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TETRAHEDRAL C^m INTERPOLATION BY RATIONAL FUNCTIONS[*1)]

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Abstract

A general local $C^m(m \geq 0)$ tetrahedral interpolation scheme by polynomials of degree $4m+1$ plus low order rational functions from the given data is proposed. The scheme can have either $4m+1$ order algebraic precision if C^{2m} data at vertices and C^m data on faces are given or $k + E[k/3] + 1$ order algebraic precision if C^k $(k \leq 2m)$ data are given at vertices. The resulted interpolant and its partial derivatives of up to order m are polynomials on the boundaries of the tetrahedra.

Key words: C^m interpolation, Rational functions, Tetrahedra.

1. Introduction

We consider the problem of constructing C^m $(m \geq 0)$ piecewise rational local interpolation to the data on a domain in $\mathbb{R}^3$ that is assumed to have been tessellated into tetrahedra (we denote the tessellation by $\mathcal{T}$). The scheme requires the following data: The partial derivatives of order s at each vertex for $s = 0, 1, \cdots, 2m$, partial derivatives of order s at s equally (no necessary) distributed points (excluding the end points) on each edge, and $\frac{1}{2}[(m+2s)(m+2s-1)-3s(s-1)]$ regularly distributed points on each face for $s = 0, \cdots, m$ (see section 4 for detail).

Interpolation over tetrahedra is a fundamental problem in the areas of data fitting, CAGD and finite element analysis. Many schemes have been developed for constructing C^1 interpolants. These schemes can be classified into three categories. The schemes in the first category require the interpolants to be polynomials over the given tetrahedra. In (Rescorla, [2]) a C^1 piecewise polynomial of degree 9 interpolation scheme is presented which needs C^4 data at the vertices. In general, a C^m piecewise polynomial interpolation scheme requires a polynomial of degree $8m+1$ and C^{4m} data (see [6]). It should be noted that this approach needs much higher order of data and higher degree of the polynomial than the order of smoothness that the scheme can achieve. To avoid such disadvantages, subdivision schemes, that may be classified into the second category, are developed. In these schemes, each tetrahedron is split into sub-tetrahedra using Clough-Tocher split (see Alfeld, [2], Worsey and Farin, [8] and Farin, [5]) or Powell-Sabin split (see Worsey and Piper [9]). In (Alfeld, [2]), Clough-Tocher split is used to split each tetrahedron into twelve sub-tetrahedra, and C^2 data and quintic are used to achieve C^1 continuity. An n-dimensional Clough-Tocher scheme is proposed by Worsey and Farin, [8]. In (Worsey and Piper,[9]), each tetrahedron is split into twenty-four sub-tetrahedra, and C^1 data and quadratic are used to achieve C^1 continuity. The main disadvantage of this approach is that it leads to more sub-tetrahedra hence more pieces of functions. For examples, the Clough-Tocher split may cause many thin sub-tetrahedra which may affect the stability of the interpolant. The third category of the schemes use rational form interpolants. The rational interpolants avoid the split of the tetrahedra. In (Alfeld, [1]), a transfinite C^1 scheme is proposed, and through the discretization of the transfinite scheme a finite C^1 rational interpolant is derived. In (Barnhill and Little, [4]), a C^1 BBG interpolant, which is then discretized to a 28-degrees-of-freedom C^1 scheme. However, such a discretization is rather complicated. The most general simplicial

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rational interpolation scheme is the *perpendicular interpolation* described in [3]. The C^m interpolation scheme requires C^m data at vertices, uses rational function with denominator degree $6m+12$ (for even m) or $6m+6$ (for odd m), and has order m or $m+1$ algebraic precision. To achieve the goal of using lower order polynomials, global spline interpolation methods have been proposed by Wang and Shi (see [10]) for constructing C^1 interpolants in any dimension.

In this paper, we shall use the rational form to construct locally C^m interpolant for any integer $m \geq 0$. For achieving global C^m continuity, we require C^{2m} data at the vertices and C^m data on the faces and use a polynomial of degree $4m+1$ plus a rational term with denominator degree at most $3m$. The polynomial part will interpolate up to $n := E[m/2]$ order data, while the rational part, which and its partial derivatives of up to order m are polynomials on the boundary of the tetrahedra, will interpolate higher order data. We should mention that all the parameters appeared in the interpolant in our scheme are linear. Hence the interpolants are not only useful in the CAGD area, but also suitable for the finite element analysis. The fact of the interpolant and its partial derivatives are polynomials do have some advantages. It makes the construction of the interpolant as easy as polynomial. This feature is important in some applications in which only boundary values (including derivatives) are involved. Comparing with the perpendicular interpolation of [3], the advantages of our schemes are: the interpolants use lower order rational functions, achieve higher order algebraic precisions and have polynomial boundary feature. We should point out that although the algebraic precision is not crucial in the area of scattered data interpolation, but it is important in the application of the finite element analysis, since it relates to the convergence order. The disadvantage of our scheme is that more data (face data and C^{2m} vertex data) are involved. However, we propose an approach to obtain these data when only lower order data at vertex are given.

The paper is organized as follows: Section 2 gives the notations and the forms of the rational interpolation functions. Sections 3 shows that the used rational functions are well defined and have the required smoothness and have minimal degree properties. Section 4 establishes the formulas for computing the coefficients of the interpolants. In section 5, we discuss the dimension of the interpolation function space, and in section 6 we consider the algebraic precision that the interpolant can achieve.

2. Interpolation Forms

The interpolants in this paper are locally defined on tetrahedra as trivariate polynomials plus trivariate rational functions. The polynomials used in this paper are in Bernstein-Bezier (BB) forms over tetrahedra. Let $p_i = (x_i, y_i, z_i)^T \in I\!R^3$ for $i = 1, \cdots, 4$. Then the tetrahedron, denoted by $[p_1p_2p_3p_4]$, with vertices p_i is defined by $[p_1p_2p_3p_4] = \{p \in I\!R^3 : p = \sum_{i=1}^4 \alpha_i p_i, 0 \leq \alpha_i \leq 1, \ \sum_{i=1}^4 \alpha_i = 1\}$ where $(\alpha_1, \cdots, \alpha_4)^T$ is known as barycentric coordinate of p. On a tetrahedron, a trivariate polynomial of degree n is expressed by $f(\alpha) = f(\alpha_1, \cdots, \alpha_4) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha_1, \cdots, \alpha_4)$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T \in Z_+^4$, $|\lambda| = \sum_{i=1}^4 \lambda_i$ and $B_\lambda^n(\alpha_1, \cdots, \alpha_4) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$, where Z_+^4 is the collection of the four dimensional vectors with nonnegative integer components. As a subscript, λ stands for $\lambda_1\lambda_2\lambda_3\lambda_4$ or $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Now we consider the directional derivatives of $f(\alpha)$. If we use the symbolic shift operator E_j, i.e., $E_j b_\lambda = b_{\lambda+e_j}$ for $j = 1, \cdots, 4$, where $e_j = (\delta_{jl})_{l=1}^4$ is the jth unit vector in $I\!R^4$, then $f(\alpha)$ can be expressed as $f(\alpha) = \left(\sum_{i=1}^4 \alpha_i E_i\right)^n b_0$. Let $\xi = (\xi_1, \cdots, \xi_4)^T$ be a directional vector in barycentric coordinate, that is, ξ is the difference of the barycentric coordinates of two points q_1 and q_2 in $I\!R^3$(hence $\sum_{i=1}^4 \xi_i = 0$), then directional derivative $D_\xi f(\alpha) = n\left(\sum_{i=1}^4 \alpha_i E_i\right)^{n-1}\left(\sum_{i=1}^4 \xi_i E_i\right) b_0$. It is not difficult to check that $D_{q_1-q_2}F(p) = D_\xi f(\alpha)$, where $F(p)$ is the Cartesian coordinate form of $f(\alpha)$. More generally, let $\xi_j = (\xi_1^{(j)}, \cdots, \xi_4^{(j)})^T$, $\ j = 1, 2, ..., s (s \leq n)$ be any s directional vectors, then the s-th order directional derivative is

$$D^s_{\xi_1\xi_2\ldots\xi_s} f(\alpha) = \frac{n!}{(n-s)!}\left(\sum_{i=1}^4 \alpha_i E_i\right)^{n-s} \prod_{j=1}^s \left(\sum_{i=1}^4 \xi_i^{(j)} E_i\right) b_0. \tag{2.1}$$

This equality is used frequently to compute the coefficients of a BB form polynomial around vertices, edges and faces of the given tetrahedron from its partial derivatives.

Now we give the form of the interpolation functions. For a given nonnegative integer m, which represents the smooth order of the interpolants constructed, let $n = E[m/2]$, where $E[\cdot]$ denotes taking integer part. To achieve C^m continuity, we shall use the following interpolation function:

$$I_m(\alpha) = P_m(\alpha) + R_m(\alpha) \tag{2.2}$$

where $P_m = P_m^{(1)} + P_m^{(2)}$ is a polynomial of degree $4m+1$ and $R_m(\alpha) = \sum_{s=n+1}^{m} R_m^{(s)}(\alpha)$ is a rational function. The concrete forms and their roles of $P_m^{(i)}$ and $R_m^{(s)}$ are illustrated as follows:

$$P_m^{(1)}(\alpha) = \sum_{\lambda\in\Delta_m} b_\lambda^{(0)} B_\lambda^{4m+1}(\alpha) \tag{2.3}$$

where $\Delta_m = \sum_{i=1}^4 \Delta_{mi}^{(v)} + \sum_{1\le i\le j\le 4} \Delta_{mij}^{(e)} + \sum_{i=1}^4 \Delta_{mi}^{(f)}$ with

$$\Delta_{mi}^{(v)} = \{\lambda \in Z_+^4 :\ |\lambda| = 4m+1,\ \lambda_i \ge 2m+1\}$$
$$\Delta_{mij}^{(e)} = \{\lambda \in Z_+^4 :\ |\lambda| = 4m+1,\ \lambda_i + \lambda_j \ge 3m+1,\ \lambda_i \le 2m, \lambda_j \le 2m\}$$
$$\Delta_{mi}^{(f)} = \{\lambda \in Z_+^4 :\ |\lambda| = 4m+1,\ \lambda_i \le n,\ \lambda_j \le 2m\ for\ j \ne i; \lambda_j + \lambda_k \le 3m\ for\ j,k \ne i\}$$

here $\sum$ and $+$ are used to denote the union of sets. $P_m^{(1)}(\alpha)$ will interpolate partial derivatives of up to order $2m$ of the data at vertices and partial derivatives order s at s points on each edge for $s = 1, \cdots, m$, and normal directional derivatives of order s at $M_s := \frac{1}{2}[(m+2s)(m+2s-1) - 3s(s-1)]$ points on each face for $s = 0, \cdots, n$.

$$P_m^{(2)}(\alpha) = \sum_{|\lambda|=4m+1,\ \lambda\notin\Delta_m} b_\lambda^{(0)} B_\lambda^{4m+1}(\alpha) \tag{2.4}$$

is free which is specified to make the algebraic precision of the interpolant as high as possible.

$$R_m^{(s)}(\alpha) = \frac{1}{\sum_{i=1}^4 \prod_{j=1,j\ne i}^4 \alpha_j^s} \sum_{i=1}^4 \alpha_i^s P_{m+2(s-1)}^{(i)}(\alpha\setminus\alpha_i) \prod_{j=1,j\ne i}^4 \alpha_j^{m+1} \tag{2.5}$$

with

$$P_{m+2(s-1)}^{(i)}(\alpha\setminus\alpha_i) = \sum_{\substack{|\lambda\setminus\lambda_i| = m+2(s-1),\\ \lambda_j \le m+s-1,\ j\ne i}} b_{\lambda\setminus\lambda_i}^{(s)} B_{\lambda\setminus\lambda_i}^{m+2(s-1)}(\alpha\setminus\alpha_i)$$

where $\alpha\setminus\alpha_i$ means α_i being deleted from $\alpha = (\alpha_1, \cdots, \alpha_4)^T$. For example, $\alpha\setminus\alpha_1 = (\alpha_2, \alpha_3, \alpha_4)^T$. The meaning of $\lambda\setminus\lambda_i$ is the same. $R_m^{(s)}(\alpha)$ will make I_m interpolate normal directional derivatives of order s at M_s points on each face for $s = n+1, \cdots, m$. It should be noted that, for achieving C^m continuity, only $P_m^{(1)}$ and R_m are absolutely necessary.

In choosing the rational function $R_m^{(s)}$ in (2.5), we have made its denominator degree as low as possible. An alternative is to choose all $R_m^{(s)}$ have the same denominator for $s = n+1, \cdots, m$. That is

$$R_m^{(s)}(\alpha) = \frac{1}{\sum_{i=1}^4 \prod_{j=1,j\ne i}^4 \alpha_j^m} \sum_{i=1}^4 \alpha_i^s P_{m+2(s-1)}^{(i)}(\alpha\setminus\alpha_i) \prod_{j=1,j\ne i}^4 \alpha_j^{2m+1-s} \tag{2.6}$$

The role of this $R_m^{(s)}$ is exactly the same as the previous one. Hence we do not distinguish them in notation. But they are obviously not equivalent. In practice, the former often behave a little better than the latter in the sense of approximation error. This is the reason we prefer to use low order rational functions.

3. Properties of the Interpolation Functions

We assume that the values of the rational functions and their partial derivatives are defined by their limits at the edges of the tetrahedra of $\mathcal{T}$ at where the denominators of the rational functions vanish. It is not difficult to show that these limit exists and hence the interpolation functions are well defined. In fact, we have the following

Theorem 3.1. *The function I_m defined in (2.2) is m times differentiable on edges, $2m$ times differentiable at vertices of the tetrahedron considered.*

Following the proof steps of the Theorem 3.4 of paper [11], we can prove this theorem similarly. We omit the proof here.

It is well known that computing high order partial derivatives for a rational function is a rather complicated task, while it is easy for polynomials. The following theorem tells us that, the partial derivatives of the rational parts of I_m can be computed as easy as polynomials.

Theorem 3.2. *For a given integer i $(1 \le i \le 4)$, and nonnegative integers l_j with $\sum_{j=1}^4 l_j \le m$*

$$\left.\frac{\partial^{l_1+l_2+l_3+l_4} R_m(\alpha)}{\partial\alpha_1^{l_1}\partial\alpha_2^{l_2}\partial\alpha_3^{l_3}\partial\alpha_4^{l_4}}\right|_{\alpha_i=0} = \left.\frac{\partial^{l_1+l_2+l_3+l_4} Q_m^{(i)}(\alpha)}{\partial\alpha_1^{l_1}\partial\alpha_2^{l_2}\partial\alpha_3^{l_3}\partial\alpha_4^{l_4}}\right|_{\alpha_i=0} \tag{3.1}$$

where $Q_m^{(i)}(\alpha) = \sum_{s=n+1}^{m} \alpha_i^s P_{m+2(s-1)}^{(i)}(\alpha \setminus \alpha_i) \prod_{j=1,j\neq i}^{4} \alpha_j^{m+1-s}$.

Proof. Theorem 3.1 tells us that the function R_m is m times differentiable. Then the partial derivatives of R_m can be calculated for $\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_3 > 0$ and then be extended to the edges at which $\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_3 = 0$. Without loss of generality, we assume $i = 4$. It is easy to see that, the first three terms in the sum of (2.5) contain a factor α_4^{m+1}. Hence their partial derivatives of order s, for $s = 0, \cdots, m$, are zero on the face $\alpha_4 = 0$. Therefore, we need only to consider the last term in the sum (2.5). This term can be written as:

$$\begin{aligned}\frac{(\alpha_1\alpha_2\alpha_3)^{m+1}\alpha_4^s P_{m+2(s-1)}^{(4)}(\alpha \setminus \alpha_4)}{\sum_{i=1}^4 \prod_{j=1,j\neq i}^4 \alpha_j^s} &= (\alpha_1\alpha_2\alpha_3)^{m+1-s}\alpha_4^s P_{m+2(s-1)}^{(4)}(\alpha \setminus \alpha_4) \\ &\quad - \frac{(\alpha_1\alpha_2\alpha_3)^{m+1-s}\alpha_4^{2s}\sum_{i=1}^3 \prod_{j=1,j\neq i}^3 \alpha_j^s P_{m+2(s-1)}^{(4)}(\alpha \setminus \alpha_4)}{\sum_{i=1}^4 \prod_{j=1,j\neq i}^4 \alpha_j^s}\end{aligned} \tag{3.2}$$

For $\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_3 > 0$, the second term of the right-handed side of the above equality is any times differentiable. Since it contains a factor α_4^{2s} with $2s \ge m+1$, its partial derivatives of up to order m are zero over $\alpha_4 = 0, \alpha_1\alpha_2\alpha_3 > 0$. Then by the continuity of these partials, they are also zero at $\alpha_4 = 0, \alpha_1\alpha_2\alpha_3 = 0$. Therefore, (3.1) holds. If $R_m^{(s)}$ is defined by (2.6), the proof is similar.

Since the $2D$ (two dimensional) triangular C^m interpolation scheme proposed by Xu et al. (see [11]) requires a polynomial of degree $2m+1$ plus a rational term and C^m data, one may expect that the similar $3D$ interpolant may use a trivariate polynomial of degree $2m+1$ and require C^m data. However, this is not true. To illustrate this, suppose the $3D$ triangulation $\mathcal{T}$ contains a $2D$ triangulation $\mathcal{T}_2$. Then the restriction of the $3D$ C^m interpolant over $\mathcal{T}$ to $\mathcal{T}_2$ is a C^m $2D$ polynomial interpolant. It follows from Farin's result (see [5]) that, such an interpolant has degree at least $4m+1$ and requires C^{2m} data. Therefore, we have

Proposition 3.3. *The polynomial degree $4m+1$ of the interpolant I_m is minimal.*

Now we show that the degree of the rational parts of I_m is also minimal. Consider a more general form of $R_m^{(s)}$

$$R_m^{(s)}(\alpha) = \frac{1}{\sum_{i=1}^4 \prod_{j=1,j\neq i}^4 \alpha_j^k} \sum_{i=1}^{4} \alpha_i^s P_{m+2(s-1)}^{(i)}(\alpha \setminus \alpha_i) \prod_{j=1,j\neq i}^{4} \alpha_j^{m+k+1-s}$$

which includes (2.5) and (2.6) as two special cases. Suppose we use this form $R_m^{(s)}$ in the proof of Theorem 3.2, then the claim "*the first three terms in the sum of (2.5) contain a factor α_4^{m+1}*" requires that $m+k+1-s \ge m+1$. That is, $k \ge s$ for $s = n+1 \cdots, m$. Therefore, if we allow k varying with s, then the smallest k satisfying $k \ge s$ is $k = s$. This is the case defined by (2.5). If we let k be fixed, then the smallest k is $k = m$. This is the case defined by (2.6). Therefore, we have

Proposition 3.4. *The denominator degree $3(n+1)$ of the rational function (2.5) and the denominator degree $3m$ of the rational function (2.6) are minimal.*

We should point out that the validity of the two minimal degree properties of the interpolants above is under the assumptions that the interpolant is polynomial on the boundary of tetrahedron and the rational functions have the given form.

4. Computation of the Interpolants

Suppose we are given the following type of data (other types of data are discussed in section 6):

(**a**). At each vertex, partial derivatives of order s of some function for $s = 0, 1, \cdots, 2m$.

(**b**). On each edge, partial derivatives of order s at s equally (no necessary) distributed points for $s = 0, 1, \cdots, m$.

(**c**). On each face, normal directional derivatives of order s at M_s regularly (necessary) distributed points for $s = 0, 1, \cdots, m$.

On a face of the tetrahedron $[p_1p_2p_3p_4]$, say $[p_1p_2p_3]$, the M_s regularly distributed points are the points whose barycentric coordinates are $(i, j, k, 0)^T/(m + 2s - 2)$ with $i + j + k = m + 2s - 2,\ \ i, j, k \le m + s - 1$. This regularity requirement will make the interpolation problem always have a unique solution. We refer the data (**a**)–(**c**) in this paper as C^{2m} data over $\mathcal{T}$. Now we determine the coefficients $b_\lambda^{(s)}$ from these data so that the composite function is C^m.

a. The coefficients of the $P_m^{(1)}$

Consider first the computation of the coefficients $b_\lambda^{(0)}$ for $\lambda \in \Delta_{m1}^{(v)}$. Let $d_i = p_i - p_1$, $i = 2, 3, 4$ be three directions whose barycentric coordinates are $\xi_i = e_i - e_1$. Then by (2.1), the directional derivative of order $s := u + v + w$ of P_m at p_1 in the direction d_2, d_3 and d_4 with orders u, v and w, respectively, is

$$\begin{aligned} D^s_{\xi_2^u\xi_3^v\xi_4^w} P_m(\alpha)|_{\alpha_1=1} &= \frac{(4m+1)!}{(4m+1-s)!} E_1^{n-s}(E_2 - E_1)^u (E_3 - E_1)^v (E_4 - E_1)^w b_0^{(0)} \\ &= \frac{(4m+1)!}{(4m+1-s)!} \sum_{i=0}^{u} \sum_{j=0}^{v} \sum_{k=0}^{w} \frac{(-1)^{s-i-j-k} u!v!w!}{i!j!k!(u-i)!(v-j)!(w-k)!} b^{(0)}_{4m+1-i-j-k,i,j,k} \end{aligned} \tag{4.1}$$

Using this formula repeatedly, we could determine the coefficients $b^{(0)}_{4m+1-s,u,v,w}$ for $s \le 2m$ from C^{2m} data at p_1. The coefficients $b_\lambda^{(0)}$ for $\lambda \in \Delta_{mi}^{(v)}$ and $i = 2, 3, 4$ are similarly determined from the C^{2m} data at p_2, p_3 and p_4, respectively. It is not difficult to show the following lemma:

Lemma 4.1. *If the coefficients $b_\lambda^{(0)}$ of P_m, for $\lambda \in \Delta_{mi}^{(v)}$, $i = 1, \cdots, 4$, are determined by (4.1) from the vertex data, then P_m interpolates the directional derivative of order s in any directions for any $s (0 \le s \le 2m)$ at the vertices.*

The coefficients $b_\lambda^{(0)}$ for $\lambda \in \Delta_{mij}^{(e)}$ for $1 \le i \le j \le 4$ are determined by formula (2.1) from the order s data on the edges for $s = 1, 2, \cdots, m$. For example, on the edge $\alpha_1 + \alpha_2 = 1$, we take two directions, that are perpendicular to $p_2 - p_1$, to be $d_3 = (p_3 - p_2) + a(p_2 - p_1)$, $d_4 = (p_4 - p_2) + b(p_2 - p_1)$ whose barycentric coordinates are $\xi_3 = (-a, a-1, 1, 0)^T$ and $\xi_4 = (-b, b-1, 0, 1)^T$, respectively, where $a = \frac{(p_2-p_3)^T(p_2-p_1)}{\|p_2-p_1\|^2}$, $b = \frac{(p_2-p_4)^T(p_2-p_1)}{\|p_2-p_1\|^2}$. Then, for the given integers u and v with $s = u + v$, by (2.1) we have

$$\begin{aligned} \frac{(4m+1-s)!}{(4m+1)!} D_{\xi_3^u\xi_4^v} P_m \Big|_{\alpha_1+\alpha_2=1} &= (\alpha_1 E_1 + \alpha_2 E_2)^{4m+1-s} \left[\sum_{i=1}^{4} \xi_i^{(3)} E_i\right]^u \left[\sum_{i=1}^{4} \xi_i^{(4)} E_i\right]^v b_0^{(0)} \\ &= \sum_{i+j=4m+1-s} B_{ij}^{4m+1-s}(\alpha_1, \alpha_2) \sum_{|\lambda|=u} \sum_{|\kappa|=v} B_\lambda^u(\xi_3) B_\kappa^v(\xi_4) b^{(0)}_{i+\lambda_1+\kappa_1, j+\lambda_2+\kappa_2, \lambda_3, \kappa_4} \end{aligned} \tag{4.2}$$

Equality (4.2) is used to determine $b^{(0)}_{i+\lambda_1+\kappa_1, j+\lambda_2+\kappa_2, \lambda_3, \kappa_4}$ for $\lambda_3 = u,\ \kappa_4 = v$ iteratively, that is $b^{(0)}_{ijuv}$ with $s = u + v$. For any $s (0 \le s \le m)$, since $b^{(0)}_{ijuv}$ have been determined by the vertex data at p_1 and p_2 for $i \ge 2m + 1$ or $j \ge 2m + 1$ (that is, $i \ge 2m + 1$ or $i \le 2m - s$), the remaining coefficients to be determined are $b^{(0)}_{ijuv}$ for $2m - s + 1 \le i \le 2m$. That is, there are s coefficients to be determined.

From the given s data on the edge, these coefficients are uniquely defined by solving a linear system of equations of order s. Hence, the coefficients b_λ for $\lambda \in \Delta_{m12}^{(e)}$ are obtained. It should be noted that the coefficient matrix of the system is independent of u and v. One should take this advantage in solving these systems. The coefficients b_λ for λ in the other $\Delta_{mij}^{(e)}$ are similarly determined from the data on the other edges. The derivation above gives the following lemma:

Lemma 4.2. *If the coefficients $b_\lambda^{(0)}$ of P_m, for $\lambda \in \Delta_{mij}^{(e)}$, $1 \le i \le j \le 4$, are determined by (4.2) from the edge data, then P_m interpolates the directional derivative of order s in any directions that is perpendicular to the edge at s data points of the edge and, or for $s = 0, \cdots, m$.*

Now we determine the coefficients b_λ for $\lambda \in \Delta_{mi}^{(f)}$ for $i = 1, \cdots, 4$ from the face data. Assume $i = 4$. Consider the directional derivatives of order s of $P_m^{(1)}$ at the direction

$$d_4 = (p_4 - p_3) + a(p_3 - p_1) + b(p_3 - p_2) \tag{4.3}$$

whose barycentric coordinate is $\xi_4 = (-a, -b, -1 + a + b, 1)^T$, where a and b are so defined that d_4 is perpendicular to the face $[p_1p_2p_3]$. It follows from (2.1) that

$$\begin{aligned} \frac{(4m+1-s)!}{(4m+1)!} D_{\xi_4^s} P_m \Big|_{\alpha_4=0} &= (\alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3)^{4m+1-s} (\textstyle\sum_{i=1}^4 \xi_i^{(4)} E_i)^s b_0^{(0)} \\ &= \sum_{i+j+k=4m+1-s} B_{ijk}^{4m+1-s}(\alpha \setminus \alpha_4) \sum_{|\lambda|=s, \lambda_4<s} B_\lambda^s(\xi_4) b_{i+\lambda_1, j+\lambda_2, k+\lambda_3, \lambda_4}^{(0)} \\ &\quad + \sum_{i+j+k=4m+1-s} B_{ijk}^{4m+1-s}(\alpha \setminus \alpha_4) b_{ijks}^{(0)} \end{aligned} \tag{4.4}$$

We use (4.4) to determine $b_{ijks}^{(0)}$ iteratively for $s = 0, 1, \cdots, n$. For any s, since $b_{ijks}^{(0)}$ have been determined by the vertex data at p_1, p_2 and p_3 for $i \ge 2m+1$ or $j \ge 2m+1$ or $k \ge 2m+1$, and the edge data on the edges $[p_1p_2]$, $[p_1p_3]$ and $[p_2p_3]$ for $i + j \ge 3m + 1$ or $j + k \ge 3m + 1$ or $i + k \ge 3m + 1$, the remaining coefficients to be determined are $b_{ijks}^{(0)}$ for $i, j, k \le 2m, i + j, j + k, i + k \le 3m$. That is, there are M_s coefficients to be determined. From the given M_s data on the face, these coefficients are uniquely determined by solving a linear system of equations of order M_s. Hence, the coefficients $b_\lambda^{(0)}$ for $\lambda \in \Delta_{m4}^{(f)}$ are defined. The coefficients $b_\lambda^{(0)}$ for λ in the other $\Delta_{mi}^{(f)}$ are similarly determined from the data on the other faces.

Lemma 4.3. *If the coefficients $b_\lambda^{(0)}$ of P_m, for $\lambda \in \Delta_{mi}^{(f)}$, $i = 1, \cdots, 4$, are determined by (4.4) from the face data, then P_m interpolates the normal directional derivative order s at M_s data points on the faces for $s = 0, \cdots, n$.*

b. The coefficients of the $P_m^{(2)}$

The coefficients of $P_m^{(2)}$, that is $b_\lambda^{(0)}$ for $|\lambda| = 4m + 1$ and $\lambda \notin \Delta_m$, are so chosen that P_m has recovery property. That is, if the given data are computed from a polynomial of degree $4m + 1$, then P_m coincides with that polynomial. As before, we use (4.4) for $s = n + 1, \cdots, m$ to determine these coefficients. Now the linear system of equations derived is over-determined. We solve it in the least square sense.

It should be noted that the coefficients of $P_m^{(2)}$ are multiply determined from the different face data. We take their average as the final result. However, if the data come from a polynomial of degree $4m + 1$, the least square approximation gives exact solution and the average gives the exact coefficients of the given polynomial.

c. The coefficients of the R_m

Computing the normal directional derivative $D_{\xi_4^r} I_m = (\sum_{i=1}^4 \xi_i^{(4)} \frac{\partial}{\partial \alpha_i})^r I_m$ of order $r (n + 1 \le r \le m)$ in the direction d_4 defined by (4.3) on the face $\alpha_4 = 0$, we have by (3.1)

$$\begin{aligned} D_{\xi_4^r} I_m \,|_{\alpha_4=0} - D_{\xi_4^r} P_m \Big|_{\alpha_4=0} &= D_{\xi_4^r} R_m \Big|_{\alpha_4=0} \\ &= \sum_{s=n+1}^{r-1} \sum_{i+j+k=r-s} B_{ijks}^r(\xi_4) \frac{s! \partial^{r-s} [(\alpha_1\alpha_2\alpha_3)^{m+1-s} P_{m+2(s-1)}^{(4)}(\alpha \setminus \alpha_4)]}{\partial \alpha_1^i \partial \alpha_2^j \partial \alpha_3^k} \\ &\quad + r! (\alpha_1\alpha_2\alpha_3)^{m+1-r} P_{m+2(r-1)}^{(4)}(\alpha \setminus \alpha_4) \end{aligned} \tag{4.5}$$

where $D_{\xi_4^r} P_m \Big|_{\alpha_4=0}$ is known and can be computed by (4.4). Now we use (4.5) to determine $P_{m+2(r-1)}^{(4)}$ iteratively for $r = n + 1, \cdots, m$ by interpolating the directional derivatives $D_{\xi_4^r} I_m$ on the face $\alpha_4 = 0$ at M_r points. Again, this leads to a linear system of M_r equations.

Lemma 4.4. *If the coefficients $b_\lambda^{(r)}$ of R_m are determined by (4.5) from the face data, then I_m interpolates the normal directional derivative of order r at M_r data points on the faces for $r = n+1, \cdots, m$.*

Now we are in the position to show that the composite function that consists of the interpolants defined in this section is C^m. We note first that the composite function is well defined even on the faces of $\mathcal{T}$, since it is C^0. To see this, one should note that the coefficients, that are determined by (4.1), (4.2) and (4.4), of $P_m^{(1)}$ on a face (one λ_i is zero) depend on only the directional derivatives on that face.

Theorem 4.5. *For a given space tetrahedral tessellation $\mathcal{T}$ and C^{2m} data on $\mathcal{T}$, let F_m be a piecewise rational function over $\mathcal{T}$ such that F_m has the form (2.2) and its coefficients are defined by step* **a**–**c** *above on each tetrahedron. Then F_m is C^m continuous on $\mathcal{T}$.*

Proof. On each tetrahedron of $\mathcal{T}$, F_m is locally C^m (see Theorem 4.4). Hence, we need to prove F_m is C^m at vertices, edges and faces of $\mathcal{T}$. Since the partial derivatives of up to order $2m$ of R_m are zero at the vertices, by Lemma 4.1 we know that I_m interpolates the partial derivatives of up to order $2m$ at the vertices. Hence F_m is C^{2m} at vertices. In order to show F_m is C^m continuous on edges, it is sufficient to prove that the k directional derivatives of F_m at an edge is uniquely defined, from the data on that edge, in two directions d_1 and d_2 that are perpendicular to the edge for $k = 0, \cdots, m$. For a given edge, let $I_m^{(i)}$ be the interpolants over the tetrahedra that share the common edge. Then, by Theorem 3.2, $D_{d_1^u d_2^v} I_m^{(i)}$ are polynomials of degree $4m+1-s$ on the edges, where $u+v=s$. They interpolate, by Lemma 4.1, directional derivatives of order $s, \cdots, 2m$ at the two end points of the edge, and interpolate, by Lemma 4.2, directional derivatives of order s at s points on the edge. These directional derivatives (totaled $4m+2-s$) uniquely determine $D_{d_1^u d_2^v} I_m^{(i)}$ on the edge. That is, $D_{d_1^u d_2^v} I_m^{(i)}$ coincide with each other on the edge.

If I_m and I_m' are two interpolants defined on two tetrahedra that share a common face, then we can similarly prove, by Lemma 4.3 and Lemma 4.4, that $D_{d^k} I_m$ coincide with $D_{d^k} I_m'$ on that face for $k = 0, \cdots, m$, where d is a direction that is perpendicular to that face.

5. Dimension of the Interpolating Space

The interpolation functions in this paper are linear combinations of polynomials and rational functions. Hence, the collection of the interpolation functions forms a linear function space.

Theorem 5.1. *On the tetrahedron $[p_1p_2p_3p_4]$, the functions in the following two sets are linearly independent:* $\{B_\lambda^{4m+1}(\alpha) : \lambda \in \Delta_m\}$; $\sum_{i=1}^4 \sum_{s=n+1}^m \{\prod_{j=1, j\neq i}^4 \alpha_j^{m+1} \alpha_i^s B_{\lambda\setminus\lambda_i}^{m+2(s-1)}(\alpha \setminus \alpha_i) / \sum_{l=1}^4 \prod_{j=1, j\neq l}^4 \alpha_j^s : |\lambda \setminus \lambda_i| = m + 2(s-1),\ \lambda_j \le m+s-1\}$

Proof. Let I_m be a linear combination of the functions in the sets above. Then I_m can be written as the form (2.2) with $P_m^{(2)} = 0$. That is, $I_m = P_m^{(1)} + R_m$. Now suppose $I_m \equiv 0$ on the tetrahedron, we need to prove that the combination coefficients $b_\lambda^{(s)}$ in I_m must be zero. It follows from the definition of I_m that the partial derivatives of up to order m and $2m$ of $P_m^{(1)}$ on edges and at vertices are zero, respectively. Hence the coefficients $b_\lambda^{(0)}$ of $P_m^{(1)}$ are zero for $\lambda \in \Delta_{mij}^{(e)}$ and $\lambda \in \Delta_{mi}^{(v)}$. Consider the function I_m on the face, say $\alpha_4 = 0$. Since $I_m \equiv 0$, we have $P_m^{(1)}(\alpha_1, \alpha_2, \alpha_3, 0) \equiv 0$. Hence $b_{ijk0}^{(0)} = 0$. Similarly, $b_{ij0k}^{(0)} = b_{i0jk} = b_{0ijk}^{(0)} = 0$. Therefore, $P_m^{(1)}$ has a factor $\alpha_1\alpha_2\alpha_3\alpha_4$. Remove this factor from each term of the I_m, then by the same argument, we have $b_{ijk1}^{(0)} = b_{ij1k}^{(0)} = b_{i1jk}^{(0)} = b_{1ijk}^{(0)} = 0$. Repeat this procedure n times, we have $b_\lambda^{(0)} = 0$ for $\lambda \in \Delta_{mi}^{(f)}$. Therefore, $b_\lambda^{(0)} = 0$ for $\lambda \in \Delta_m$. Continue this step m times, we have all the coefficients in R_m are zero.

Corollary 5.2. *The interpolation function space consists of the functions $I_m = P_m^{(1)} + R_m$ has dimension $14m(m+1)^2 + 4(m+1)$.*

Proof. It is easy to see that the cardinality of $\Delta_{mi}^{(v)}$ is $\frac{1}{6}(2m+3)(2m+2)(2m+1)$ for $i = 1, \cdots, 4$. The total of the four is $\frac{1}{3}(16m^3 + 48m^2 + 44m + 12)$. The cardinality of $\Delta_{mij}^{(e)}$ is $1 \cdot 2 + 2 \cdot 3 + \cdots m(m+1) = \frac{1}{3}m(m+1)(m+2)$. Hence the total of the six is $2m^3 + 6m^2 + 4m$. The cardinality of $\Delta_{mi}^{(f)}$ plus the degrees of freedom of $P_{m+2(s-1)}^{(s)}(\alpha \setminus \alpha_i)$ for $s = n+1, \cdots, m$ is

$\frac{1}{2}\sum_{s=0}^{m}[(m+2s)(m+2s-1)-3s(s-1)]$. It can be calculated that this number is $\frac{1}{6}(10m^3+9m^2-m)$. The sum of the four is $\frac{2}{3}(10m^3+9m^2-m)$. Put these degrees of freedom together, we get the corollary.

6. Algebraic Precision

The algebraic precision of an interpolant is the largest integer k for which the interpolation function recovers the polynomial P_k of degree k if the given data is extracted from P_k.

Theorem 6.1. *If the C^{2m} data over $\mathcal{T}$ are computed from a polynomial of degree $4m+1$, then the interpolation function I_m defined in (2.2) recovers the polynomial.*

Proof. Let P be a given polynomial of degree $4m+1$, then by the definition of P_m, $P_m = P$ if the data is extracted from P. From (4.5) we know that $P^{(i)}_{m+2(s-1)}(\alpha \setminus \alpha_i) \equiv 0$ for $s = n+1, \cdots, m$. Hence, $R_m \equiv 0$. Therefore, $I_m = P$.

For most applications, the data are given only at the vertices. In this case, we need to compute the data on the edges and faces required. Suppose we are given C^k data at the vertices of $\mathcal{T}$ with $k \le 2m$. Now we give a simple way to generate the required C^{2m} data over $\mathcal{T}$ in the following two steps:

(a). For each tetrahedron determine a trivariate polynomial $P_{N_k}(\alpha) = \sum_{|\lambda|=N_k} b_\lambda B_\lambda^{N_k}(\alpha)$ of degree $N_k := k + E[k/3] + 1$ from the C^k data at four vertices by formula (4.1). If a coefficient is multiple determined, we take their average as the required value.

(b). For each vertex, edge and face, compute the required partial derivatives of $P^{(i)}_{N_k}$ and then take their average as the required partial derivatives, here $P^{(i)}_{N_k}$ are defined as above on the tetrahedra that share the common vertex, edge and face, respectively.

Theorem 6.2. *If the $C^k (k \le 2m)$ data at the vertices are computed from a polynomial of degree N_k and the C^{2m} data over $\mathcal{T}$ are determined as above, then the interpolant I_m has algebraic precision N_k.*

Proof. If the data at the vertices are computed from a given polynomial P of degree N_k, then the determined polynomial P_{N_k} in step (a) coincides with P. Hence the C^{2m} data computed from P_{N_k} are the same as the C^{2m} data of P. Hence all the data come from the same polynomial P. Then by Theorem 6.1, we have $I_m = P$.

One special case of the theorem is that we are given C^{2m} data at the vertices of $\mathcal{T}$, then the algebraic precision is $2m + E[2m/3] + 1$. Another case is we are given C^m data at the vertices of $\mathcal{T}$, then the algebraic precision is $m + E[m/3] + 1$.

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