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CONSTRAINED RATIONAL CUBIC SPLINE AND ITS APPLICATION*1)

Qi Duan Huan-ling Zhang Xiang Lai Nan Xie
(*Department of Applied Mathematics, Shandong University of Technology, Jinan 250061, China*)

Fu-hua (Frank) Cheng
(*Department of Computer Science, University of Kentucky, Lexington, KY 40506, USA*)

Abstract

In this paper, a kind of rational cubic interpolation function with linear denominator is constructed. The constrained interpolation with constraint on shape of the interpolating curves and on the second-order derivative of the interpolating function is studied by using this interpolation, and as the consequent result, the convex interpolation conditions have been derived.

Key words: Rational spline, Constrained design, Constrained interpolation, Convex interpolation, Shape control.

1. Introduction

Design of high quality, manufacturable surfaces, such as the outer shape of a ship, car or aeroplane, is an important yet challenging task in today's manufacturing industries. Although significant progress has been made in the last decade in developing and commercializing production quality CAD tools, demand for more effective tools is still high due to the ever increase in model complexity and the needs to address and incorporate manufacturing requirements in the early stage of surface design. Within this content, *constrained design* has been identified as one of the surface design problems that need to be solved [1]. This problem deals with control of the bound of curve/surface, the shape and the curvature in the design process.

Spline interpolation is a useful tool for curve and surface design [2,3,4). But in general, the common spline interpolation, such as B-spline, cubic spline, is a kind of fixed interpolation. It means that the shape of the interpolating curve is fixed for the given interpolating data. If one wishes to modify the shape of the interpolating curve, the interpolating data need to be changed. How the shape of the curve can be modified under the condition that the given data are not changed? In recent years, based on the idea of adding the parameters in the interpolating function, the rational spline have been of interest[6,7,8,9]. But because of those rational cubic splines with quadratic or cubic denominators, it is not convenient to do the tasks such as constraining the interpolating curves to be in the given region or constraining them to be concave or convex. In [10], a rational cubic spline with linear denominators based on function values has been constructed, and it can be used to do the constraint in some case. In this paper, a kind of rational cubic interpolation function with linear denominator based on the function values and the derivatives is constructed. The method proposed is C^1 continuous but under certain conditions, a C^2 rational spline can be got.

This paper considers the constrained curve interpolation problem both on the shape and the second-order derivative of the interpolant. The sufficient conditions for the interpolating curves to be above (or below) a straight line and/or a quadratic curve in an individual knot interval

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and the necessary and sufficient conditions for constraining the second-order derivative of the interpolating function are derived. In fact, not only necessary and sufficient conditions are given, but the existence conditions for this constrained rational cubic interpolation have been derived also. More interested is that because the second-order derivative of the interpolant function can be controlled by this method, the convex interpolation conditions can be got easily as the consequent result.

This paper is arranged as follows. In Section 2, the general form of the rational cubic spline curves is given, and some of its properties is considered, including the tri-diagonal system of equations for the construction of such a C^1 rational cubic curve. Section 3 is about the shape constraint problem. Section 4 is about second-order derivative control problem. The convex interpolation conditions which are the consequent result of the second-order derivative constraint and its existence are considered in Section 5.

2. Rational Cubic Interpolation

Let $\{f_i, i = 0, 1, \ldots n\}$ be a given set of data points, where $f_i = f(t_i)$ and $t_0 < t_1 < \cdots < t_n$ is the knot spacing. Also, let $\{d_i, i = 0, 1, \ldots n\}$ denote the first-order derivatives of the being interpolated function $f(t)$ at the knots. Define the C^1-continuous, piecewise rational cubic function by

$$P(t)\big|_{[t_i, t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \tag{1}$$

where

$$\begin{aligned} p_i(t) &= (1-\theta)^3 \alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta) W_i + \theta^3 \beta_i f_{i+1}, \\ q_i(t) &= (1-\theta)\alpha_i + \theta \beta_i, \\ \theta &= (t - t_i)/h_i, \\ h_i &= t_{i+1} - t_i, \end{aligned}$$

and

$$\begin{aligned} V_i &= (2\alpha_i + \beta_i) f_i + \alpha_i h_i d_i, \\ W_i &= (\alpha_i + 2\beta_i) f_{i+1} - \beta_i h_i d_{i+1}, \end{aligned}$$

with $\alpha_i, \beta_i > 0$. $P(t)$ satisfies $P(t_i) = f_i, \;\; P'(t_i) = d_i, \qquad i = 0, 1, \cdots, n.$

$P(t)$ is the standard cubic Hermite interpolant if $\alpha_i = \beta_i$. If $d_i, i = 0, 1, \cdots, n$, are not fixed, $P(t)$ can be a C^2 rational cubic spline by requiring

$$P''(t_i+) = P''(t_i-)$$

for $i = 1, 2, \cdots, n-1$. This condition leads to the following tri-diagonal system of linear equations:

$$\begin{aligned} & h_i \frac{\alpha_{i-1}}{\beta_{i-1}} d_{i-1} + \left(h_i\left(1 + \frac{\alpha_{i-1}}{\beta_{i-1}}\right) + h_{i-1}\left(1 + \frac{\beta_i}{\alpha_i}\right)\right) d_i + h_{i-1} \frac{\beta_i}{\alpha_i} d_{i+1} \\ = \;& h_{i-1}\left(1 + 2\frac{\beta_i}{\alpha_i}\right)\Delta_i + h_i\left(1 + 2\frac{\alpha_{i-1}}{\beta_{i-1}}\right)\Delta_{i-1}; i = 1, 2, \cdots, n-1, \end{aligned} \tag{2}$$

where

$$\Delta_i = (f_{i+1} - f_i)/h_i.$$

If the knots are equally spaced, equation (2) becomes

$$\begin{aligned} & \frac{\alpha_{i-1}}{\beta_{i-1}}d_{i-1} + (2 + \frac{\alpha_{i-1}}{\beta_{i-1}} + \frac{\beta_i}{\alpha_i})d_i + \frac{\beta_i}{\alpha_i}d_{i+1} \\ = & \ (1 + 2\frac{\beta_i}{\alpha_i})\Delta_i + (1 + 2\frac{\alpha_{i-1}}{\beta_{i-1}})\Delta_{i-1}; \qquad i = 1, 2, \cdots, n-1. \end{aligned} \tag{3}$$

Furthermore, if $\alpha_i = \beta_i$, then (3) becomes the well known tri-diagonal system for cubic spline

$$d_{i-1} + 4d_i + d_{i+1} = 3(\Delta_i + \Delta_{i-1}); \qquad i = 1, 2, \cdots, n-1. \tag{4}$$

3. Constrained Interpolation on Shape Control

Given a function $g(t)$ and a data set $\{(t_i, f_i, d_i) : i = 0, 1, \cdots, n\}$ with

$$f_i \geq g(t_i), \qquad i = 0, 1, \cdots, n$$

let $P(t)$ be a rational cubic function defined by (1), if $P(t) \geq g(t)$ for all $t \in [t_0, t_n]$, then $P(t)$ is called a *constrained interpolant* above $g(t)$. Within this content, consider the following two cases.

Case 1. Let $g(t)$ be the piecewise linear function defined on $[t_0, t_n]$ with joints at the partition $\Delta : t_0 < t_1 < \cdots < t_n$ and

$$f_i \geq g(t_i), \qquad i = 0, 1, \ldots, n.$$

Since $q_i(t) > 0$ for $t \in [t_i, t_{i+1}]$, then

$$P(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

is equivalent to

$$p_i(t) - q_i(t)g(t) \geq 0.$$

Let

$$U_i(t) = p_i(t) - q_i(t)g(t), \tag{5}$$

it follows

$$\begin{aligned} U_i(t) = & (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1} \\ & - ((1-\theta)\alpha_i + \theta\beta_i)((1-\theta)g_i + \theta g_{i+1}) \geq 0, \end{aligned} \tag{6}$$

where g_i, g_{i+1} represent $g(t_i), g(t_{i+1})$ respectively. Since

$$\begin{aligned} & ((1-\theta)\alpha_i + \theta\beta_i)((1-\theta)g_i + g_{i+1}\theta) \\ = & \ (1-\theta)^2\alpha_i g_i + \theta(1-\theta)(\alpha_i g_{i+1} + \beta_i g_i) + \theta^2\beta_i g_{i+1} \\ = & \ (1-\theta)^3\alpha_i g_i + \theta(1-\theta)^2(\alpha_i g_{i+1} + \beta_i g_i + \alpha_i g_i) + \theta^2(1-\theta)(\alpha_i g_{i+1} + \beta_i g_i + \beta_i g_{i+1}) \\ & +\theta^3\beta_i g_{i+1}, \end{aligned}$$

(6) becomes

$$U_i(t) = (1-\theta)^3\alpha_i(f_i - g_i) + \theta(1-\theta)^2 A_i + \theta^2(1-\theta)B_i + \theta^3\beta_i(f_{i+1} - g_{i+1}) \geq 0, \tag{7}$$

where

$$\begin{aligned} A_i &= V_i - (\alpha_i g_{i+1} + \beta_i g_i + \alpha_i g_i) \\ &= \alpha_i(2f_i - g_{i+1} - g_i + h_i d_i) + \beta_i(f_i - g_i), \\ B_i &= W_i - (\alpha_i g_{i+1} + \beta_i g_i + \beta_i g_{i+1}) \\ &= \beta_i(2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1}) + \alpha_i(f_{i+1} - g_{i+1}). \end{aligned}$$

If $A_i \geq 0$, and $B_i \geq 0$, since

$$\begin{aligned} U_i(t_i) &= \alpha_i(f_i - g_i) \geq 0, \\ U_i(t_{i+1}) &= \beta_i(f_{i+1} - g_{i+1}) \geq 0, \end{aligned}$$

then $U_i(t) \geq 0$ for all $t \in [t_i, t_{i+1}]$. Hence, the following theorem holds.

Theorem 3.1. *Given $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ with $f_i \geq g_i$, the sufficient condition for the rational cubic curve $P(t)$ to lie above the piecewise linear curve $g(t)$ is that the positive parameters α_i, β_i satisfy the following linear inequality:*

$$A_i = \alpha_i(2f_i - g_{i+1} - g_i + h_i d_i) + \beta_i(f_i - g_i) \geq 0, \tag{8}$$
$$B_i = \alpha_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1}) \geq 0. \tag{9}$$

For a given data set $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ the corresponding A_i, B_i in the above theorem are called the *criterion numbers* for the rational cubic interpolant above the straight line in the subinterval $[t_i, t_{i+1}]$.

Case 2. Let $g(t)$ be a quadratic function, and $f_i \geq g(t_i)$. Since

$$g_i(t) = (1-\theta)^2 g_i + \theta(1-\theta)(2g_i + g_i' h_i) + \theta^2 g_{i+1}, \quad t \in [t_i, t_{i+1}]$$

where

$$g_i = g(t_i),\ g_{i+1} = g(t_{i+1}),\ g_i' = g'(t_i),$$

then

$$P(t) = \frac{p_i(t)}{q_i(t)} \geq g(t)$$

is equivalent to

$$U_i(t) = (1-\theta)^3 \alpha_i(f_i - g_i) + \theta(1-\theta)^2 C_i + \theta^2(1-\theta) D_i + \theta^3 \beta_i(f_{i+1} - g_{i+1}) \geq 0, \tag{10}$$

where

$$\begin{aligned} C_i &= (2\alpha_i + \beta_i)(f_i - g_i) + \alpha_i h_i(d_i - g_i') \\ &= \alpha_i(2f_i - 2g_i + h_i d_i - h_i g_i') + \beta_i(f_i - g_i), \end{aligned} \tag{11}$$
$$\begin{aligned} D_i &= (2\beta_i + \alpha_i) f_{i+1} - \alpha_i g_{i+1} - 2\beta_i g_i - \beta_i h_i(d_{i+1} + g_i') \\ &= \alpha_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - 2g_i - h_i d_{i+1} - h_i g_i'). \end{aligned} \tag{12}$$

For a given data set $\{(t_i, f_i, d_i) i = 0, 1, \cdots, n\}$, and a given quadratic (maybe piecewise) function g(t), C_i and D_i are called the *criterion numbers* for the rational cubic interpolant above the quadratic curve in the subinterval $[t_i, t_{i+1}]$.

In the same way as in case 1, there is the sufficient condition theorem for Case 2.

Theorem 3.2. *Let $\{(t_i, f_i, d_i), i = 0, 1, \cdots, n\}$ be a given data set, and $g(t)$ be a given quadratic function satisfying $f_i \geq g_i$, then the sufficient condition for the rational cubic curve $P(t)$ defined by (1) to lie above the quadratic curve $g(t)$ is that the parameters α_i, β_i satisfy*

$$\alpha_i(2f_i - 2g_i + h_i d_i - h_i g_i') + \beta_i(f_i - g_i) \geq 0, \tag{13}$$
$$\alpha_i(f_{i+1} - g_{i+1}) + \beta_i(2f_{i+1} - 2g_i - h_i d_{i+1} - h_i g_i') \geq 0. \tag{14}$$

The existence of a constrained rational cubic interpolant $p(t)$ satisfying the constraints in $[t_i, t_{i+1}]$ depends on the existence of the solution paramenters α_i, β_i of the inequality system (8)&(9) or (13)&(14) for case 1 or case 2, respectively. For simplicity of notations, we shall write the system as follows:

$$a_1\alpha_i + b_1\beta_i \geq 0 \tag{15}$$
$$a_2\alpha_i + b_2\beta_i \geq 0 \tag{16}$$

with a_1, b_1, a_2, and b_2 defined as follows for (8)&(9)

$$\begin{aligned} a_1 &= 2f_i - g_{i+1} - g_i + h_i d_i \\ b_1 &= f_i - g_i \\ a_2 &= f_{i+1} - g_{i+1} \\ b_2 &= 2f_{i+1} - g_{i+1} - g_i - h_i d_{i+1} \end{aligned}$$

and defined as follows for (13)&(14).

$$\begin{aligned} a_1 &= 2f_i - 2g_i + h_i d_i - h_i g_i' \\ b_1 &= f_i - g_i \\ a_2 &= f_{i+1} - g_{i+1} \\ b_2 &= 2f_{i+1} - 2g_i - h_i d_{i+1} - h_i g_i' \end{aligned}$$

By elemetary analytic geometry, it is easy to get the following existence conditions for the interpolation.

Theorem 3.3. *For the constrained rational cubic interpolation discussed in section 3,*

1) *if $f_i > g_i$ for all $i = 0, 1, \ldots n$ (i.e. $b_1 > 0, a_2 > 0$), then the set of positive solution parameters α_i, β_i of the inequality system* (15)&(16) *is nonempty except when $a_1 < 0, b_2 < 0$ and $a_1 b_2 > a_2 b_1$.*

2) *if $f_i = g_i$ and $f_{i+1} > g_{i+1}$ (i.e. $b_1 = 0$ and $a_2 > 0$) for interval $[t_i, t_{i+1}]$, then the set of positive solution parameters α_i, β_i of the inequality system* (15)&(16) *is nonempty except when $a_1 < 0$.*

3) *if $f_i > g_i$ and $f_{i+1} = g_{i+1}$ (i.e. $b_1 > 0$ and $a_2 = 0$) for interval $[t_i, t_{i+1}]$, then the set of positive solution parameters α_i, β_i of the inequality system* (15)&(16) *is nonempty except when $b_2 < 0$.*

4) *if $f_i = g_i$ for all $i = 0, 1, \ldots n$ (i.e. $b_1 = a_2 = 0$) then the set of positive solution parameters α_i, β_i of the inequality system* (15)&(16) *is nonempty if and only if $a_1 \geq 0$ and $b_2 \geq 0$.*

The techniques used in this sections for constrained interpolation above a straight line or a quadratic curve can be used for the "below" case as well. Therefore, one may actually consider constrained interpolation between two curves.

4. Constraint on the Second-order Derivative of the Interpolant

The second-order derivative of an interpolant has been used in estimating the strain energy and, consequently, smoothness of the interpolant. Smaller energy generally implies smoother shape. However, it is possible that the overall energy of an interpolant is small while great enough to generate abnormal shape at some points or even some small intervals. A better way would be to control the second-order derivative directly. When $t \in [t_i, t_{i+1}]$, from (1) it is easy to get

$$\begin{aligned} P''(t) &= (h_i^2((1-\theta)\alpha_i+\theta\beta_i)^3)^{-1}\cdot \\ &\{((1-\theta)\alpha_i+\theta\beta_i)^2(6(1-\theta)\alpha_i f_i+(6\theta-4)V_i+(2-6\theta)W_i+6\theta\beta_i f_{i+1}) \\ &-2(\beta_i-\alpha_i)((1-\theta)\alpha_i+\theta\beta_i)(-3(1-\theta)^2\alpha_i f_i+(1-4\theta+3\theta^2)V_i+(2\theta-3\theta^2)W_i+ \\ &3\theta^2\beta_i f_{i+1})+2(\beta_i-\alpha_i)^2((1-\theta)^3\alpha_i f_i+\theta(1-\theta)^2V_i+\theta^2(1-\theta)W_i+\theta^3\beta_i f_{i+1})\}. \end{aligned}$$

Let $P''(t)\le M$, then

$$\begin{aligned} Q(\theta) &= Mh_i^2((1-\theta)\alpha_i+\theta\beta_i)^3+ \\ &\{-((1-\theta)\alpha_i+\theta\beta_i)^2(6(1-\theta)\alpha_i f_i+(6\theta-4)V_i+(2-6\theta)W_i+6\theta\beta_i f_{i+1}) \\ &+2(\beta_i-\alpha_i)((1-\theta)\alpha_i+\theta\beta_i)\cdot \\ &(-3(1-\theta)^2\alpha_i f_i+(1-4\theta+3\theta^2)V_i+(2\theta-3\theta^2)W_i+3\theta^2\beta_i f_{i+1}) \\ &-2(\beta_i-\alpha_i)^2((1-\theta)^3\alpha_i f_i+\theta(1-\theta)^2V_i+\theta^2(1-\theta)W_i+\theta^3\beta_i f_{i+1})\}\ge 0. \end{aligned}$$

Note that

$$Q'(\theta)=[(1-\theta)\alpha_i+\theta\beta_i]^2[3(\beta_i-\alpha_i)Mh_i^2+6(\alpha_i f_i-V_i+W_i-\beta_i f_{i+1})],$$

hence, $Q(\theta)$ is monotone in $[0,1]$. On the other hand,

$$\begin{aligned} Q(0) &= 2\alpha_i^2\beta_i(2f_i-2f_{i+1}+h_id_i+h_id_{i+1})+\alpha_i^3(Mh_i^2+2f_i-2f_{i+1}+2h_id_i), \\ Q(1) &= \beta_i^3(Mh_i^2-2f_i+2f_{i+1}-2h_id_{i+1})+\alpha_i\beta_i^2(4f_{i+1}-4f_i-2h_id_{i+1}-2h_id_i), \end{aligned}$$

thus, it follows

Theorem 4.1. *For the rational cubic interpolant function $P(t)$ defined by (1) , the second-order derivative $P''(t)$ is less than or equal to M in $[t_i,t_{i+1}]$ if and only if the positive parameters α_i,β_i satisfy the following inequality system*

$$2\beta_i(2f_i-2f_{i+1}+h_id_i+h_id_{i+1})+\alpha_i(Mh_i^2+2f_i-2f_{i+1}+2h_id_i)\ge 0, \tag{17}$$

$$\beta_i(Mh_i^2-2f_i+2f_{i+1}-2h_id_{i+1})+2\alpha_i(2f_{i+1}-2f_i-h_id_{i+1}-h_id_i)\ge 0. \tag{18}$$

As is known that when $\alpha_i=\beta_i$, $P(t)$ is the standard cubic Hermite interpolate function $H(x)$. Obviously, for Theorem 4.1, the following corollary holds.

Corollary 4.1. *For a standard cubic Hermite interpolante function $H(t)$, the sufficient and necessary condition for the second-order derivative $H''(t)$ to be less than or equal to M in $[t_i,t_{i+1}]$ is that the given data $\{f_i,f_{i+1},d_i,d_{i+1}\}$ satisfy the following conditions:*

$$\begin{aligned} Mh_i^2-6f_{i+1}+6f_i+4h_id_i+2h_id_{i+1}\ge 0, \\ Mh_i^2+6f_{i+1}-6f_i-4h_id_{i+1}-2h_id_i\ge 0. \end{aligned}$$

As far as the existence condition for the second-order derivative of $P(t)$ in $[t_i,t_{i+1}]$ to be less than or equal to a given number M is concerned, from Theorem 4.1, by setting $\lambda_i=\beta_i/\alpha_i$, where $\lambda_i>0$, and let

$$\begin{aligned} a &= 4f_i-4f_{i+1}+2h_id_i+2h_id_{i+1}, \\ b &= Mh_i^2-2f_i+2f_{i+1}-2h_id_{i+1}, \\ c &= Mh_i^2+2f_i-2f_{i+1}+2h_id_i, \end{aligned}$$

then, the conditions (17) and (18) become

$$a\lambda_i+c\ge 0, \tag{19}$$

$$b\lambda_i-a\ge 0. \tag{20}$$

Obviously, the following theorem holds.

Theorem 4.2. *For the given* $\{(f_i, d_i), i = 1, 2, \cdots\}$, *there must exist the parameters* $\alpha_i > 0, \beta_i > 0$ *for the interpolation function* $P(t)$ *defined by (1) such that the second-order derivative of* $P(t)$ *in* $[t_i, t_{i+1}]$ *keeps less than or equal to the given number* M *except the following cases:*

1) $a < 0$ *and* $c \leq 0$; *or*
2) $a = 0$ *and* $c < 0$; *or*
3) $a = 0$ *and* $b < 0$; *or*
4) $a > 0$ *and* $b \leq 0$.

Similarly, the techniques used in this section 4 can be used for the case that the second-order derivative is greater than or equal to a given number N. Thus one may also consider constrained interpolation in which the second-order derivative of the interpolating function is bounded in a given interval $[N, M]$.

5. Convex Interpolation Condition

Engineering practice usually demands the interpolating function to retain the shape shown by the given data. For instance, if the inequalities $f_{i-1} + f_{i+1} - 2f_i > 0 (< 0)$ hold, the interpolating function is required to construct a convex (concave) function in the whole interval or subinterval, according to need, this is called the convex interpolation. Because of the flexibility of the rational cubic interpolation, this task can be carried out just by selected the suitable values of the parameters α_i, β_i. The condition to be met by the parameters α_i, β_i to keep the shape of the given data is called the convex interpolating condition.

As done in Section 4, it is easy to get the conditions for the second-order derivative of the interpolant function $P(t)$ defined by (1) to be greater or equal to zero in $[t_i, t_{i+1}]$, it is the following convex condition theorem.

Theorem 5.1. *The sufficient and necessary condition for the rational cubic interpolation function defined by (1) to keep convex in the interval* $[t_i, t_{i+1}]$ *is that the parameters* β_i, α_i *satisfy the following conditions.*

$$\left\{ \begin{array}{l} \beta_i(2f_{i+1} - 2f_i - h_i d_i - h_i d_{i+1}) + \alpha_i(f_{i+1} - f_i - h_i d_i) \geq 0, \\ \beta_i(f_i - f_{i+1} + h_i d_{i+1}) + \alpha_i(2f_i - 2f_{i+1} + h_i d_{i+1} + h_i d_i) \geq 0. \end{array} \right. \tag{21}$$

The related problem with Theorem 5.1 is that whether the parameters α_i and β_i exist to satisfy (21) such that the interpolation (1) keeps convex in the subinterval $[t_i, t_{i+1}]$? let

$$\begin{aligned} a^* &= f_{i+1} - f_i - h_i d_i, \\ b^* &= f_{i+1} - f_i - h_i d_{i+1}, \end{aligned}$$

the conditions (21) becomes

$$\left\{ \begin{array}{l} (a^* + b^*)\lambda_i + a^* \geq 0, \\ -b^*\lambda_i - (a^* + b^*) \geq 0. \end{array} \right. \tag{22}$$

If the inequality (22) has the solution $\lambda_i > 0$, there are some α_i and β_i to satisfy (21). The following Theorem 5.2 is about the existence of the parameter λ_i.

Theorem 5.2. *For the given* $\{(f_i, d_i), i = 1, 2, \cdots\}$, *there must exist the parameters* $\alpha_i > 0, \beta_i > 0$ *for the interpolation function* $P(t)$ *defined by (1) such that the interpolation function* $P(t)$ *keeps convex in the subinterval* $[t_i, t_{i+1}]$ *except the following cases:*

1) $a^* + b^* < 0$ *and* $a^* \leq 0$; *or*
2) $a^* = -b^* < 0$; *or*
3) $a^* + b^* > 0$ *and* $b^* \geq 0$.

The correspoding sufficient and necessary conditions for cancave interpolation can be deduced similarly.

References

[1] Y. Chen, P. Steward, Ford Research Laboratory, private communication, 1996.

[2] B.A. Narsky, The Beta-spline: A local Representation Based on Shape Parameters and Fundamental Geomatrc Mensure. Ph.D thesis, University of Utah, 1981.

[3] P. Dierck, B. Tytgat, Generating the Bézier points of β-spline curve, *Comp. Aided Geom. Design*, **6** (1989), 279-291.

[4] T.A. Foley, Local control of interval tension using weighted splines, *Comp. Aided Geom. Design*, **3** (1986), 281-294.

[5] G.M. Nielson, Rectangular ν-splines, *IEEE comp. Graph. Appl.*, **6** (1986), 35-40.

[6] M. Sarfraz, Cubic spline curves with shape control, *Comp & Graph*, **18** (1994), 707-713.

[7] J.W. Schmidt, W. Hess, Positivity of cubic polynomials on intervals and positive spline interpolation, *BIT*, **28** (1988), 340–352.

[8] M. Sarfraz, Interpolatory rational cubic spline with biased point and interval tension control, *Comp & Graph*, **16** (1992), 427–430.

[9] J.A. Gregory, M. Sarfraz, P.K. Yuen, Interactive curve design using C^2 rational splines, *Comp & Graph*, **18** (1994), 153–159.

[10] Qi Duan, K. Djidjeli, W.G. Price, E.H. Twizell, Rational cubic spline based on function values, *Comp & Graph*, **22** (1998), 479-486.