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MATRIX ANALYSIS ON LEADING TERM OF CONDITION NUMBER FOR ADDITIVE SCHWARZ METHODS[*1)]

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Abstract

It is well known the order of preconditioned matrix by using additive Schwarz methods. In order to estimate the resulted PCG iteration counts, the related leading term brfore the order is given in this paper.

Key words: Additive Schwarz method, PCG, Matrix analysis.

1. Introduction

Let us consider the following second order elliptic boundary value problem:

$$\mathcal{L}u = f \quad in \quad \Omega \tag{1}$$

$$u = 0 \quad on \quad \partial\Omega \tag{2}$$

where $\mathcal{L}$ is a self-adjoint positive operator and

$$\Omega \subset \mathcal{R}^d \quad (1 \le d \le 3)$$

is a polyhedral domain.

Using weak solution it leads to a discrete equation

$$Au = f \tag{3}$$

with

$$A = (\alpha_{ij}), \quad \alpha_{ij} = \mathcal{A}(\phi_i, \phi_j) \tag{4}$$

where $\{\phi_i\}$ could be nodal basis consisting of piece-wise linear functions or other spline functions. It is well known that the coefficient matrix A is symmetry positive definite matrix with condition number

$$\kappa(A) := \frac{\lambda_{max}(A)}{\lambda_{min}(A)} = O(h^{-2}) \tag{5}$$

Furthermore, under rectangle uniform mesh it is easy to obtain the leading term of Discrete Laplacian condition number before the above order as follows which is independent on dimension.

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Dimension	*Scheme*	λ_{max}	λ_{min}	κ
$1-D$	$3-point$	$2-(\pi h)^2$	$(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$
$2-D$	$5-point$	$4-2(\pi h)^2$	$2(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$
$3-D$	$7-point$	$6-3(\pi h)^2$	$3(\pi h)^2$	$\frac{2}{\pi^2}h^{-2}$

When we use conjugate gradient algorithm for solving the system, for a given tolerance, the iteration number will proportional to h^{-1}. This convergent rate is really slow for a large scale problems. It is our aim to study how better is better the preconditioner by using additive Schwarz method.

Suppose there are two subdomain partitions, one is without overlapping

$$\Omega = \cup\Omega_i \quad with \quad \Omega_i \cap \Omega_j = \phi \quad if \quad i \neq j$$

and another is with overlapping

$$\Omega = \cup\hat{\Omega}_i \quad with \quad \hat{\Omega}_i \cap \hat{\Omega}_j \neq \phi \quad if \quad i \neq j$$

and denote A_i be a matrix representation which is the restriction of the original operator A over the subdomain Ω_i. Without considering unknowns permutation appropriately, then the related preconditioner can be written as

$$B_i = \begin{bmatrix} A_i^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and the corresponding matrix A has the following block form

$$A = \begin{bmatrix} A_i & A_{ij} \\ A_{ij}^t & A_j \end{bmatrix}, \qquad B_i A = \begin{bmatrix} I & A_i^{-1}A_{ij} \\ 0 & 0 \end{bmatrix}$$

Hence, in general we obtain a whole subdomain preconditioner B_s as follows:

$$B_s = \sum B_i, \quad with \quad B_i = R_i^t A_i^{-1} R_i \tag{6}$$

where R_i is a truncated permutation matrix from Ω to the subdomain Ω_i.

Some questions now are arisen on the above algorithm as a PCG preconditioner . What is efficiency of the above parallel ASM ? How fast the related PCG iteration does converge? How does the eigenvalues of preconditioned matrix B_sA distribute ? How to estimate the largest and smallest eigenlavues of B_sA ?

This estimation can be reduced to special generalized eigen-decomposition

$$Au = \lambda B_s^{-1} u$$

From finite element theory, it has been known by using energy norm estimation

$$\lambda_{min}(B_sA) = O((hH)^{-1}), \quad \lambda_{max}(B_sA) = O(1).$$

where h is the smallest mesh size on subdomains, and H is the subdomain width.

In practical, it is need to estimation the leading term before the order to estimate the PCG iteration counts more precisely for a given tolerance.

2. Some Properties of Related Matrices Q_s and Q_Γ

With our assumption, A is a M-matrix. Hence, matrix each B_i is non-negative and B_iA is a matrix with all non-positive off-diagonals, so does B_sA. Based on discrete maximum principle, it is not hard to show that

$$\lambda_{max}(B_sA) \le 1 + max\{d_i\},$$

where d_i – overlapping multiple index among subdomains.

In order to estimate the smallest eigenvalue of preconditioned matrix B_sA more clearly, we turn to study a related matrix $Q_s = I - B_sA$. It is clear that after a proper permutation Q_s can be represented in terms of

$$Q_s = \begin{bmatrix} D & * \\ 0 & Q_\Gamma \end{bmatrix}$$

where D is diagonal with non-positive integer

$$D = diag\{1 - d_i\},$$

We call Q_Γ as equivalent interior boundary matrix to Q_s. It is clear that Q_Γ consists from Q_s deleting related columns with zero on all non-diagonals and related rows. The size of Q_Γ is much smaller than Q_s. In fact Q_Γ can be seen as a restriction on the discrete overlapping region of Q_s. All eigenvalues of Q_Γ belong to $\sigma(Q_s)$. In particular, if λ_{Max} is the largest eigenvalue of Q_Γ, then $\mu_{min} = 1 - \lambda_{Max}$ is the smallest eigenvalue of B_sA.

Hence, the smallest eigenvalue of B_sA is just the same as the largest eigenvalue of Q_Γ which is only defined on the much smaller discrete overlapping region. Furthermore, it is easy to know Q_Γ is non-negative with all zeroes along the main diagonals. The eigenvector u_Γ, corresponding λ_{Max}, is non-negative, too. In another word, the matrix Q_Γ has unique positive eigenvector u_Γ, and $Q_\Gamma u_\Gamma, = \lambda_{Max} u_\Gamma$. In the matrix point of view, to get an estimation of λ_{Max} $\mu \le \lambda_{Max} \le \lambda$, it is sufficient to find a positive vector z such that $\mu z \le Q_\Gamma z \le \lambda z$,

3. 1-D Case: Model Problem

$$-(pu')' = f, \quad u(0) = u(1) = 0 \tag{7}$$

where $p(x) \ge p_0 > 0$. For a given mesh partition

$$\Delta : 0 = x_0 < x_1 < ... < x_{N+1} = 1$$

with δ mesh size overlapping subdomain partition

$$s_1 = n_1, s_2 = n_1 + n_2, s_m = \sum_{k=1}^{m} n_k = N, \quad with \;\; n_k > 1, k = 1, 2, ..., m$$

$H_1 = x_{s_1+\delta} - x_0, \quad H_k = x_{s_k+\delta} - x_{s_{k-1}-\delta}, \quad (s = 2, ..., m-1) \quad H_m = x_1 - x_{s_{m-1}-\delta}.$

By using finite element method or difference method, the resulting stiffness matrix A is tridiagonal of order N and its coefficients are equal to

$$\alpha_{ij} = \mathcal{A}(\phi_i, \phi_j) = \{ \begin{array}{cc} 0 & if \quad |i-j| > 1 \\ -p_{i-1/2}h_i^{-1} & if \quad j = i-1 \\ p_{i-1/2}h_i^{-1} + p_{i+1/2}h_{i+1}^{-1} & if \quad j = i \\ -p_{i+1/2}h_{i+1}^{-1} & if \quad j = i+1 \end{array} \tag{8}$$

where $p_{i-1/2} = (p_{i-1} + p_i)/2, \quad p_{i+1/2} = (p_{i+1} + p_i)/2.$

If we introduce so-called generalized mesh $\hat{h}_i = \frac{h_i}{p_{i-1/2}}$, the general variable coefficient case can be taken as a non-uniform mesh with constant coefficients.

Using ASM to the above matrix A leads to

$$B_k A = \begin{bmatrix} 0 & 0 & & ... \\ ... & ... & & ... \\ ... & A_{k,k}^{-1}A_{k,k-1} & I & A_{k,k}^{-1}A_{k,k+1} \\ 0 & 0 & & ... \\ ... & ... & & ... \end{bmatrix},$$

where $rank(A_{11}) = n_1 + \delta, \quad rank(A_{kk}) = n_k + 2\delta, \quad k = 2, ..., m-1, \quad rank(A_{mm}) = n_m + \delta.$

Let $A_{k,k}^{-1}A_{k,k-1} = \{0, 0, ..., 0, u^{k-}\}, \quad A_{k,k}^{-1}A_{k,k+1} = \{u^{k+}, 0, ..., 0, 0\}$. where u^{k-} and u^{k+} are the solution of related two-point boundary value problems as follows

$$\mathcal{L}u = 0, \quad u(x_{n_k-\delta}) = 1, \quad u(x_{n_k+\delta}) = 0;$$

and

$$\mathcal{L}u = 0, \quad u(x_{n_k+\delta}) = 1, \quad u(x_{n_k-\delta}) = 0;$$

The interior boundary matrix $Q_\Gamma = (I - B_s A)|_\Gamma$ becomes

$$Q_\Gamma = \begin{bmatrix} 0_2 & q_{12} & 0_2 & ... & & & \\ q_{21} & 0_2 & q_{23} & 0_2 & ... & & \\ & ... & & & ... & & \\ & ... & q_{j,j-1} & 0_2 & q_{j,j+1} ... & & \\ & ... & & & ... & & \\ & & & ... & 0_2 & q_{m+1,m} & 0_2 \end{bmatrix},$$

where 0_2 – 2x2 zero matrix, caused from two-point boundary value problem.

$$0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad q_{j,j-1} = \begin{bmatrix} 0 & -(1-\beta_{j-1}) \\ 0 & -\alpha_j \end{bmatrix}, \quad q_{j,j+1} = \begin{bmatrix} -\beta_{j-1} & 0 \\ -(1-\alpha_j) & 0 \end{bmatrix}$$

In the uniform mesh case $\alpha_j = \alpha, \beta_j = \beta$

$$\Delta_k(\lambda) = \{\lambda^2 - (1 - \alpha - \beta)\}\Delta_{k-1}(\lambda) - \lambda^2\alpha\beta\Delta_{k-2}(\lambda),$$

$$\Delta_0(\lambda) = 1, \qquad \Delta_1(\lambda) = \lambda^2 - (1-\alpha)(1-\beta).$$

A general representation of $\Delta_k(\lambda)$ was given by Jiachang Sun and Tony Chan in [1] as follows

$$\Delta_m(\lambda) = \frac{(\alpha\beta)^{m/2}\lambda^{m-1}}{\sin\phi}\{\lambda\sin(m+1)\phi - (\alpha\beta)^{1/2}\sin m\phi\}, \tag{9}$$

where $\cos\phi = \frac{\lambda^2 - (1-\alpha-\beta)}{2(\alpha\beta)^{1/2}\lambda}$. Thus, there is an estimation of eigenvalues for small α, β

$$\lambda_k = (\alpha\beta)^{1/2}\cos\phi_k \pm \sqrt{\alpha\beta(\cos\phi_k)^2 + (1-\alpha-\beta)}, \tag{10}$$

where $\phi_k = \frac{k\pi}{m+1}, (k = 1, 2, ..., m)$.

In particular, the minimum eigenvalue is equal to [1]

$$\lambda_{min}(B_s A) \approx \frac{\pi^2}{2}Hh.$$

For non-uniform case, it is not difficult to show that the above argument still holds if H and h are resplaced by the corresponding average mesh width of subdomain H_o and minimal generalized mesh width $\hat{h}_o$ of overlapping among subdomains, respectively:

$$\lambda_{max}(Q_s) = \lambda_{max}(Q_\Gamma) \approx 1 - \frac{\pi^2}{2} H_o \hat{h}_o, \tag{11}$$

It is worth to point that eigenvalues of Q_s depend on partition of subdomains in the overlapping area only. They are independent upon partition within non-overlapping of any subdomains.

4. Main Idea for Reducing Dimension Procedure

In this section we describe the main idea how to estimate a high-dimension eigenvalue problem via related lower dimension case, if the original domain can decomposed by tensor product of related lower dimension.

Step 1: Find related lower dimension interior boundary eigenvalue.

Step 2: Extended the interior eigenvector to whole lower dimension by solving piecewise discrete boundary value problems on each subdomain with overlapping.

Step 3: Expand the lower dimension eigenvector to the higher dimension eigenvector via tensor product.

Step 4: Localization required eigenvalue on original dimension by using comparison theorem based on maximum principle.

There are some special cases as follows:

Case 1. Reducing dimension degree: from 2-D to 1-D .
Suppose a 2-D discrete operator can be decomposed into 1-D

$$\mathcal{L} = \mathcal{L}_x + \mathcal{L}_y$$

and the discrete domain can also be decomposed as

$$\Omega^h = \Omega_x^h \otimes \Omega_y^h$$

. In this case we can decompose the discrete matrix via Kronecker product

$$A = A_x \otimes I_y + I_x \otimes A_y$$

and we may take an overlapping preconditioner as

$$B_s^{-1} = B_{sx}^{-1} \otimes I_y + I_x \otimes B_{sy}^{-1}.$$

Case 2. Reducing dimension degree: from 3-D to 1-D directly.
Suppose a 3-D discrete operator can be decomposed into 1-D

$$\mathcal{L} = \mathcal{L}_x + \mathcal{L}_y + \mathcal{L}_z.$$

and the discrete domain can also be decomposed as

$$\Omega^h = \Omega_x^h \otimes \Omega_y^h \otimes \Omega_z^h.$$

In this case we can decompose the discrete matrix via Kronecker product

$$A = A_x \otimes (I_y \otimes I_z) + I_x \otimes A_y \otimes I_z + (I_x \otimes I_y) \otimes I_z.$$

and we may take an overlapping preconditioner as

$$B_s = B_{sx}^{-1} \otimes (I_y \otimes I_z) + I_x \otimes B_{sy}^{-1} \otimes I_z + (I_x \otimes I_y) \otimes B_{sz}^{-1}.$$

Case 3. Reducing dimension degree: from 3-D to 2-D and 1-D.
Suppose a 3-D discrete operator can be decomposed into 2-D plus 1-D

$$\mathcal{L} = \mathcal{L}_{x,y} + \mathcal{L}_z.$$

and the discrete domain can also be decomposed as

$$\Omega^h = \Omega^h_{x,y} \otimes \Omega^h_z.$$

In this case we can decompose the discrete matrix via Kronecker product

$$A = A_{x,y} \otimes I_z + +I_{x,y} \otimes A_z.$$

and we may take an overlapping preconditioner as

$$B_s = B_{s_{x,y}}^{-1} \otimes I_z + I_{x,y} \otimes B_{sz}^{-1}$$

5. Some Examples with First Boundary Value Condition

We consider rectangle domain in the following examples.

Example 1.

$$\mathcal{L}u = -\frac{\partial}{\partial x} a_1(x) \frac{\partial u(x,y)}{\partial x} - \frac{\partial}{\partial y} a_2(y) \frac{\partial u(x,y)}{\partial y}, \tag{12}$$

$$u|_{\partial\Omega} = f$$

$$\mathcal{L}_x u = -\frac{\partial}{\partial x} a_1(x) \frac{\partial u(x,y)}{\partial x},$$

$$\mathcal{L}_y u = -\frac{\partial}{\partial y} a_2(y) \frac{\partial u(x,y)}{\partial y}$$

where $a_1(x) > 0, a_2(y) > 0$.

Suppose the discrete domain can also be decomposed as

$$\Omega^h = \Omega^h_x \otimes \Omega^h_y.$$

Thus, via Kronecker product we can decompose the discrete $M-$ matrix as

$$A = A_x \otimes I_y + I_x \otimes A_y$$

and the related overlapping preconditioner as

$$B_s^{-1} = B_{sx}^{-1} \otimes I_y + I_x \otimes B_{sy}^{-1}.$$

Step 1: Find related 1-D interior boundary largest eigenvalue.

$$Q_{\Gamma x} u_{\Gamma x} = \lambda_x u_{\Gamma x}, \quad u_{\Gamma x} \geq 0,$$

$$Q_{\Gamma y}u_{\Gamma y} = \lambda_y u_{\Gamma y}, \quad u_{\Gamma y} \geq 0,$$

where both eigenvectors $u_{\Gamma x}$ and $u_{\Gamma y}$, corresponding to own largest eigenvalue of each positive matrix $Q_{\Gamma x}$ and $Q_{\Gamma y}$, are positive.

Step 2: Extended the interior eigenvector to whole 1-D via piecewise discrete boundary value problems.

$$\mathcal{L}_x u_x = 0, \quad in \quad \Omega^h_{x,i},$$
$$u_x = u_{\Gamma x} \quad on \quad \partial\Omega^h_{x,i}$$

and

$$\mathcal{L}_y u_y = 0, \quad in \quad \Omega^h_{y,j},$$
$$u_y = u_{\Gamma y} \quad on \quad \partial\Omega^h_{y,j}$$

Note that, under the assumption, the above two vectors u_x and u_y keep positive based on maximum principle. And vectors $B^{-1}_{sx}u_x$ and $B^{-1}_{sy}u_y$ are all non-negative.

Step 3: Expand the 1-D vector to a 2-D vector via Kronecker product. Set a 2-D positive vector u such that $u = u_x \otimes u_y$

Then

$Au = (A_x \otimes I_y + I_x \otimes A_y)(u_x \otimes u_y) = (A_x u_x) \otimes u_y + u_x \otimes (A_y u_y)$
$= \lambda_x(B^{-1}_{sx}u_x) \otimes u_y + \lambda_y u_x \otimes (B^{-1}_{sy}u_y)$

Note that all values and vectors appeared in the above formula are non-negative.

Step 4: Localization of the original smallest eigenvalue.

$$min(\lambda_x, \lambda_y)B^{-1}_s u \leq Au \leq max(\lambda_x, \lambda_y)B^{-1}_s u$$

Finally, it leads to an estimation of the smallest eigenvalue for $B_s A$ as follows

$$min(\lambda_x, \lambda_y) \leq \lambda_{min}(B_s A) \leq max(\lambda_x, \lambda_y)$$

Example 2. Consider general 2-D second order self-adjoint case

$$\mathcal{L}u = -\frac{\partial}{\partial x}a_1(x,y)\frac{\partial u(x,y)}{\partial x} - \frac{\partial}{\partial y}a_2(x,y)\frac{\partial u(x,y)}{\partial y} \tag{13}$$

$$\mathcal{L}_x u = -\frac{\partial}{\partial x}a_1(x,y)\frac{\partial u(x,y)}{\partial x}, \quad \mathcal{L}_y u = -\frac{\partial}{\partial y}a_2(x,y)\frac{\partial u(x,y)}{\partial y}$$

where $a_1(x,y) \geq a_{10} > 0, a_2(x,y) \geq a_{20} > 0$.

Let

$$\lambda_1 = min_{y\in\Gamma}\{\lambda_{min}(B_1 A_1(y))\},$$
$$\lambda_2 = min_{x\in\Gamma}\{\lambda_{min}(B_2 A_2(x))\}$$

then

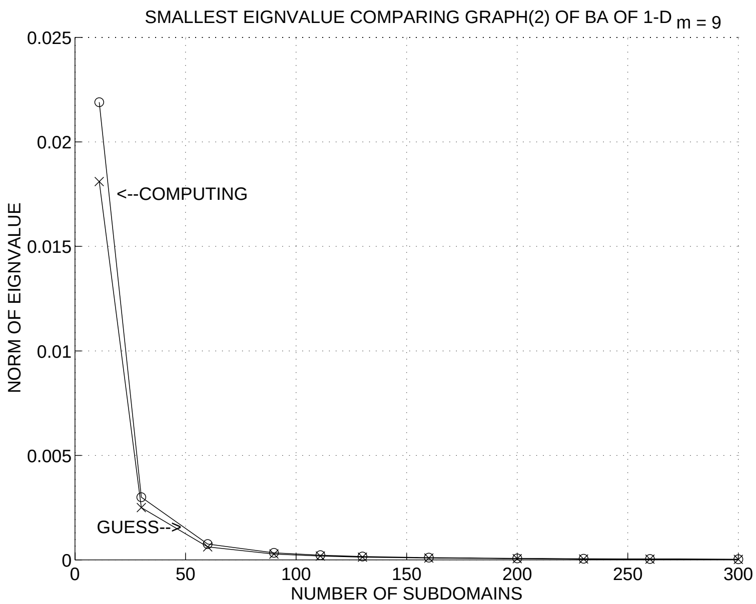
$$mini\{\lambda_1, \lambda_2\} \leq \lambda_{min}(BA) \leq max\{\lambda_1, \lambda_2\}.$$

6. Numerical Tests

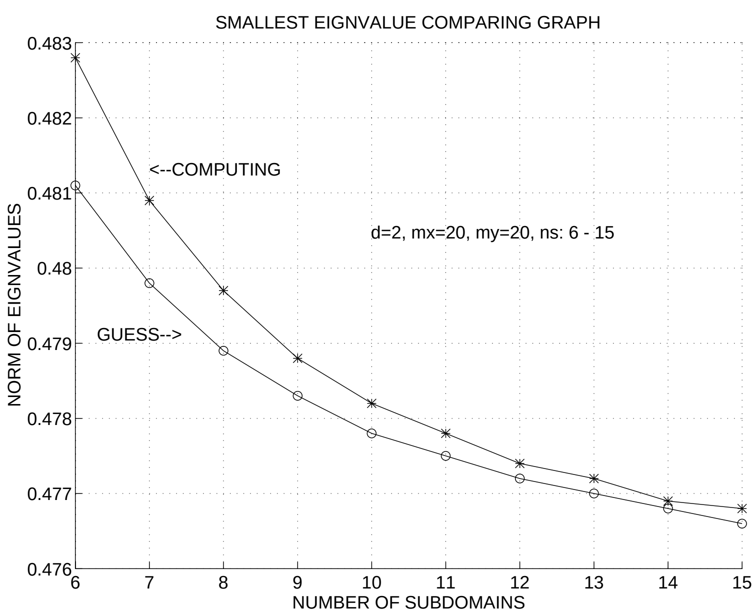
The following numerical tests have been done by Pan Feng and He Wei-jun with whom we have helpful discussion.

In following figures, *computing* means the smallest eigenvalue computed by MATLAB and *GUESS* means the related estimation based on our analysis.

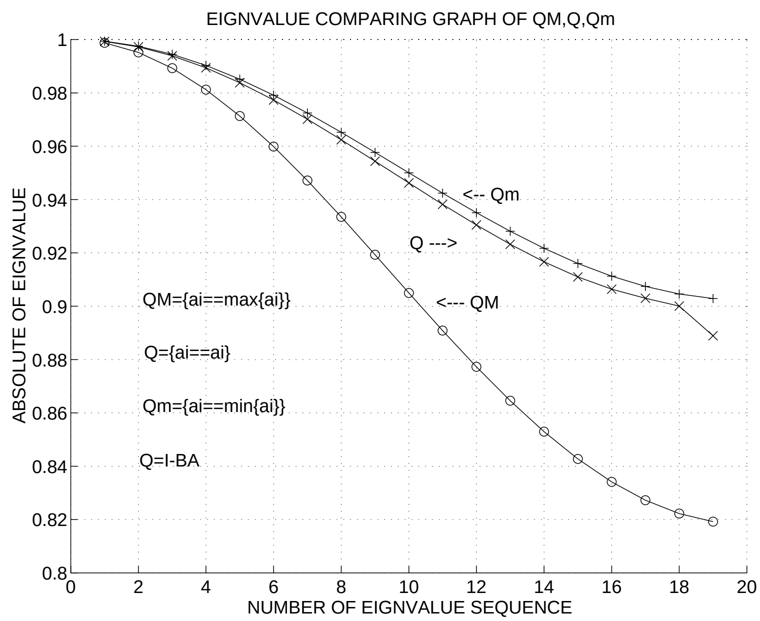
Test 1. 1-D model problem with uniform mesh $-u" = f, \quad u(o) = u(1) = 0$, ns – number of subdomains, m – mesh number within a subdomain, $h^{-1} = m \times ns + 1$ d – overlapping mesh size =2.



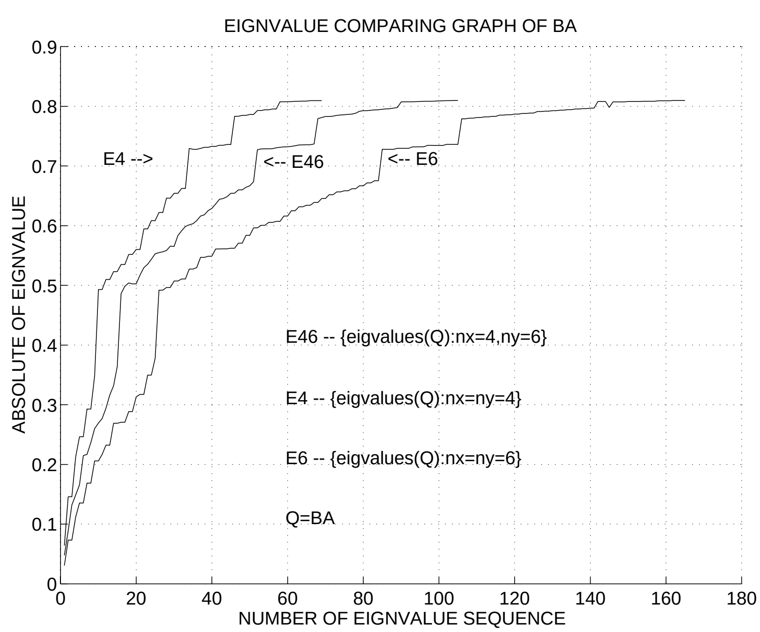
Test 2. 1-D non-uniform case



Test 3. 2-D Laplace equation with rectangle partition and block subdomains



Test 4. 2-D Laplace equation with rectangle partition and string subdomain



7. Conclusion

Under a block-rectangle subdomain partitions, in this paper we have got some main results on the smallest eigenvalue estimation of B_sA as follows

$$\lambda_{min}(B_sA) \approx \frac{\pi^2}{2} H_o \hat{h}_o, \tag{14}$$

H_o – Average mesh width of related 1-D subdomain,
$\hat{h}_o$ – Minimal generalized mesh width of overlapping among 1-D related subdomains.

Analogy, for a strip subdomain partitions the estimation should be modifies as

$$\lambda_{min}(B_sA) \approx \frac{\pi^3}{2} H_o \hat{h}_o, \tag{15}$$

The detailed analysis will be appeared later.

The above leading term estimations can be used for many high dimension problem decomposed from lower dimension.

It seems for strip case the leading term is π times bigger than for block case. However, it does not mean strip decomposition is better than block decomposition for high dimension. Because string subdomain solver is much expensive!

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