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ORTHOGONAL PIECE-WISE POLYNOMIALS BASIS ON AN ARBITRARY TRIANGULAR DOMAIN AND ITS APPLICATIONS*1)

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

This paper present a way to construct orthogonal piece-wise polynomials on an arbitrary triangular domain via barycentric coordinates. A boundary value problem for Laplace equation and its eigenvalue problem can be solved as two applications of this approach.

Key words: Orthogonal piece-wise polynomials, Triangular domain, Eigen-decomposition.

1. Introduction

It is well known that the concept of orthogonal polynomials plays a key role in numerical analysis. How to generalize orthogonal polynomials into higher dimension, without using tensor product, is still an open problem. As we know, the original orthogonal polynomial has been studied in univariable case. Strictly, the tensor product approach is still staying in the one dimension level via decreasing dimension. The result only can use in rectangular domain.

There are some different ways to construct orthogonal polynomials in univariable case, such as three-term recurrence, generalized function, tridiagonal matrices, Gram orthogonal procedure, etc. Unfortunately, most of them are only suitable for univariate case. However, so-call Sturm-Liouville approach constructs orthogonal polynomial via eigen-decomposition of an ordinary differential operator with two-points boundary values. The eigen-decomposition approach may extend to high dimension by solving the related partial differential equation. Moreover, in high dimension simplex domains, barycentric coordinates and Bernstein form of multivariate polynomial or B net have essential advantage doing analysis than the usual Cartesian coordinates.

The eigen-decomposition for the Laplace operator in two dimensions is a classical problem in mathematics and physics. Especially for Dirichlet and Neumann boundary conditions, there are many theoretical results on the eigenvalues and eigenfunctions that have some application to numerical methods, e.g. [1]. If there were an effective asymptotic expression for the nth eigenvalue λ_n, only a finite number of the lowest eigenvalues would need to be calculated for any given region.

In practical case, one needs to calculate a lot of eigenvalue problems, such as large scale linear solver arisen from oil-reservoir simulation, preconditioning technique deal with high ill-condition, unstructured mesh in oil-reservoir simulation software, high precision algorithm for solving partial sum of eigenvalues of Schrodinger equation arisen from material science and so on.

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In this paper we consider Laplace operator with Dirichlet boundary condition on a triangular domain Ω as:

$$\begin{cases} -\Delta u = \lambda u, & u \text{ in } \Omega \\ u = 0, & u \text{ on } \partial\Omega \end{cases}$$

where the side h_1, h_2, h_3 of Ω satisfy $h_1 \le h_2 \le h_3$.

This problem may represent the vibration of a fixed membrane and the propagation a wave down a waveguide[1,2]. Pinsky[3] presented the exactly formulae for the eigenvalues and eigenfunctions as $h_1 = h_2 = h_3$, and Prager[4] studied them for the 30-60-90 case. In this paper, we discuss an approximate method to calculate the eigenvalues and eigenfunctions of Laplace operator with Dirichlet boundary condition in an arbitrary triangular domain. By using barycentric coordinates and Bernstein polynomials, an orthogonal piece-wise polynomials basis on the triangle via approximating eigen-decomposion are constructed, 4-16 approximating eigenvalues and eigen-vectors for 2-D Laplacian operator on an arbitrary triangle are obtained. As an example, one approximates solution for Poisson equation on a triangle domain is given out at last.

2. Barycentric Coordinates and Bernstein Polynomials on High Dimension Simplex Domain

For a given n-D simplex with $n+1$ vertices

$$\{P_1, P_2, \cdots, P_{n+1}\}$$

the barycentric coordinate system is defined as

$$t = (t_1, t_2, \cdots, t_{n+1}),$$

with

$$t_k = \text{Vol}\,[P_1, P_2, \cdots, P_{k-1}, P, P_{k+1}, \cdots, P_{n+1}]/V$$

where the volume

$$V := \text{Vol}\,[P_1, P_2, \cdots, P_{n+1}].$$

It is obvious that

$$\sum_{k=1}^{n+1} t_k = 1, \qquad 0 \le t_k \le 1.$$

Set $k = (k_1, k_2, \cdots, k_{n+1})$ with $k! = k_1! k_2! \cdots k_{n+1}!$, $|k| = k_1 + k_2 + \cdots + k_{n+1}$, $t^k = t_1^{k_1} t_2^{k_2} \cdots t_{n+1}^{k_{n+1}}$.

Then so-called B-form of multivariate polynomials f(t) is defined as

$$f(t) = \sum_{|k|=m} f_k b_k^m(t)$$

where f_k is called B-net of the m-degree polynomial over $n-$ dimension simplex domain, and

$$b_k^m(t) = \frac{m!}{k!} t^k$$

forms the related B-B basis, or Bernstei-Bézier basis.

Especially, $f(t) = 0$ or $f(t) = 1$ if and only if all $f_k = 0$ or all $f_k = 1$, respectively.

If $f(t)$ is a piecewise polynomial with some global continuous conditions, we may represent it in terms of the related piecewise B-net to keep the required continuous conditions, e.g. see [5], [7] or [8].

3. A construction of orthogonal piecewise polynomial basis on a given triangular domain Ω

Now we turn back to a triangular domain. Given a function f defined on the triangle, there are three barycentric coordinates, only two are independent. Hence, one may have three kinds of representation as

$$f = f_1(t_2,t_3) = f_2(t_3,t_1) = f_3(t_2,t_1).$$

Just like in univariate case, f can be divided into three different orthogonal functions

$$f = \frac{1}{6}(f_{1s}(t_2,t_3) + f_{2s}(t_3,t_1) + f_{3s}(t_2,t_1)) + \frac{1}{6}f_{1a}(t_3,t_2) + \frac{1}{6}(f_{2a}(t_1,t_3) + f_{3a}(t_2,t_1)) \quad (1)$$

where

$$\begin{aligned} f_{1s} &= f_1(t_2,t_3) + f_1(t_3,t_2), & f_{1a} &= f_1(t_2,t_3) - f_1(t_3,t_2), \\ f_{2s} &= f_2(t_3,t_1) + f_2(t_1,t_3), & f_{2a} &= f_2(t_3,t_1) - f_2(t_1,t_3), \\ f_{3s} &= f_3(t_1,t_2) + f_3(t_2,t_1), & f_{3a} &= f_3(t_1,t_2) - f_3(t_2,t_1). \end{aligned}$$

The above functions with subscript -s and -a are symmetry with respect to the related two variables, respectively. The following Lemma is obvious and useful later.

Lemma 1. *Let a function $f(r,t) \in L_2$ be given on a triangle domain Ω. Decompose f into the symmetric part $f_s(r,t) = f(r,t) + f(t,r)$ and the asymmetric part $f_a(r,t) = f(r,t) - f(t,r)$, then they are orthogonal in the usual integration form.*

$$(f_s, f_a) = 0.$$

Definition 1. *An operator L is called isotopic(or asymmetry,or symmetry) preserved if for any equilateral triangle case $L\mathbf{u}$ is also isotopic (or asymmetry, or symmetry) for any given isotopic (or asymmetry,or symmetry) $\mathbf{u}$.*

Example. For equilateral domain, laplacian operator preserves isotopic a symmetry and symmetry.

Theorem 1. *If $\mathbf{u}$ is isotopic, $\mathbf{v}$ is asymmetry along a direction, then the inner product $(\mathbf{u},\mathbf{v}) = 0$ and the energy product $(-\Delta\mathbf{u},\mathbf{v}) = 0$.*

To construct a set of orthogonal piecewise polynomial basis over a given triangular domain, we consider three-direction mesh partition ([5],[7]). This partition is a natural extension of uniform partition for univariate case. For simplicity, we restrict ourselves to use piecewise cubic polynomials. And to deal with eigen decomposition for second order elliptic problems, we add first order continuous into the space. Hence we define a functional space $S_3^{1,0}(\Delta_N)$ as follows

$$S_3^{1,0}(\Delta_N) = \{s| \;\; s \in \Pi_3(\Delta_N), \;\; s \in C^{1,0}(\Omega)\}$$

where Δ_N denotes a partition divided into N uniform segments at each edge.

Lemma 2.

$$\dim(S_3^{1,0}(\Delta_N)) = N^2 \quad (2)$$

Here $N_T = N^2$ is the number of total subtriangles on Ω.

Proof. By using the dimension formula of [6], we obtain

$$\dim(S_3^1(\Delta_N)) = 10 + 3L + 2V = N^2 + 6N + 3 \quad (3)$$

where $L = 3(N-1)$ and $V = (N-1)(N-2)/2$ are number of partition lines and inner vertices in the triangle, respectively. According to the homogeneous boundary condition, the number of independent constrains there equals to $3(2N+1)$.

Hence, we may take the all center B-net coefficients as the freedom of the space. This may simplify to construct orthogonal piecewise polynomials greatly.

Combining Lemma 2 and Lemma 1 leads to the following result:

Theorem 2. *There exists an orthogonal decomposion*

$$S_3^{1,0}(\Delta_N) = S_{3,I}^{1,0}(\Delta_N) \oplus S_{3,II}^{1,0}(\Delta_N) \oplus S_{3,III}^{1,0}(\Delta_N) \tag{4}$$

where

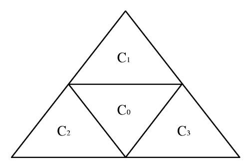
$S_{3,I}^{1,0}(\Delta_N)$ − −−	isotropic
$S_{3,II}^{1,0}(\Delta_N)$ − −−	asymmetry along one direction
$S_{3,III}^{1,0}(\Delta_N)$ − −−	symmetry along the direction and asymmetry along other two directions

Moreover,

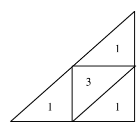
$$dim(S_{3,II}^{1,0}(\Delta_N)) = N(N-1)/2;$$
$$dim(S_{3,I}^{1,0}(\Delta_N)) + dim(S_{3,III}^{1,0}(\Delta_N)) = N(N+1)/2;$$

3.1. Case 1: $N = 2$

In the case of $N = 2$, $\dim(S_3^{1,0}(\Delta_2)) = 4$. There are four sub-triangles with notations $\Omega_{100}, \Omega_{010}, \Omega_{001}$ and Ω_{000}. We take the related four B-net as coefficients C_1, C_2, C_3 and C_0, respectively. With these notations, we found four orthogonal basis as following:

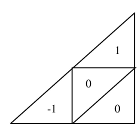


1-st basic function: isotopic and convex



$$C_0 = 3, C_1 = C_2 = C_3 = 1$$

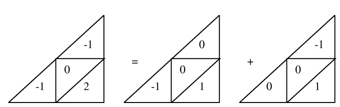
2-nd basic function: asymmetry along a direction



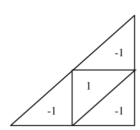
$$C_0 = 0, C_1 = 1, C_2 = 0, C_3 = -1$$

3-rd basic function: symmetry along the direction and asymmetry along other directions

$$C_0 = 0, C_1 = -1, C_2 = 2, C_3 = -1$$



4-st basic function: isotopic with higher frequency



$$C_0 = 1, C_1 = -1, C_2 = -1, C_3 = -1$$

To verify the orthogonality among the above four piecewise polynomial $\phi_j (j = 1, 2, 3, 4)$, we note that ϕ_1 and ϕ_4 belong to isotropic, and ϕ_2 and ϕ_3 are asymmetry along to different direction. By Lemma 1, we have

$$(\phi_1, \phi_j) = 0, \quad (\phi_4, \phi_j) = 0, \quad j = 2, 3.$$

Hence, it is only need to check the orthogonality between ϕ_1 and ϕ_4, where

$$\phi_1 = \begin{cases} \phi_{100} & = b^3_{111} + 2b^3_{021} + 2b^3_{012} & \in \Omega_{100} \\ \phi_{010} & = b^3_{111} + 2b^3_{102} + 2b^3_{201} & \in \Omega_{010} \\ \phi_{001} & = b^3_{111} + 2b^3_{120} + 2b^3_{210} & \in \Omega_{001} \\ \phi_{000} & = 3b^3_{111} + 2b^3_{021} + 2b^3_{012} + 2b^3_{102} + 2b^3_{201} & \\ & \quad +2b^3_{120} + 2b^3_{210} & \in \Omega_{001} \end{cases}$$

$$\phi_{000} = \phi_{100} + \phi_{010} + \phi_{001},$$

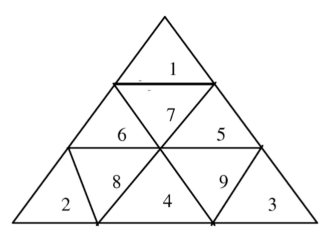
$$\phi_4 = \begin{cases} b^3_{111} & \in \Omega_{100} \\ b^3_{111} & \in \Omega_{010} \\ b^3_{111} & \in \Omega_{001} \\ -b^3_{111} & \in \Omega_{001} \end{cases}$$

Hence

$$(\phi_1, \phi_4)_\Omega = (b^3_{111}, \phi_{100} + \phi_{010} + \phi_{001} - \phi_{000})_{\Omega_{100}} = 0.$$

3.2. Case 2: $N = 3$

In the case of $N = 2$, $\dim(S^{1,0}_3(\Delta_3)) = 9$. There are nine sub-triangles divided into two kinds. The first kind of sub-triangles, the shape of which is the same as Δ, are notated with $\Omega_{200}, \Omega_{020}, \Omega_{002}$ and $\Omega_{110}, \Omega_{101}, \Omega_{011}$. The second kind of sub-triangles, the shape of which is like ∇, are notated with $\Omega_{100}, \Omega_{010}, \Omega_{001}$. We take the related four B-net as coefficients a_1, a_2, a_3, b_1, b_2, b_3 and c_1, c_2, c_3, respectively. With these notations, we set up an orthogonal basis as following:



First kind: 3 isotopic functions

$$u_1 = (1, 1, 1, b_1, b_1, b_1, c_1, c_1, c_1)$$

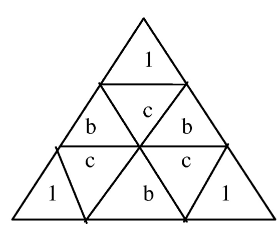
$$b_1 = \frac{113 + \sqrt{16069}}{33}, \qquad c_1 = b_1 + 1,$$

$$u_4 = (1, 1, 1, b_4, b_4, b_4, c_4, c_4, c_4)$$

$$b_4 = \frac{113 - \sqrt{16069}}{33}, \qquad c_4 = b_4 + 1,$$

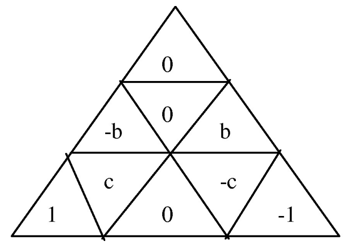
$$u_9 = (1, 1, 1, b_9, b_9, b_9, c_9, c_9, c_9)$$

$$b_9 = 1, \qquad c_9 = -1.$$



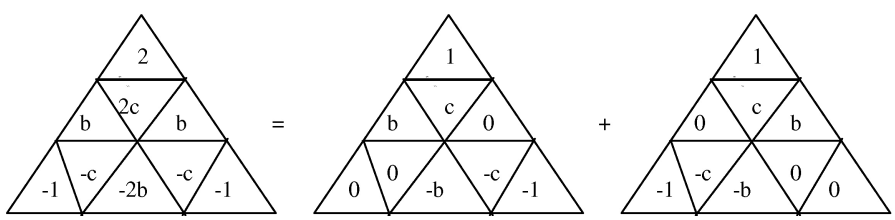
Second kind: 3 asymmetry functions

$$u_2 = (0, 1, -1, 0, -b_2, b_2, 0, c_2, -c_2)$$
$$b_2 = 2\cos\frac{2\pi}{9}, \qquad c_2 = 1 + 2\cos\frac{\pi}{9},$$
$$u_5 = (0, 1, -1, 0, -b_5, b_5, 0, c_5, -c_5)$$
$$b_5 = 2\cos\frac{\pi}{9}, \qquad c_5 = 1 + 2\cos\frac{4\pi}{9}$$
$$u_8 = (0, 1, -1, 0, -b_8, b_8, 0, c_8, -c_8)$$
$$b_8 = 2\cos\frac{4\pi}{9}, \qquad c_8 = 1 + 2\cos\frac{2\pi}{9}.$$



Third kind: 3 partial symmetry functions

$$u_3 = (2, -1, -1, -2b_3, b_3, b_3, 2c_3, -c_3, -c_3),$$
$$b_3 = 2\cos\frac{2\pi}{9}, \qquad c_3 = 1 + 2\cos\frac{\pi}{9},$$
$$u_6 = (2, -1, -1, -2b_6, b_6, b_6, 2c_6, -c_6, -c_6),$$
$$b_6 = 2\cos\frac{\pi}{9}, \qquad c_6 = 1 + 2\cos\frac{4\pi}{9},$$
$$u_7 = (2, -1, -1, -2b_7, b_7, b_7, 2c_7, -c_7, -c_7),$$
$$b_7 = 2\cos\frac{4\pi}{9}, \qquad c_7 = 1 + 2\cos\frac{2\pi}{9}.$$



By Lemma 1, it is easy to see the above three kinds function are orthogonal each other. As an example, now we show how to use B-net to verify the orthogonality of the above first three functions which are all isotropic.

Let us denote one isotropic function as

$$\phi_{IS,3} = \begin{cases} b_{111}^3 & \in \Omega_{200},\ \ \Omega_{020},\ \ \Omega_{002},\ \ \Omega_{110},\ \ \Omega_{101},\ \ \Omega_{011} \\ -b_{111}^3 & \in \Omega_{100},\ \ \Omega_{010},\ \ \Omega_{001} \end{cases}$$

and other two isotropic functions as

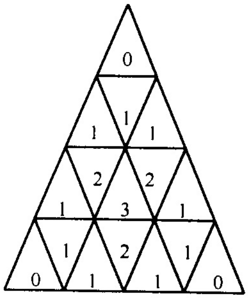
$$\phi_{IS,c} = \begin{cases} \phi_{200} & = b^3_{111} + (b^3_{210} + b^3_{120})(1 + c/2) & \in \Omega_{200} \\ \phi_{020} & = b^3_{111} + (b^3_{102} + b^3_{201})(1 + c/2) & \in \Omega_{020} \\ \phi_{002} & = b^3_{111} + (b^3_{120} + b^3_{210})(1 + c/2) & \in \Omega_{002} \\ \phi_{110} & = c * b^3_{111} + 3c/2 * (b^3_{003} + b^3_{012} + b^3_{102}) & \\ & \quad +(1 + c/2) * (b^3_{021} + b^3_{201}) & \in \Omega_{110} \\ \phi_{101} & = c * b^3_{111} + 3c/2 * (b^3_{030} + b^3_{021} + b^3_{120}) & \\ & \quad +(1 + c/2) * (b^3_{210} + b^3_{012}) & \in \Omega_{101} \\ \phi_{011} & = c * b^3_{111} + 3c/2 * (b^3_{300} + b^3_{210} + b^3_{201}) & \\ & \quad +(1 + c/2) * (b^3_{120} + b^3_{102}) & \in \Omega_{011} \\ \phi_{100} & = (1 + c) * b^3_{111} + 3c/2 * (b^3_{300} + b^3_{210} + b^3_{201}) & \\ & \quad +(1 + c/2) * (b^3_{120} + b^3_{102} + b^3_{021} + b^3_{012}) & \in \Omega_{100} \\ \phi_{010} & = (1 + c) * b^3_{111} + 3c/2 * (b^3_{030} + b^3_{021} + b^3_{120}) & \\ & \quad +(1 + c/2) * (b^3_{012} + b^3_{210} + b^3_{102} + b^3_{201}) & \in \Omega_{010} \\ \phi_{001} & = (1 + c) * b^3_{111} + 3c/2 * (b^3_{003} + b^3_{102} + b^3_{012}) & \\ & \quad +(1 + c/2) * (b^3_{201} + b^3_{021} + b^3_{210} + b^3_{120}) & \in \Omega_{001} \end{cases}$$

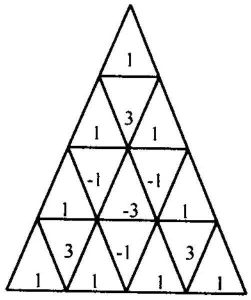
Hence,

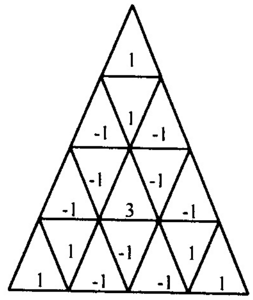
$$\begin{aligned} (\phi_{IS,3}, \phi_{IS,c})_\Omega &= (b^3_{111}, \phi_{200} + \phi_{020} + \phi_{002} + \phi_{110} + \phi_{101} + \phi_{011})_{\Omega_{200}} \\ &\quad -(b^3_{111}, \phi_{100} + \phi_{010} + \phi_{001})_{\Omega_{200}} \\ &= (b^3_{111}, 0)_{\Omega_{200}} = 0. \end{aligned}$$

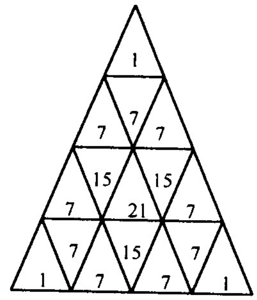
3.3. Case 3: $N = 4$

For $N = 4$, $\dim(S^{1,0}_3(\Delta_4)) = 16$, sixteen piecewise polynomials basis in terms of B-form, shown at the next page, include five isotropic functions (see IS-1 to IS-5, six asymmetry functions along t_1 direction (see A-1 to A-6) and five symmetry functions along this direction and asymmetry along other two directions(see S-1 to S-5). For easy to use, we put the B-net basis as simple as possible. According to Lemma 1, any pair of functions, belonged to two different kind among the three function groups are orthogonal each other. In the first group, all functions are orthogonal each other, except the fourth IS-4 which is not orthogonal to the first one IS-1. In fact, two of them (IS-2 and IS-5) come from the spase $(S^{1,0}_3(\Delta_2))$ directly, obvious they are orthogonal. By using block B-net, it is not difficult to verify IS-3 is orthogonal to IS-2 and IS-5. Of course, it is easy to get a linear combination of IS-4 and IS-1 to orthogonal to IS-1. In the second group, any linear combination of A-1 and A-3 or A-2 and A-6 is orthogonal to other functions.Similar case occurs for the third group. Any partial symmetry function basis can be obtained from the related asymmetry basis, just like done above for space $(S^{1,0}_3(\Delta_2))$ and $(S^{1,0}_3(\Delta_3))$. Note that in space $(S^{1,0}_3(\Delta_4))$ case, dimension number of asymmetry function space is one more than partial symmetry function space. Because one partial symmetry basis reduces to isotropic.The detail of proof is omitted here.


IS-1


IS-2


IS-3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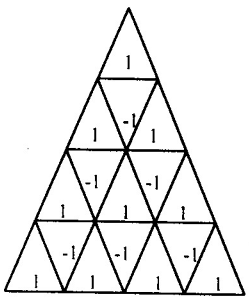
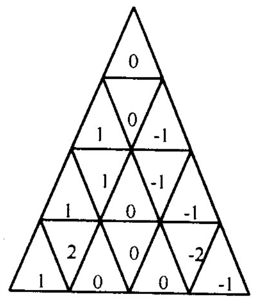
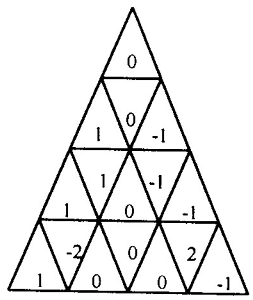

IS-5


A-1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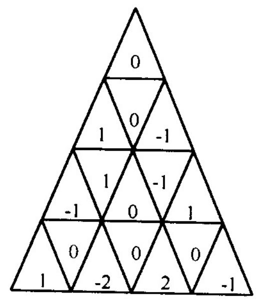
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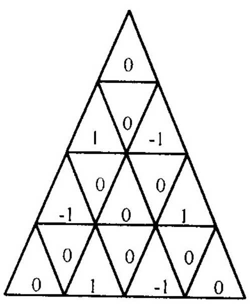
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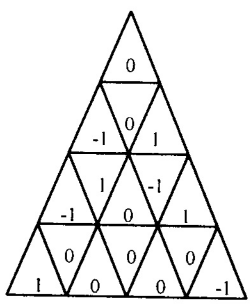
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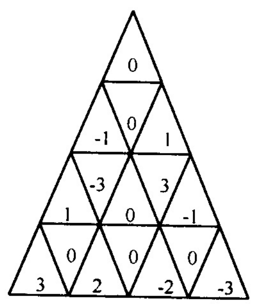
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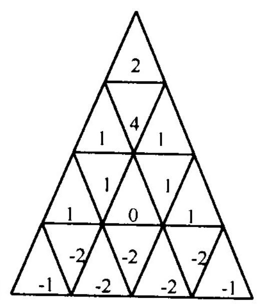
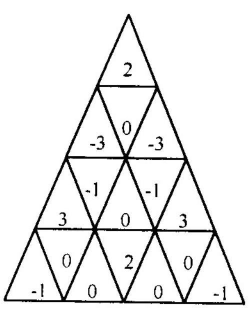
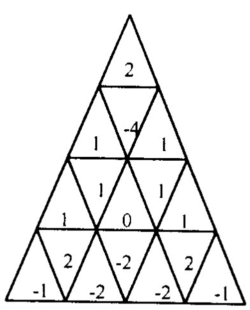

S-1


S-2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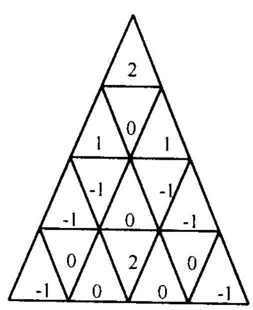
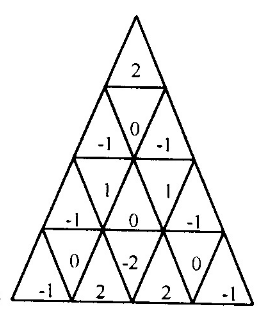

S-4


S-5

4. Applications and Numerical Tests

Application 1. Spectral method

Consider to solve the following elliptic boundary value problem on a triangle domain

$$Lu = f, \qquad u|_{\Omega} = 0$$

Set

$$u = \sum_{i=1}^{N^2} c_I u_i$$

which leads to solve a system

$$MC = g$$

with

$$g = [g_i], \qquad g_i := \int_{\Omega} g * u_i d\Omega$$

$$K = [< u_i, u_j >], \qquad < u_i, u_j > := \int_{\Omega} Lu_i * u_j d\Omega$$

Note that if $L = -\Delta$ the stiffness matrix K is diagonal when the triangle domain Ω is equilateral, and K is diagonal dominant for general triangle domain.

Numerical examples: Poisson equation on triangle domain

$$\begin{cases} -\Delta u = 1, & u \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where Ω denotes the triangle with length $h_1 \leq h_2 \leq h_3$, S be the area of Ω.

Approximating solution is

$$u = \sum_{i=1} c_i u_i, \qquad c_i = \frac{g_i}{\mu_i}$$

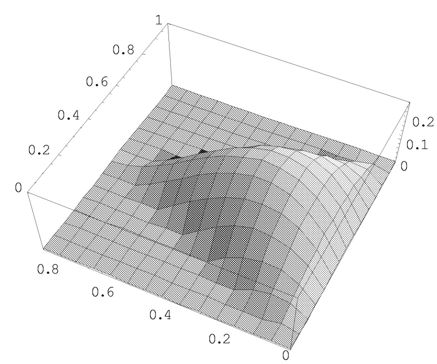
where

$$g_i := \int_{\Omega} g * u_i d\Omega, \qquad \mu_i = \int_{\Omega} -\Delta u_i * u_j d\Omega$$

with $c_i \neq 0$ iff u_i is isotopic and

$$c_1 = \frac{13S}{93(h_1^2 + h_2^2 + h_3^2)}, \qquad c_4 = \frac{39S}{128(h_1^2 + h_2^2 + h_3^2)},$$

$$c_9 = \frac{2S}{9(h_1^2 + h_2^2 + h_3^2)}.$$



In particular, if the domain Ω is an equilateral triangle, the above spetral method gives the exact solution $u = \frac{3}{4} t_1 t_2 t_3$.

Application 2. Eigen Decomposition

Now we turn back to discuss the following eigen problem for Laplace operator with Dirichlet boundary conditions

$$-\Delta u = \lambda u, \qquad u|_{\partial\Omega} = 0.$$

For an equilateral triangle, Pinsky [3] and Prager [4] gave exact formulas of eigenvalue and related eigenfunction for Laplace operator with Dirichlet and Neumman boundary conditions. Their expressions seem too complicated for real computing. And as we know, so far there is no theoretical results on general triangle domain. Therefore, it is reasonable to take the above orthogonal piecewise polynomials as eigenfunction approximation. The related eigenvalue estimation can be obtained by computing so-called Rayleigh quotient.

The following table list estimation for the first four eigenvalue for equilateral triangle case $(h_1 = h_2 = h_3 = 1)$

0	λ_1	$\lambda_2 = \lambda_3$	λ_4
$N = 2$	53.3333	132.741	224
$N = 3$	53.6939	124.330	217.14
$N = 4$	52.6516	123.549	213.33
Theoretical value	52.6379	122.822	210.5516

Moreover, for general case: $h_1 \leq h_2 \leq h_3$, we list the first nine eigenfunction approximation, based on the orthogonal decomposition of space $(S_3^{1,0}(\Delta_3)$, as follows

$$\lambda_1 = \frac{3.29337 * (h_1^2 + h_2^2 + h_3^2)}{S^2},$$

$$\lambda_2 = \frac{9.84295 * (h_1^2 + h_2^2) + 3.62602 * h_3^2}{S^2},$$

$$\lambda_3 = \frac{11.9153 * h_1^2 + 5.69833 * (h_2^2 + h_3^2)}{S^2},$$

$$\lambda_4 = \frac{11.5715 * (h_1^2 + h_2^2 + h_3^2)}{S^2},$$

$$\lambda_5 = \frac{12.3367 * h_1^2 + 16.1553 * (h_2^2 + h_3^2)}{S^2},$$

$$\lambda_6 = \frac{5.23974 * h_1^2 + 19.0785 * (h_2^2 + h_3^2)}{S^2},$$

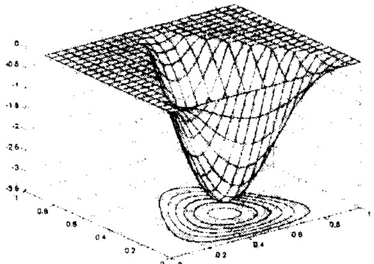
$$\lambda_7 = \frac{42.9718 * h_1^2 + 13.8518 * (h_2^2 + h_2^2)}{S^2},$$

$$\lambda_8 = \frac{4.14511 * h_1^2 + 33.2651 * (h_2^2 + h_3^2)}{S^2},$$

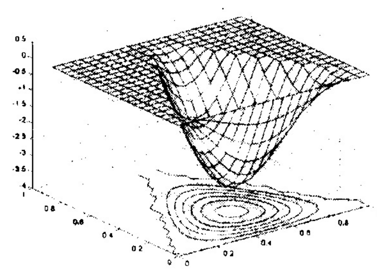
$$\lambda_9 = \frac{31.5 * (h_1^2 + h_2^2 + h_3^2)}{S^2}.$$

Pictures, listed in the next page, compare the first four eigen-functions between our orthogonal piece-wise polynomial in spase $S_3^{1,0}(\Delta_4)$ and related theoretical eigenfunction in [3]. The L_2 norm of all function is 1. Their shape and scaling are similar in both global and local meaning. To show convergence, we list their difference in maximum norm for space $S_3^{1,0}(\Delta_2)$ and $S_3^{1,0}(\Delta_4)$, cooresponding $h = \frac{1}{2}$ and $h = \frac{1}{4}$.

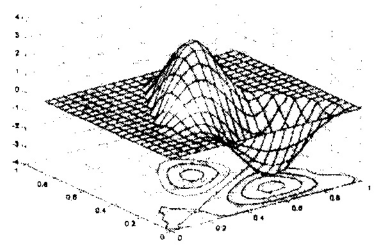
N	IS-1	A-1	S-1	IS-2
2	0.1377	0.8272	0.9442	0.3947
4	0.0134	0.1637	0.1812	0.1377



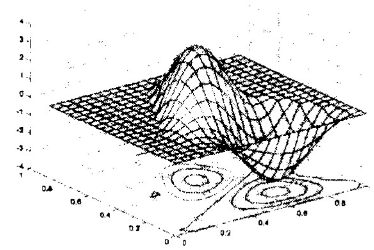
IS-1



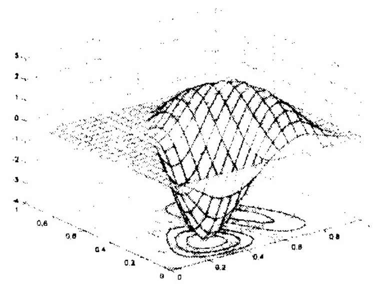
$\Phi_{3,3}$



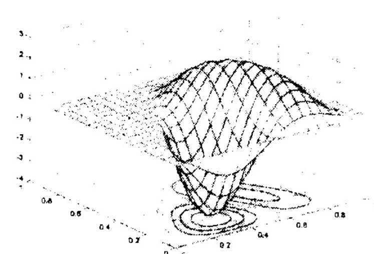
A-1



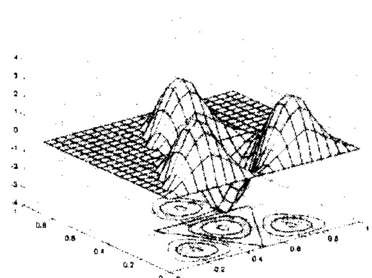
$\Phi_{4,5}-\Phi_{5,4}$



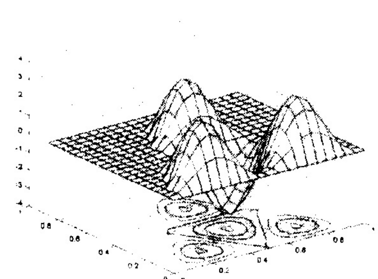
S-1



$\Phi_{4,5}+\Phi_{5,4}$



IS-2



$\Phi_{6,6}$

Approximation eigenfunciton

Exact eigenfunction

5. Conclusion

Today multivariate Bernstein polynomial is one of the main tools for multivariate approximation theory, it was proved that each Bernstein polynomial can be approximated to any degree by its Bézier nets —a sequence of piecewise linear functions consisted from the original B-net[5]. By using the barycentric coordinates and B-B representation, we have constructed an orthogonal piecewise polynomials basis on a triangle via approximating eigen-decomposition. In equilateral triangular case the basis is also orthogonal with energy inner product. This may be an efficient algorithms to construct the orthogonal basis on an arbitrary triangle, and extend to more general polygon domain in 2-D combined with DDM and simplex domain on 3-D.

Applied to spectral method, the first sixteen orthogonal basis and an approximate solution for Poisson equation on a triangle domain ($N = 4$) is obtained. It also may be applied to numerical integration, wavelet fast transformation, mathematical software, etc.

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