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THE STABILITY OF LINEAR MULTISTEP METHODS FOR LINEAR SYSTEMS OF NEUTRAL DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the numerical solution of initial value problems for systems of neutral differential equations

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau), y'(t-\tau)) \quad t > 0, \\ y(t) &= \phi(t) \quad t < 0, \end{aligned}$$

where $\tau > 0$, f and ϕ denote given vector-valued functions. The numerical stability of a linear multistep method is investigated by analysing the solution of the test equations $y'(t) = Ay(t) + By(t-\tau) + Cy'(t-\tau)$, where A, B and C denote constant complex $N \times N$-matrices, and $\tau > 0$. We investigate the properties of adaptation of the linear multistep method and the characterization of the stability region. It is proved that the linear multistep method is NGP-stable if and only if it is A-stable for ordinary differential equations.

Key words: Numerical stability, Linear multistep method, Delay differential equations.

1. Introduction

For a large class of electrical networks containing lossless transmission lines the describing equations can be reduced to a system of neutral differential equations.

This paper deals with the numerical solution of initial value problems for systems of neutral differential equations

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau), y'(t-\tau)) \quad t > 0, \\ y(t) = \phi(t) \quad t < 0, \end{cases} \tag{1}$$

where f and ϕ denote given vector-valued functions with $f(t, x, y, z) \in C^N$ (whenever $t \in \mathbb{R}^+$, $y(t) \in C^N$, $y(t-\tau) \in C^N$), $\phi(t) \in C^N$, $\tau > 0$ and $y(t) \in C^N$is unknown for $t > 0$.

The purpose of the present paper is to investigate the stability properties of the linear multistep methods (LMMs) based on the following test system

$$\begin{cases} y'(t) = Ay(t) + By(t-\tau) + Cy'(t-\tau) \quad t > 0, \\ y(t) = \phi(t) \quad t < 0, \end{cases} \tag{2}$$

where A, B and C denote constant complex $N \times N$-matrices and $\tau > 0$. The solution of (2) is called asymptotically stable if

$$\lim_{t \to \infty} y(t) = 0. \tag{3}$$

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For any matrix X, denote its determinant by $\det[X]$, its spectrum by $\sigma[X]$ and its spectral radius by $\rho[X]$.

In [11], a new and simple criterion on the matrices A, B and C such that the solution of (2) is asymptotically stable has been derived.

Lemma 1.1. *(see [11]) Let $\| C \| < 1$. Then all exact solutions to (2) are asymptotically stable if*

$$\forall\lambda \in \sigma[A] \Longrightarrow \Re(\lambda) < 0, \tag{4}$$

$$\sup_{\Re(\xi)=0} \rho[(\xi I - A)^{-1}(\xi C + B)] < 1, \tag{5}$$

where $\| A \| = \sup_{\|x\|=1} \| Ax \|$, $\| x \|^2 = \langle x, x\rangle$, $x \in C^N$.

Consider the ordinary differential equations

$$\begin{aligned} x'(t) &= f(t, x(t)) \qquad t > 0, \\ x(0) &= x_0, \end{aligned}$$

where $x(t)$ and $f(t, x(t))$ are vector-valued functions. If $h > 0$ denotes a given stepsize, the gridpoint t_n is given by $t_n = nh$, and x_n denotes an approximation to $x(t_n)$. A linear multistep method can be written as

$$\sum_{j=0}^{k} \alpha_j x_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, x_{n-j}) \quad (n = k+1, k+2, \ldots). \tag{6}$$

Here α_j and $\beta_j (j = 0, 1, 2, \cdots, k)$ denote the coefficients of a LMM. Let $\rho(z)$ and $\sigma(z)$ be the usual characteristic polynomials

$$\begin{aligned} \rho(z) &= \sum_{j=0}^{k} \alpha_j z^{k-j}, \\ \sigma(z) &= \sum_{j=0}^{k} \beta_j z^{k-j}. \end{aligned}$$

A linear multistep method (ρ, σ) is called A-stable if all roots z of $\rho(z) - \lambda\sigma(z) = 0$ satisfy $|z| < 1$ whenever $\Re(\lambda) < 0$. Then we easily obtain the following result.

Lemma 1.2. *The linear multistep method is A-stable if and only if $\rho(z)I - \sigma(z)A$ is invertible (whenever all eigenvalues λ of A satisfy $\Re(\lambda) < 0$ and $|z| \geq 1$).*

In section 2, we will present adaptations of the linear multistep methods for the numerical solution of (1). In section 3, we investigate the stability region of the methods with respect to the test system (2) and some equivalent conditions are established.

2. Adaptations of Linear Multistep Methods

In order to adapt the linear multistep methods (6) to (1), we introduce unknowns $u(t)$ and $v(t)$ as

$$u(t) = y(t - \tau) \text{ and } v(t) = y'(t - \tau). \tag{7}$$

Then (1) can be converted into the following form

$$\begin{cases} y'(t) = f(t, y(t), u(t), v(t)) \quad t > 0, \\ u'(t) = v(t) \quad t > 0, \\ y(t) = \phi(t) \quad t < 0. \end{cases} \tag{8}$$

Application of the linear multistep methods (6) to (8) yields

$$\sum_{j=0}^{k} \alpha_j y_{n-j} = h \sum_{j=0}^{k} \beta_j f(t_{n-j}, y_{n-j}, u_{n-j}, v_{n-j}), \tag{9}$$

and

$$\sum_{j=0}^{k} \alpha_j u_{n-j} = h \sum_{j=0}^{k} \beta_j v_{n-j}. \tag{10}$$

Let $\tau = (m - \delta)h$ with the integer $m \geq 1$ and $\delta \in [0, 1)$. Then u_{n-j} can be computed by Lagrange interpolation as

$$u_{n-j} = \sum_{k=-r}^{s} L_k(\delta) y_{n-j-m+k} \ (\text{ whenever } n \geq m+1, m \geq s+1). \tag{11}$$

Here r and s denote given nonnegative integers, and

$$L_k(\delta) = \prod_{\substack{j = -r \\ j \neq k}}^{s} \frac{\delta - j}{k - j}. \tag{12}$$

To gain insight into the stability behaviour of (6) in the numerical solutions of the general equations (1), we apply (9), (10) and (11) to the test system (2) and obtain

$$\begin{aligned} \sum_{j=0}^{k} \alpha_j y_{n-j} &= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{j=0}^{k} \beta_j \bar{B} u_{n-j} + h \sum_{j=0}^{k} \beta_j C v_{n-j} \\ &= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{j=0}^{k} \beta_j \bar{B} u_{n-j} + \sum_{j=0}^{k} C \alpha_j u_{n-j} \\ &= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{j=0}^{k} (\beta_j \bar{B} + \alpha_j C) u_{n-j} \\ &= \sum_{j=0}^{k} \beta_j \bar{A} y_{n-j} + \sum_{j=0}^{k} (\beta_j \bar{B} + \alpha_j C) \sum_{i=-r}^{s} L_i(\delta) y_{n-j-m+i}, \end{aligned} \tag{13}$$

where $\bar{A} = hA$, $\bar{B} = hB$.

3. Linear Stability of Linear Multistep Method

Let

$$\begin{aligned} P(z; \delta) &= (\sigma(z)\bar{B} + \rho(z)C)\alpha(z; \delta), \\ Q(z) &= \rho(z)I - \sigma(z)\bar{A}, \\ \alpha(z; \delta) &= \sum_{i=-r}^{s} L_i(\delta) z^{i+r}. \end{aligned}$$

Then the characteristic equation of (13) can be written as

$$\det[z^{m+r}Q(z) - P(z;\delta)] = 0.$$

In view of Lemma 1.1, we introduce the set

$$H = \left\{ \begin{array}{ll} (\bar{A}, \bar{B}, C) \in C^{N\times N} \times C^{N\times N} \times C^{N\times N} : & \| C \| < 1, \forall \lambda \in \sigma[\bar{A}], \Re(\lambda) < 0, \\ & \sup_{\Re(\xi)=0} \rho[(\xi I - \bar{A})^{-1}(\xi C + \bar{B})] < 1 \end{array} \right\}.$$

Definition 3.1. *Let* $(\bar{A}, \bar{B}, C)$ *be given and* $\delta \in [0, 1)$*. Then the process (13) is* δ*-stable at* $(\bar{A}, \bar{B}, C)$ *if and only if* $\lim_{n\to\infty} y_n = 0$ *whenever* $m \geq s + 1$.

The δ-stability region of the process (13) is defined by

$$S_\delta(r, s) = \{(\bar{A}, \bar{B}, C) \in C^{N\times N} \times C^{N\times N} \times C^{N\times N} : \text{the process (13) is } \delta\text{-stable at } (\bar{A}, \bar{B}, C)\}.$$

The stability region S of the process (13) is defined by

$$S(r, s) = \bigcap_{0\leq\delta<1} S_\delta(r, s).$$

Definition 3.2. *(see [11])*
(i) *The process (13) is NP-stable if* $H \subseteq S_0(r, s)$.
(ii) *The process (13) is NGP-stable if* $H \subseteq S(r, s)$.

It easily follows from the well-known results on linear recurrence relations that the process (13) is δ-stable at $(\bar{A}, \bar{B}, C)$ if and only if the characteristic polynomial of (13) satisfies

$$\det[z^{m+r}Q(z) - P(z;\delta)] = 0 \Rightarrow |z| < 1, (\text{whenever } z \in C, m \geq s + 1). \tag{14}$$

By application of a theorem which was derived in [10] for the conditions of type (10), we obtain

Lemma 3.3. *The process (13) is* δ*-stable at* $(\bar{A}, \bar{B}, C)$ *whenever* $m \geq s + 1$ *if and only if*

$$Q(z) \text{ is invertible (whenever } |z| \geq 1), \tag{15}$$

$$\sup_{|z|=1} \rho[Q(z)^{-1}P(z;\delta)] \leq 1, \tag{16}$$

$$\det[z^{m+r}Q(z) - P(z;\delta)] \neq 0, \tag{17}$$

$$\text{whenever } m \geq s + 1, |z| = 1, \rho[Q(z)^{-1}P(z;\delta)] = 1.$$

Lemma 3.4. *Assume the linear multistep method* (ρ, σ) *is A-stable for ODEs, then* $|z| \geq 1$ *and* $\sigma(z) \neq 0 \Rightarrow \Re\frac{\rho(z)}{\sigma(z)} \geq 0$.

Finally, we focus on the polynomial α. Consider the condition

$$|\alpha(z;\delta)| \leq 1, \text{whenever } |z| = 1 \text{ and } 0 \leq \delta < 1. \tag{18}$$

Lemma 3.5. *(see [9]) The condition (18) is equivalent to the condition* $r \leq s \leq r + 2$*. Moreover, if* $r + s > 0$, $r \leq s \leq r + 2$, $|z| = 1$, $0 < \delta < 1$*, then* $|\alpha(z;\delta)| = 1$ *if and only if* $z = 1$.

The following theorem deals with the characterization of the stability region.

Theorem 1. *For any given integers $r \leq s \leq r+2$, then*

$$S_0(r,s) = S(r,s).$$

Proof. The proof is analogous to that of [14].

In order to investigate the statement (16), we consider the regions Σ and Γ given respectively by

$$\Sigma = \{\xi : \xi \in C, \left|\frac{1+\xi/2}{1-\xi/2}\right| < 1\}, \tag{19}$$

$$\Gamma = \{\xi : \xi \in C, \left|\frac{1+\xi/2}{1-\xi/2}\right| = 1\}, \tag{20}$$

Lemma 3.6. *(see [11]) Assume $\sigma[\bar{A}] \subseteq \Sigma$. Then $\rho[(\xi I - \bar{A})^{-1}(\xi C + \bar{B})] \leq \sup_{\xi\in\Gamma} \rho[(\xi I - \bar{A})^{-1}(\xi C + \bar{B})]$, whenever $\xi \notin \Sigma$.*

The following theorem forms the main result of the paper.

Theorem 2. *The process (13) is NGP-stable if and only if $r \leq s \leq r+2$ and the linear multistep method (ρ, σ) is A-stable for ODEs.*

Proof.

(a) Assume the method (ρ,σ) is A-stable for ODEs. Let $(\bar{A}, \bar{B}, C) \in H$, from Lemma 1.2, it follows that $Q(z)$ is invertible (whenever $|z| \geq 1$). Hence, (15) is fulfilled.

(i)If $\sigma(z) = 0$ for some z with $|z| = 1$, then $\rho[Q(z)^{-1}P(z;\delta)] \leq \rho[C] \leq \| C \| < 1$.

(ii)If $\sigma(z) \neq 0$ with $|z| = 1$, then it follows, from Lemma 3.4, that

$$\Re\frac{\rho(z)}{\sigma(z)} \geq 0.$$

(1)When $\Re\frac{\rho(z)}{\sigma(z)} = 0$, then from the definition of the set of H, we have

$$\rho[Q(z)^{-1}P(z;\delta)] = \rho[(\frac{\rho(z)}{\sigma(z)}I - \bar{A})^{-1}(\frac{\rho(z)}{\sigma(z)}C + \bar{B})] < 1.$$

(2)When $\Re\frac{\rho(z)}{\sigma(z)} > 0$, then $\frac{\rho(z)}{\sigma(z)} \notin \Sigma$. From Lemma 3.6 and $\sigma[\bar{A}] \subseteq \Sigma$, we arrive at

$$\begin{aligned}\rho[Q(z)^{-1}P(z;\delta)] &= \rho[(\frac{\rho(z)}{\sigma(z)}I - \bar{A})^{-1}(\frac{\rho(z)}{\sigma(z)}C + \bar{B})] \\ &\leq \sup_{\xi\in\Gamma}(\xi I - \bar{A})^{-1}(\xi C + \bar{B})] < 1.\end{aligned}$$

Hence, $\sup_{|z|=1} \rho[Q(z)^{-1}P(z;\delta)] < 1$. From Lemma 3.3, we conclude that $H \subseteq S(r,s)$.

(b) Assume $H \subseteq S(r,s)$. Let $(\bar{A}, \bar{B}, C) \in H \subseteq S(r,s)$. Then all eigenvalues λ of $\bar{A}$ satisfy $\Re(\lambda) < 0$. From Lemma 3.3, $Q(z)$ is invertible (whenever $|z| \geq 1$). It follows, from Lemma 1.2, that the linear multistep method is A-stable for ODEs.

Let δ, z be given with $0 \leq \delta < 1$, $|z| = 1$, $\sigma(z) \neq 0$. Let $\bar{A} = xI$, $\bar{B} = yI$ and $C = 0$ with $x < 0$, $|y| = (1-\epsilon)(-x)$ and $0 < \epsilon < 1$, where I denotes an identity $N \times N$-matrix. It is easily verfied that $(\bar{A}, \bar{B}, C) \subseteq H$. Then, from Lemma 3.3

$$\rho[Q(z)^{-1}P(z;\delta)] = \frac{|\alpha(z;\delta)||\sigma(z)y|}{|\rho(z) - \sigma(z)x|} \leq 1.$$

Thus $|\alpha(z;\delta)| \leq \frac{|\rho(z)-\sigma(z)x|}{|\sigma(z)|(1-\epsilon)(-x)} = \frac{|\frac{\rho(z)}{\sigma(z)}-x|}{(1-\epsilon)(-x)}$. Let $x \to -\infty$ and $\epsilon \to 0$, it follows that $|\alpha(z;\delta)|$ is bounded by 1. From the continuity of α for z, this bound is also valid when $\sigma(z)=0$, and hence $|\alpha(z,\delta)| \leq 1$ (whenever $|z|=1$, $0 \leq \delta < 1$). It follows from Lemma 3.5 that $r \leq s \leq r+2$.

This completes the proof of this theorem.

From Theorem 1 and Theorem 2, we obtain the following theorem.

Theorem 3. *Suppose $r \leq s \leq r+2$, then the following statements are equivalent:*
(1) *the linear multistep method (ρ, σ) is A-stable;*
(2) *the process (13) is NP-stable for (2);*
(3) *the process (13) is NGP-stable for (2).*

References

[1] A. N. Al-Mutib, Stability properties of numerical methods for solving delay differential equations, *J. Comput. Appl. Math.,* **10**:1 (1984), 71-79.

[2] V. K. Barwell, Special stability problem for functional differential equations, *BIT,* **15** (1975), 130-135.

[3] M. Calvo, T. Grande, On the asymptotic stability of the θ-methods for delay differential equations, *Numer. Math.,* **54**, (1988), 257-269.

[4] C.W. Cryer, Highly stable multistep methods for retarded differential equations, *SIAM J. Numer.Anal.,* **11** (1974), 788-797.

[5] R.D. Driver, Ordinary and Delay Differential Equations, Springer Verlag, New York, 1977.

[6] R.T. Duffin, Algorithms for classical stability problems, *SIAM Review,* **11** (1969), 196-213.

[7] J.K. Hale, Theory of Functional Differential Equations, Sringer Verlag, New York, 1977.

[8] K.J. in 't Hout, M.N. Spijker, Stability analysis of numerical methods for delay differential equations, *Numer. Math.,* **59** (1991), 807-814.

[9] K.J. in 't Hout, A new interpolation procedure for adapting Runge-Kutta methods for delay differential equations, *BIT*, **32** (1992), 634-649.

[10] K.J. in 't Hout, The stability of θ-methods for systems of delay differential equations, Report Series, No. 282, ISSN, 0112-4021, (1993).

[11] J.X. Kuang, J.X. Xiang, H.J. Tian, The asymptotic stability of one-parameter methods for neutral differential equations, *BIT*, **34** (1994), 400-408.

[12] P. Lancaster, M. Tismenetsky, The Theory of Matrices, Second edition, Academic Press, Oriando San Diego New York, (1985).

[13] M.Z. Liu, M.N. Spijker, The stability of θ- methods in the numerical solution, *IMA. Numer. Anal.*, **10**:1 (1990), 31-48.

[14] H.J. Tian, J.X. Kuang, The stability of linear multistep methods for systems of delay differential equations, *A journal of Chinese University, Numerical Mathematics*, **2**, (1995).

[15] P.J. van der Houwen, B.P. Sommijer, Stability in linear multistep methods for pure delay equations, *J. Comput. Appl. Math.*, **10** (1984), 55-63.

[16] D.S. Watanabe, M.G. Roth, The stability of difference formulas for delay differential equations, *SIAM J. Numer. Anal.*, **22** (1985), 132-145.

[17] M. Zennaro, P-stability properties of Runge-Kutta methods for delay differential equations, *Numer. Math.,* **49** (1986), 305-318.