}) - F(t_n, Z^{(n)}, \bar{Y}^{(n)})]. \tag{3.11}$$

Now we give some estimates for the interpolation procedures.

Theorem 3.5. *For the interpolation procedure (2.7), we have*

$$\sum_{k=0}^{n}\|\bar{Y}^{(k)} - Y(t_k - \tau)\|^2 \le d_3\sum_{k=0}^{n}\|y_k - z(t_k)\|^2 + \gamma_1(n+1)(h^{2(p+1)} + h^{2(\mu+\nu+1)}), \tag{3.12}$$

where

$$\begin{aligned} L_0 &= \max_{-\mu \le i \le \nu}\sup_{x \in [0,1)}|L_i(x)|, \\ d_3 &= 2s(\mu+\nu+2)(\mu+\nu+1)L_0^2, \\ \gamma_1 &= s(\mu+\nu+2)\max(2(\mu+\nu+1)L_0^2\sigma_1^2 M_{p+1}^2, M_{\mu+\nu+1}^2). \end{aligned}$$

Proof. It follows from (2.7) that

$$\begin{aligned} \|\bar{Y}_j^{(k)} - y(t_k + c_j h - \tau)\| \le &\|\sum_{i=-\mu}^{\nu} L_i(x_j)(y_{k+l_j-m+i} - y(t_{k+l_j-m+i}))\| \\ &+ \|\sum_{i=-\mu}^{\nu} L_i(x_j)y(t_{k+l_j-m+i}) - y(t_{k+l_j-m} + x_j h)\|. \end{aligned} \tag{3.13}$$

From the remainder estimate of Lagrange's interpolation formula, we have

$$\begin{aligned} \|\sum_{i=-\mu}^{\nu} L_i(x_j)y(t_{k+l_j-m+i}) - y(t_{k+l_j-m} + x_j h)\| &\le \frac{M_{\mu+\nu+1}}{(\mu+\nu+1)!}h^{\mu+\nu+1}\prod_{i=-\mu}^{\nu}|x_j - i| \\ &\le M_{\mu+\nu+1}h^{\mu+\nu+1}. \end{aligned} \tag{3.14}$$

Therefore, from Cauchy inequality we further obtain

$$\|\bar{Y}_j^{(k)}-y(t_k+c_jh-\tau)\|^2 \le (\mu+\nu+2)[L_0^2\sum_{i=-\mu}^{\nu}\|y_{k+l_j-m+i}-y(t_{k+l_j-m+i})\|^2+M_{\mu+\nu+1}^2h^{2(\mu+\nu+1)}] \tag{3.15}$$

which gives

$$\sum_{k=0}^{n}\|\bar{Y}_j^{(k)}-y(t_k+c_jh-\tau)\|^2 \le (\mu+\nu+2)[(\mu+\nu+1)L_0^2\sum_{k=0}^{n}\|y_k-y(t_k)\|^2 +(n+1)M_{\mu+\nu+1}^2h^{2(\mu+\nu+1)}], \tag{3.16}$$

where we have used $y_k=y(t_k)$ for $k<0$. From (3.4), it is easily seen that

$$\sum_{k=0}^{n}\|\bar{Y}_j^{(k)}-y(t_k+c_jh-\tau)\|^2 \le (\mu+\nu+2)[2(\mu+\nu+1)L_0^2\sum_{k=0}^{n}(\|y_k-z(t_k)\|^2 +\sigma_1^2M_{p+1}^2h^{2(p+1)})+(n+1)M_{\mu+\nu+1}^2h^{2(\mu+\nu+1)}].$$

Then (3.12) holds, which completes the proof of Theorem 3.5.

Theorem 3.6. *Suppose the method (A,b,c) is BSI-stable. Then for the interpolation procedure (2.9), there exists positive constant h_3 such that*

$$\sum_{k=0}^{n}\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2 \le 6d_2d_3\sum_{k=0}^{n}[s\|y_k-z(t_k)\|^2+\|R^{(k)}\|^2]+2\gamma_1(n+1)h^{2(\mu+\nu+1)}, \quad h\in(0,h_3]. \tag{3.17}$$

Proof. It follows from (2.9) that

$$\|\bar{Y}_j^{(k)}-y(t_k+c_jh-\tau)\| \le \|\sum_{i=-\mu}^{\nu}L_i(\delta)(Y_j^{(k-m+i)}-y(t_{k-m+i}+c_jh))\| +\|\sum_{i=-\mu}^{\nu}L_i(\delta)y(t_{k-m+i}+c_jh)-y(t_{k-m}+c_jh+\delta h)\|. \tag{3.18}$$

From the remainder estimate of Lagrange's interpolation formula, we have

$$\|\sum_{i=-\mu}^{\nu}L_i(\delta)y(t_{k-m+i}+c_jh)-y(t_{k-m}+c_jh+\delta h)\| \le M_{\mu+\nu+1}h^{\mu+\nu+1}.$$

Then, from Cauchy inequality we further obtain

$$\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2 \le s(\mu+\nu+2)[L_0^2\sum_{i=-\mu}^{\nu}\|Y^{(k-m+i)}-Y(t_{k-m+i})\|^2+M_{\mu+\nu+1}^2h^{2(\mu+\nu+1)}]$$

which gives

$$\sum_{k=0}^{n}\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2 \le d_3\sum_{k=0}^{n}\|Y^{(k)}-Y(t_k)\|^2+\gamma_1(n+1)M_{\mu+\nu+1}^2h^{2(\mu+\nu+1)}, \tag{3.19}$$

where we have used $Y^{(k)}=Y(t_k)$ for $k<0$.

On the other hand, a combination of (2.11a) and (3.2a) leads to

$$Y^{(k)}-Y(t_k)=hA[F(t_k,Y^{(k)},\bar{Y}^{(k)})-F(t_k,Y(t_k),\bar{Y}^{(k)})] +hA[F(t_k,Y(t_k),\bar{Y}^{(k)})-F(t_k,Y(t_k),Y(t_k-\tau))]+e(y_k-z(t_k))-R^{(k)}.$$

It follows from Lemma 3.4, the condition (2.3) and Cauchy inequality that

$$\|Y^{(k)}-Y(t_k)\|^2 \le d_2\|hA[F(t_k,Y(t_k),\bar{Y}^{(k)})-F(t_k,Y(t_k),Y(t_k-\tau))]+e(y_k-z(t_k))-R^{(k)}\|^2 \le 3d_2[h^2\|A\|^2\beta^2\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2+s\|y_k-z(t_k)\|^2+\|R^{(k)}\|^2], h\in(0,h_2]. \tag{3.20}$$

Substituting (3.20) into (3.19), we obtain

$$\sum_{k=0}^{n}\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2 \le 3d_2d_3\sum_{k=0}^{n}[h^2\|A\|^2\beta^2\|\bar{Y}^{(k)}-Y(t_k-\tau)\|^2$$

$$+s\|y_k - z(t_k)\|^2 + \|R^{(k)}\|^2] + \gamma_1(n+1)M^2_{\mu+\nu+1}h^{2(\mu+\nu+1)}, h \in (0, h_2]$$

Let

$$h_3 = \begin{cases} h_2, & \beta = 0, \\ \min(h_2, (6d_2d_3\|A\|^2\beta^2)^{-1/2}), & \beta \neq 0, \end{cases} \tag{3.21}$$

then (3.17) holds, which completes the proof of Theorem 3.6.

Theorem 3.7. *Suppose the method (A, b, c) is algebraically stable, BSI-stable and BS-stable. Then there exist nonnegative constants γ_2, γ_3 and γ_4 such that*

$$\|y_{n+1} - z(t_{n+1})\|^2 \leq \|y_0 - z(t_0)\|^2 + \sum_{k=0}^{n}[\gamma_2 h\|y_k - z(t_k)\|^2 + \gamma_3 h\|\bar{Y}^{(k)} - Y(t_k - \tau)\|^2$$
$$+\gamma_4 h^{-1}(\|R^{(k)}\| + \|R_0^{(k)}\|)^2], h \in (0, h_0], \tag{3.22}$$

where $h_0 = \min(h_1, h_2, 1)$.

Proof. It follows from (2.11) and (3.7) that

$$Y^{(n)} - \hat{Y}^{(n)} = e(y_n - z(t_n)) + hAQ^{(n)}, \tag{3.23a}$$
$$y_{n+1} - \hat{y}_{n+1} = y_n - z(t_n) + hb^TQ^{(n)}, \tag{3.23b}$$

where

$$Q^{(n)} = F(t_n, Y^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau)).$$

In view of algebraic stability of the method, we have(cf.[3])

$$\|y_{n+1} - \hat{y}_{n+1}\|^2 - \|y_n - z(t_n)\|^2 - 2h\mathrm{Re}\langle Y^{(n)} - \hat{Y}^{(n)}, BQ^{(n)}\rangle$$
$$= -\langle Q^{(n)}, (BA + A^TB - bb^T)Q^{(n)}\rangle \leq 0. \tag{3.24}$$

On the other hand, (3.23a)can be written in the form

$$Y^{(n)} - \hat{Y}^{(n)} = hA[F(t_n, Y^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, \bar{Y}^{(n)})]$$
$$+hA[F(t_n, \hat{Y}^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))] + e(y_n - z(t_n)).$$

It follows from Lemma 3.4, the condition (2.3) and Cauchy inequality that

$$\|Y^{(n)} - \hat{Y}^{(n)}\|^2 \leq d_2\|hA[F(t_n, \hat{Y}^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))] + e(y_n - z(t_n))\|^2$$
$$\leq 2d_2[h^2\|A\|^2\beta^2\|\bar{Y}^{(n)} - Y(t_n - \tau)\|^2 + s\|y_n - z(t_n)\|^2], h \in (0, h_2]. \tag{3.25}$$

Let $\alpha_0 = \max(0, \alpha)$.Using the conditions (2.2) and (2.3), we further obtain

$$2\mathrm{Re}\langle Y^{(n)} - \hat{Y}^{(n)}, B[F(t_n, Y^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))]\rangle$$
$$= 2\mathrm{Re}\langle Y^{(n)} - \hat{Y}^{(n)}, B[F(t_n, Y^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, \bar{Y}^{(n)})]\rangle$$
$$+2\mathrm{Re}\langle Y^{(n)} - \hat{Y}^{(n)}, B[F(t_n, \hat{Y}^{(n)}, \bar{Y}^{(n)}) - F(t_n, \hat{Y}^{(n)}, Y(t_n - \tau))]\rangle$$
$$\leq 2\alpha_0\|B\|\|Y^{(n)} - \hat{Y}^{(n)}\|^2 + 2\beta\|B\|\|Y^{(n)} - \hat{Y}^{(n)}\| \cdot \|\bar{Y}^{(n)} - Y(t_n - \tau)\|$$
$$\leq (2\alpha_0 + \beta)\|B\|\|Y^{(n)} - \hat{Y}^{(n)}\|^2 + \beta\|B\|\|\bar{Y}^{(n)} - Y(t_n - \tau)\|^2$$
$$\leq \gamma_5\|y_n - z(t_n)\|^2 + \gamma_6\|\bar{Y}^{(n)} - Y(t_n - \tau)\|^2, \quad h \in (0, h_2], \tag{3.26}$$

where

$$\gamma_5 = 2sd_2(2\alpha_0 + \beta)\|B\|, \qquad \gamma_6 = (1 + 2d_2h_2^2\|A\|^2\beta(2\alpha_0 + \beta))\beta\|B\|.$$

Substituting (3.26) into (3.24), we get

$$\|y_{n+1} - \hat{y}_{n+1}\|^2 \leq (1 + \gamma_5 h)\|y_n - z(t_n)\|^2 + h\gamma_6\|\bar{Y}^{(n)} - Y(t_n - \tau)\|^2, h \in (0, h_2]. \tag{3.27}$$

On the other hand,

$$\|y_{n+1} - z(t_{n+1})\| \leq \|y_{n+1} - \hat{y}_{n+1}\| + \|\hat{y}_{n+1} - z(t_{n+1})\|.$$

Hence

$$\|y_{n+1} - z(t_{n+1})\|^2 \leq (1 + h)\|y_{n+1} - \hat{y}_{n+1}\|^2 + (1 + h)h^{-1}\|\hat{y}_{n+1} - z(t_{n+1})\|^2. \tag{3.28}$$

A combination of (3.28),(3.27) and (3.9) leads to

$$\|y_{n+1} - z(t_{n+1})\|^2 \leq (1 + \gamma_2 h)\|y_n - z(t_n)\|^2 + \gamma_3 h\|\bar{Y}^{(n)} - Y(t_n - \tau)\|^2$$

$$+\gamma_4 h^{-1}(\|R^{(n)}\| + \|R_0^{(n)}\|)^2, \quad h \in (0, h_0], \tag{3.29}$$

where

$$\gamma_2 = 2\gamma_5 + 1, \quad \gamma_3 = 2\gamma_6, \quad \gamma_4 = 2d_1^2.$$

By induction we can easily obtain that (3.22) holds. The proof is completed.

In the following, in order to avoid the complexity of expression, we use notation $O(h^p)$ to designate that there exists a constant C such that $O(h^p) \le Ch^p$.

For the process (2.4) with interpolation procedure (2.7), we have the following results.

Theorem 3.8. *Suppose the method* (A, b, c) *is algebraically stable and diagonally stable, and it has stage order* p. *Then*

(i) *the process (2.4)-(2.7) is D-convergent of order at least* $\min(p, \mu+\nu+1)$, *and the global error satisfies (3.32);*

(ii) *if* $\sigma_0 = 0$ *and there exists a real number* λ *such that* $\sigma = \lambda e$, *then the process (2.4)-(2.7) is D-convergent of order at least* $\min(p+1, \mu+\nu+1)$ *and the global error satisfies (3.34).*

Proof. A combination of Theorem 3.7 and Theorem 3.5 leads to

$$\|y_{n+1}-z(t_{n+1})\|^2 \le \|y_0-z(t_0)\|^2+(\gamma_2+\gamma_3 d_3)h\sum_{k=0}^{n}\|y_k-z(t_k)\|^2+\gamma_1\gamma_3 t_{n+1}(h^{2(p+1)}+h^{2(\mu+\nu+1)})$$
$$+\gamma_4 t_{n+1}h^{-2}\max_{0\le k\le n}(\|R^{(k)}\| + \|R_0^{(k)}\|)^2, h \in (0, h_0].$$

Using discrete Bellman inequality, we have

$$\|y_{n+1}-z(t_{n+1})\|^2 \le [\|y_0-z(t_0)\|^2+\gamma_1\gamma_3 t_{n+1}(h^{2(p+1)}+h^{2(\mu+\nu+1)})$$
$$+\gamma_4 t_{n+1}h^{-2}\max_{0\le k\le n}(\|R^{(k)}\| + \|R_0^{(k)}\|)^2]\exp[(\gamma_2+\gamma_3 d_3)t_{n+1}], \quad h \in (0, h_0]. \tag{3.30}$$

In view of (3.4) and Theorem 3.1, we have

$$\begin{aligned} \|y_j - y(t_j)\| &= \|y_j - z(t_j)\| + O(h^{p+1}), \quad j = 0, 1, \ldots, n+1, \\ \|R_i^{(n)}\| &= O(h^{p+1}), \quad i = 0, 1, \ldots, s, h \in (0, h_0]. \end{aligned} \tag{3.31}$$

Therefore,

$$\|y_{n+1}-y(t_{n+1})\| \le [\|y_0-y(t_0)\|+(1+t_{n+1})O(h^p+h^{\mu+\nu+1})]\exp[\frac{1}{2}(\gamma_2+\gamma_3 d_3)t_{n+1}], \quad h \in (0, h_0]. \tag{3.32}$$

This shows that the method is D-convergent of order at least $\min(p, \mu+\nu+1)$.

If $\sigma_0 = 0$ and there exists a real number λ such that $\sigma = \lambda e$, then it follows from Theorem 3.1 that

$$\|R_i^{(n)}\| = O(h^{p+2}), i = 0, 1, \ldots, s, h \in (0, h_0]. \tag{3.33}$$

In this case, we can further obtain

$$\|y_{n+1}-y(t_{n+1})\| \le [\|y_0-y(t_0)\|+(1+t_{n+1})O(h^{p+1}+h^{\mu+\nu+1})]\exp[\frac{1}{2}(\gamma_2+\gamma_3 d_3)t_{n+1}], \quad h \in (0, h_0]. \tag{3.34}$$

This shows the method is D-convergent of order at least $\min(p+1, \mu+\nu+1)$. The proof of Theorem 3.8 is completed.

For the process (2.4) with interpolation procedure (2.9), we have the following results.

Theorem 3.9. *Suppose the method* (A, b, c) *is algebraically stable and diagonally stable, and it has stage order* p. *Then*

(i) *the process (2.4)-(2.9) is D-convergent of order at least* $\min(p, \mu+\nu+1)$, *and the global error satisfies (3.36);*

(ii) *if* $\sigma_0 = 0$ *and there exists a real number* λ *such that* $\sigma = \lambda e$, *then the process (2.4)-(2.9) is D-convergent of order at least* $\min(p+1, \mu+\nu+1)$ *and the global error satisfies (3.37).*

Proof. A combination of Theorem 3.7 and Theorem 3.6 leads to

$$\|y_{n+1}-z(t_{n+1})\|^2 \le \|y_0-z(t_0)\|^2 + (\gamma_2+6s\gamma_3 d_2 d_3)h\sum_{k=0}^{n}\|y_k-z(t_k)\|^2 + 2\gamma_1\gamma_3 t_{n+1}h^{2(\mu+\nu+1)}$$
$$+(\gamma_4+6\gamma_3 d_2 d_3 h^2)t_{n+1}h^{-2}\max_{0\le k\le n}(\|R^{(k)}\|+\|R_0^{(k)}\|)^2, h\in(0,h_4],$$

where $h_4=\min(h_0,h_3)$. Using discrete Bellman inequality, we have

$$\|y_{n+1}-z(t_{n+1})\|^2 \le \{\|y_0-z(t_0)\|^2 + 2\gamma_1\gamma_3 t_{n+1}h^{2(\mu+\nu+1)} + (\gamma_4+6\gamma_3 d_2 d_3 h^2)t_{n+1}h^{-2}$$
$$\times \max_{0\le k\le n}(\|R^{(k)}\|+\|R_0^{(k)}\|)^2\}\exp[(\gamma_2+6s\gamma_3 d_2 d_3)t_{n+1}], \quad h\in(0,h_4]. \tag{3.35}$$

In view of (3.4) and (3.31), we have

$$\begin{aligned}\|y_{n+1}-y(t_{n+1})\| \le {} & [\|y_0-y(t_0)\| + (1+t_{n+1})O(h^p+h^{\mu+\nu+1})]\\ & \exp[\frac{1}{2}(\gamma_2+6s\gamma_3 d_2 d_3)t_{n+1}], \quad h\in(0,h_0].\end{aligned} \tag{3.36}$$

This shows that the method is D-convergent of order at least $\min(p,\mu+\nu+1)$.

If $\sigma_0=0$ and there exists a real number λ such that $\sigma=\lambda e$, then by (3.33) we can further obtain

$$\begin{aligned}\|y_{n+1}-y(t_{n+1})\| \le {} & [\|y_0-y(t_0)\| + (1+t_{n+1})O(h^{p+1}+h^{\mu+\nu+1})]\\ & \exp[\frac{1}{2}(\gamma_2+6s\gamma_3 d_2 d_3)t_{n+1}], \quad h\in(0,h_0].\end{aligned} \tag{3.37}$$

This shows the method is D-convergent of order at least $\min(p+1,\mu+\nu+1)$. The proof of Theorem 3.9 is completed.

Finally, for interpolation procedure (2.10) we note that a method is said to be natural if $b_j(c_i)=a_{ij}(1\le i,j\le s)$(cf.[14,15]). In this case, it is easily seen that $\bar{Y}_j^{(n)}=Y_j^{(n-m)}$ if stepsize h satisfies the constraint (2.6). Application of Theorem 3.9 leads to the following results.

Theorem 3.10. *Suppose the method (A,b,c) is algebraically stable and diagonally stable, and it has stage order p. Then*

(i) *the natural Runge-Kutta method (2.4)-(2.10) with the constrained mesh (2.6) is D-convergent of order at least p;*

(ii) *if $\sigma_0=0$ and there exists a real number λ such that $\sigma=\lambda e$, then the natural Runge-Kutta method (2.4)-(2.10) with the constrained mesh (2.6) is D-convergent of order at least $p+1$.*

Remark 3.11. Specializing the above three theorems to the case of $\beta=0$, we obtain immediately the well known related B-convergence results presented by Frank et al. [7,8] and Burrage and Hundsdorfer [4].

Remark 3.12. It is well known that, under some slight assumptions, the algebraic conditions of the above three theorems are necessary to B-convergence. In view of Proposition 2.3, they are also neccesary to D-convergence under the same assumptions.

4. Equations with Several Delays

Consider the following equation with several delays

$$\begin{cases} y'(t)=f(t,y(t),y(t-\tau_1),y(t-\tau_2),\cdots,y(t-\tau_r)), \quad t\ge 0,\\ y(t)=\phi_1(t), \quad t\le 0.\end{cases} \tag{4.1}$$

Because $\tau_1,\tau_2,\cdots,\tau_r$ are positive constants, there are no additional difficulties with respect to (4.1). We can similarly define the concepts of convergence in this case. All results given in this paper can be modified easily to this more general situation. But we do not list them here for the sake of brevity.

Acknowledgments. The authors would like to thank an anonymous referee for his many valuable comments.

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