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NON-STATIONARY STOKES FLOWS UNDER LEAK BOUNDARY CONDITIONS OF FRICTION TYPE*

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Dedicated to the 80th birthday of Professor Feng Kang

Abstract

This paper is concerned with the initial value problem for non-stationary Stokes flows, under a certain non-linear boundary condition which can be called the leak boundary condition of friction type. Theoretically, our main purpose is to show the strong solvability (i.e.,the unique existence of the L^2−strong solution) of this initial value problem by means of the non-linear semi-group theory originated with Y. Kōmura. The method of analysis can be applied to other boundary or interface conditions of friction type. It should be noted that the result yields a sound basis of simulation methods for evolution problems involving these conditions.

Key words: Stokes equation, Leak boundary condition, Nonlinear semigroup.

1. Introduction

The purpose of this paper is to consider the initial value problem for the Stokes flow under nonlinear boundary conditions of friction type, which will be described below in §2 together with our motivations arising from applications, and to show that the solvability can be obtained immediately by means of the non-linear semigroup theory (NSG theory) which had originated from the celebrated work by Y. Kōmura ([12]) in 1967 and was elaborated by many authors (for a concise explanation of the theory, we refer to Sections 6 and 7 of Chapter XIV of Yosida ([15]).

In order to apply the NSG theory for integration of the initial boundary value problem, a crucial step is to define a multi-valued operator A in a Hilbert space so that A is maximally accretive (m-monotone) and $-A$ plays the role of the generator of the NSG relevant to the initial boundary value problem. Below we shall see that the property of being multi-valued of our generator is closely related with involvement of the pressure in the Stokes equation. This observation might be interesting as a new direction of applicability of the subtle NSG theory. On the other hand, once the NSG theory applies, we could have a better insight into the mathematical structure of the problem, which would lead to a reliable basis in organizing approximating methods. In fact, we have the product formula for the NSG (the solution operator of the problem) as is stated in §5.

Our study of the Stokes equation under the boundary conditions of friction type goes back to the author's series of lectures at Collège de France in October of 1993. Since then, as to the stationary flow, i.e., to the boundary value problem, the existence and uniqueness of the H^1 class solution has been established by means of the formulation through variational inequalities by the author and his collaborators (Fujita [6], Fujita-Kawarada [7], Fujita-Kawarada- Sasamoto [8]). This will be mentioned in §3.

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Recently, Norikazu Saito succeeded in showing the H^2 regularity of the solution of the boundary value problem, following suggestions given by H. Brezis and the author ([14]). His result will be also described in §3.

N. Saito's regularity result has led the author to the definition below of the required generator to match the NSG theory. This and integration of the initial problem with the aid of NSG will be done in §5 after we recall a core part of Komura's theory of NSG in §4.

2. Description of the Target Problem

First of all, let us write down the initial boundary value problem in question (abbr. S-IVP).

Usual symbols are employed: $u = u(t, x)$ and $p = p(t, x)$ stand for the flow velocity and the pressure, respectively, at time t and point x. x ranges over a bounded domain Ω in $R^n (n = 2, 3)$ bounded by smooth boundary $\Gamma = \partial\Omega$. The positive constant ν means the viscosity.

Time-dependent Stokes system

For $t \geq 0$ and $x \in \Omega$, $\{u, p\}$ should satisfy

$$\begin{cases} \dfrac{\partial u}{\partial t} &= \nu \Delta u - \nabla p + f, \\ \operatorname{div} u &= 0, \end{cases} \tag{2.1}$$

where f stands for the given external force.

The initial condition is given by

$$u(0, \cdot) = u_0 \qquad \text{in } \Omega. \tag{2.2}$$

In order to avoid non-essential complexity in phrasing, we assume that Γ is composed of two separated compact component Γ_0 and S, and that on Γ_0 the homogeneous Dirichlet boundary condition is imposed; namely,

$$u = 0 \qquad \text{on } \Gamma_0. \tag{2.3}$$

On the other hand, we shall impose

$$\text{a certain boundary condition of friction type} \qquad \text{on } S, \tag{2.4}$$

which will be specified soon.

2.1. Motivations for the BC of Friction Type

So far almost exclusively, the Dirichlet boundary condition (adhesion to solid surfaces) has been considered for motions of viscous incompressible fluids in hydrodynamics as well as in mathematics. However, there exist some flow phenomena, modeling of which might require introduction of slip and/or leak boundary conditions in reality or apparently (or metaphorically).

As examples, we can refer to the following; (1) flow through a drain or canal with its bottom covered by sherbet of mud and pebbles. (2) flow of melted iron coming out from a smelting furnace. (3) avalanche of water and rocks. (4) blood flow in a vein of an arterial sclerosis patient. (5) polymer-polymer welding and sliding phenomena as studied by P.G.de Gennes.

Furthermore, with some of these examples one observes that some fragile state of the surface or existence of sherbet zone *allows slipping of the fluid along the surface*, while *the fluid does not slip as long as the "force of stream" is below a threshold.*

In order to form a mathematical model of such slip phenomena, introduction of nonlinear slip boundary conditions of friction type (similar to Coulomb's law of friction) seems to be suitable.

Similarly, leak boundary conditions would be required when we want to model flow problems involving a leak of the fluid through the surface or penetration into the adjacent media. For instance, (1) flow through a net or sieve, e.g., a butterfly net. (2) flow through filter, e.g., a vacuum cleaner, diapers, a coffee maker. (3) water flow in a purification plant, filtration of rain to form underground water, (4) oil flow over or beneath sand layers (e.g., Kawarada-Suito [11]).

In fact, some filters prevent the leak if the "pushing force" is below a threshold, which seems again to suggest a need of introduction of the leak boundary condition of friction type, for their modeling

2.2. Additional Symbols

In order to formulate the BC of friction type, we need some additional symbols. Firstly, we introduce a barrier function $g = g(s)$ which is a positive function on S. For simplicity, we assume that $g \in L^2(S)$. The unit outer normal to Γ is denoted by $n = n(s) = (n_i)$. The deformation tensor of velocity u is written as

$$e_{ij} = e_{ij}(u) = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}, \quad e(u) = (e_{ij}(u)),$$

while the stress tensor $T = T(u,p) = (T_{ij})$ is defined as

$$T_{ij} = T(u,p)_{ij} = -p\delta_{ij} + \nu e_{ij}(u).$$

The stress vector $T(u,p)n$ on Γ is denoted by $\sigma = \sigma(u,p) = (T_{ij}n_j)$. Here and hereafter, the summation convention is used.

Finally, the normal component of vectors on the boundary will be meant by the suffix n. For instance, we shall write as u_n, $\quad \sigma_n = \sigma(u,p)_n = -p + n \cdot e(u)n$, while the tangential component of vectors on the boundary is written as u_t, $\quad \sigma_t = \sigma(u,p)_t = \sigma(u)_t$.

2.3. Leak BC of Friction Type

There are several boundary conditions of friction type which are worthy of consideration from the view point of application. However, in this paper we shall deal mostly with the leak BC of friction type, since it involves explicitly the pressure and leads to introduction of multi-valued generators.

By the *leak boundary condition of friction type* (leak BC-fric), we mean the following three conditions.

No-slip

$$u_t = 0 \qquad \text{on } S, \tag{2.5}$$

Bounded normal stress

$$|\sigma(u,p)_n| \le g \quad \text{on } S, \tag{2.6}$$

Possible leak Everywhere on S, we impose

$$\begin{cases} |\sigma_n| < g & \Longrightarrow \quad u_n = 0, \\ |\sigma_n| = g & \Longrightarrow \quad \begin{cases} u_n = 0 \text{ or } u_n \neq 0, \\ u_n \neq 0 \Longrightarrow -\sigma_n = g\dfrac{u_n}{|u_n|}. \end{cases} \end{cases} \tag{2.7}$$

In case that the leak actually takes place under the leak BC, the pressure is uniquely determined through the flow velocity. However, if $|\sigma_n| < g$ holds everywhere on S and so leak does not occur anywhere, then the pressure admits of an additive constant which can range over a certain interval.

It should be noted that the two conditions (2.6), (2.7) of the leak BC can be jointly reduced to the following single condition, in terms of the sub-differential of $|\ |: R^1 \longrightarrow R^1$ as follows.

$$-\sigma_n \in g\partial(|u_n|) \qquad \text{on } S. \tag{2.8}$$

2.4. Some Other BCs of Friction Type

We shall briefly mention the slip BC and the total movement BC of friction type. Although we do not treat these in this paper, they are of some importance in practical applications. The analysis which we carry out with the leak BC of friction type works equally well for these slip and total movement boundary conditions.

The *slip BC of friction type* on S is written as follows ([6], [8]).

No-leak

$$u_n = 0 \qquad \text{on } S, \tag{2.9}$$

Bounded tangential stress

$$|\sigma_t| \leq g \quad \text{on } S, \tag{2.10}$$

Possible slip Every where on S we impose

$$\begin{cases} |\sigma_t| < g & \Longrightarrow \quad u_t = 0, \\ |\sigma_t| = g & \Longrightarrow \quad \begin{cases} u_t = 0 \text{ or } u_t \neq 0, \\ u_t \neq 0 \Longrightarrow -\sigma_t = g\dfrac{u_t}{|u_t|}. \end{cases} \end{cases} \tag{2.11}$$

The slip BC-fric does not involve the pressure explicitly, since σ_t does not contain the pressure.

Again, the two conditions (2.10), (2.11) of the slip BC-fric are jointly reduced to the following one, if we use the sub-differential $\partial|\ |$ of the two-dimensional (one-dimensional) Euclidean norm $|\ | : R^2 \longrightarrow R^1$ if $\Omega \subset R^3$ ($|\ | : R^1 \longrightarrow R^1$ if $\Omega \subset R^2$).

$$-\sigma_t \in g\partial(|u_t|). \tag{2.12}$$

As for the *total BC of friction type*, we simply remark that it can be written in terms of the sub-differential of the 3-dimensional Euclidean norm as follows. For theoretical analysis, this BC of friction type can be regarded as standard ([7]).

$$-\sigma \in g\partial(|u|) \qquad \text{on } S. \tag{2.13}$$

S-IVP

From now on, we restrict our consideration to the leak BC of friction type, and shall write simply S-IVP to mean the initial boundary value problem which is composed of the Stokes equation, the initial condition and the above-mentioned boundary conditions on Γ_0 and S.

3. Stationary Problems

For its own sake, and as a preparation for the study of S-IVP, we consider the relevant boundary value problem for the stationary Stokes flow under the leak BC of friction type, which will be denoted by S-BVP. Namely, S-BVP means: Find $\{u, p\}$ such that

$$\begin{cases} -\nu\Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0, \end{cases} \tag{3.1}$$

and the Dirichlet boundary condition (2.3) on Γ_0 as well as the leak BC on S. Here f is a given vector function in $X = L^2(\Omega)$.

In addition, for our later purpose, we should consider a reduced form S-BVP-r (a resolvent type) of S-BVP. Actually, S-BVP-r is given, together with the same boundary conditions as above, by

$$\begin{cases} u - \nu\Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0. \end{cases} \tag{3.2}$$

3.1. Theorems for S-BVP

The existence theorem of S-BVP and S-BVP-r in the form of H^1- theory was originally proved on the occasion of the author's lecture at Collège de France in 1993 ([5]), and has been elaborated in some directions ([6], [8],[7]). Actually, we have

Theorem 3.1. *For any $f \in L^2(\Omega)$, the solution $\{u,p\}$ of S-BVP and (S-BVP-r) exists. Actually,*

$$u \in H^1(\Omega), p \in L^2(\Omega), \sigma_n \in H^{-1/2}(S). \tag{3.3}$$

u and ∇p are unique but the uniqueness of p depends.

Proof of Theorem 3.1 was done by means of variational inequalities with a non-trivial argument, which makes use of the Hahn-Banach extension theorem. To describe these variational inequalities, S-VI, we first introduce the following class of admissible functions:

$$K = \{v \in H^1(\Omega) \,|\, \operatorname{div} v = 0, v = 0 \text{ on } \Gamma_0, u_t = 0 \text{ on } S\}. \tag{3.4}$$

Except for the outflow condition $\int_S v_n d\Gamma = 0$, the admissible functions are free on S.

Then u is sought as the solution of the following:

S-VI.

Find $u \in K$ such that

$$a(v-u,u) + j(v) - j(u) \geq (v-u,f) \qquad (\forall v \in K). \tag{3.5}$$

Here (,) stands for the $L^2(\Omega)$-inner product. Under the summation convention, it means that

$$a(u,v) = \frac{\nu}{2}\int_\Omega e_{ij}(u)e_{ij}(v)dx. \tag{3.6}$$

The barrier term is defined by

$$j(v) = \int_S g|u_n|d\Gamma \tag{3.7}$$

The variational formulation S-VI-r for S-BVP-r is obtained from S-VI, if we replace $a(v-u,u)$ by $a(v-u,u)+(v-u,u)$, the arguments for the existence and uniqueness of solutions are just the same as for S-IV.

Now we state the regularity theorem for the solution which was recently proved by N. Saito following suggestions by H. Brezis and the author ([14]).

Theorem 3.2. *Let the positive barrier function g belong to $H^1(S)$. Then we have the following regularity of $\{u,p\}$ if $f \in L^2(\Omega)$:*

$$u \in H^2(\Omega), p \in H^1(\Omega), \sigma_n \in H^{1/2}(S). \tag{3.8}$$

Let us remark that the two theorems above hold true for other BCs of friction type.

4. Review of Non-linear Semigroup Theory

With the intention to apply it to S-IVP, we review briefly the NSG theory, which originated from Y. Kōmura (1967). The theory has been extended and elaborated in various directions, say, to its Banach space version. Here we just need the original version.

4.1. Monotone Operators in Hilbert Space First of all, we state

Definition 4.1. *A multi-valued operator A in Hilbert space X is monotone (or accretive) if*

$$(f_1 - f_2, u_1 - u_2) \geq 0 \quad (\forall u_1, u_2 \in D(A), \quad \forall f_1 \in Au_1, \forall f_2 \in Au_2), \tag{4.1}$$

where $D(A)$ is the domain of definition of A.

The following definition is concerned with the maximality of monotone property.

Definition 4.2. *A monotone operator A is m-monotone (or m-accretive), if*

$$R(I + A) = \textit{Range of } (I + A) = X. \tag{4.2}$$

As for a monotone operator, the condition (4.2) is equivalent to

$$R(I + \lambda A) = X \qquad (\forall \lambda > 0). \tag{4.3}$$

If A is an m-monotone operator, then the subset Au is an non-empty closed convex set in X for each $u \in D(A)$, which enables us to introduce the following definition.

Definition 4.3. *Let A be an m-monotone operator. Then its canonical restriction A^0 is defined by assigning as A^0u the element with the smallest norm in Au.*

Sometimes, one prefers the following terminology:

Definition 4.4. *An operator B in X is dissipative if $-B$ is monotone, and is m-dissipative if $-B$ is m-monotone.*

4.2. Evolution Equations with m-dissipative Operators

Let A be an m-monotone operator. We consider the following abstract initial value problem.

abst-IVP

Let A be an m-monotone operator and let a be an element in $D(A)$.

Then, the abst-IVP is to find $u = u(t)$ which is an $X-$valued absolutely continuous function on $[0, +\infty)$ such that the evolution condition

$$\frac{du}{dt} \in -Au(t) \quad (\text{ a.e.} t), \tag{4.4}$$

and the initial condition

$$u(0) = a \tag{4.5}$$

hold true.

Then we have

Theorem 4.3. *The abst-IVP is uniquely solvable. Moreover, the solution $u(t) \in D(A)$ for every t, and it satisfies*

$$\frac{d^+u}{dt} = -A^0u(t) \quad (\forall t \in [0, +\infty)). \tag{4.6}$$

Remark. An important sub-class of m-monotone operators is the sub-differential of a convex functional defined on Hilbert space. Then we can relax the restriction for the initial values for the abst-IVP, in some practical applications, our S-IVP being a case. This will be presented in our forthcoming paper. Here we just note that the solution of the abst-IVP can be represented by certain product formula which could be useful as foundation of some approximating methods ([3], [2], [15]).

Theorem 4.4. *The solution $u = u(t)$ of the abst-IVP is represented by*

$$u(t) = s - \lim_{n\to\infty} (I + \frac{1}{n}A)^{-[nt]}a \tag{4.7}$$

5. Integration of S-IVP by NSG Theory

Let us come back to S-IVP. We want to define an m-monotone operator A in the Hilbert space $X = L^2(\Omega)$ so that its abst-IVP implies S-IVP.

To this end, there are two ways. One way is to make an explicit definition of A, making use of the regularity theorem (Theorem 3.2), while the other way is to define A as the sub-differential of a certaain covexfunctional which is closely related with S-VI. As mentioned above, in this paper we only present the way of the explicit definition of A. To this end, we now assume that g satisfies the assumption of Theorem 3.2.

To begin with, we define $\mathcal{K}$ by

$$\mathcal{K} = H^2(\Omega) \cap K \tag{5.1}$$

where K is the class of admissible functions in §3. Furthermore, for each $u \in \mathcal{K}$, let $M(u)$ be a subset of scalar $H^1(\Omega)$ defined as

$$M(u) = \{p \in H^1(\Omega) \text{ such that (2.8) holds true.}\} \tag{5.2}$$

Thus any $p \in M(u)$ satisfies the condition that where $u_n \neq 0$,

$$p = n \cdot e(u)n + g\frac{u_n}{|u_n|},$$

and, where $u_n = 0$,

$$-g \leq p - n \cdot e(u) \leq g.$$

Here the domain $D(A)$ of A is introduced as the set

$$D(A) = \{u \in \mathcal{K} \text{ with non empty } M(u)\}. \tag{5.3}$$

Finally, A is defined by setting

$$Au = \{-\nu\Delta u + \nabla p \quad | \quad p \in M(u)\} \qquad (\forall u \in D(A)). \tag{5.4}$$

Then we can verify the monotonicity of A by means of a straight forward calculation (by Green's formula and monotonicity of the sub-differential $\partial|\cdot|$). The solvability of $u + Au = f$ for all $f \in X$ follows from Theorems 3.1 and 3.2. Thus we have

Theorem 5.5. *The operator A defined by (5.3), (5.4) is m-monotone.*

Obviously, the abst-IVP with this A is a strong version of S-IVP. Now by Theorem 4.1, the following theorem is true, which is the main result of this paper.

Theorem 5.6. *For $a \in D(A)$, S-IVP is uniquely solvable. Moreover, we have (4.4) as well as (4.6), which means, in particular, that $u(t,\cdot) \in H^2(\Omega)$, $p(t,\cdot) \in H^1(\Omega)$, and that $\nabla p(t,\cdot)$ is equal to the (nonlinear) projection of $\nu\Delta u(t,\cdot)$ on $M(u(t,\cdot))$ for each t.*

References

[1] H. Brezis, Opérateurs Maximaux Monotone et Semi-groupes de Contractions dans les Espaces de Hilbert, Math. Studies **5**, North-Holland, 1973.

[2] H. Brezis, A. Pazy, Accretive sets and differential equations in Banach spaces, *Israel J. of Math.,* **8** (1970), 367-383.

[3] M.G. Crandall, T. Liggett, Generation of semi-groups of nonlinear transformations in general Banach spaces, *Amer. J. Math.,* **93,** (1971), 265-298.

[4] G. Duvaut, J.L. Lions, Les Inéquations en Méchanique et en Physique, Dunod, 1972, (English version), Springer, 1976.

[5] H. Fujita, Flow problems with unilateral boundary conditions, Leçons, Collège de France, October, 1993.

[6] H. Fujita, A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions, *RIMS Kōkyūroku* **888**, Kyoto University, 1994 (Proc. of *Conf. on Math. Fluid Dynamics and Modeling*, May 30- June 2, 1994, Research Institute of Mathematical Science, Kyoto University), 199-216.

[7] H. Fujita, H. Kawarada, Variational inequalities for the Stokes equation with the boundary conditions of friction type, *Recent Development in Domain Decomposition Methods and Flow Problems*, 15-33, Gakuto Intern. Ser. Math. Sci. Appl. **11**, Gakkotosho, Tokyo, 1998

[8] H. Fujita, H. Kawarada, A. Sasamoto, Analytical and numerical approaches to stationary flow problems with leak and slip boundary conditions, *Kinokuniya Lecture Note in Num. Appl. Anal.*, **14**(1995), pp.17-31. *Advances in Numerical Mathematics* ; Proc. of *the 2nd Japan-China Seminar on Numer. Math.*, August, 1994 in Chofu, Japan.

[9] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer series in computational physics, Springer-Verlag, 1984.

[10] T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc.Japan,* **19** (1967), 508-520.

[11] H. Kawarada, H. Suito, Influence of spilled oil to bring about on coastal ecosystem, to appear in Proc. of *the 7th Symposium on Differential Equations and their Applications* (dedicated to J.L. Lions on his 70th Birthday), 1999, Houston, USA.

[12] Y. Kōmura: Nonlinear semigroups in Hilbert space, *J. Math. Soc. Japan,* **19** (1967), 493-507

[13] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Revised English edition translated by R. A. Silverman, Gordon and Breach, 1963, (Russian original,1961).

[14] N. Saito, H. Fujita, Regularity of solutions to the Stokes equations under a certain nonlinear boundary conditions, to appear in Proc. of *the 2nd Intern. Conf. on Navier-Stokes Equations: Theory and Numerical Methods*, 2000, Varenna, Italy.

[15] K. Yosida, Functional Analysis, Springer, 1st edition 1965, 6th edition 1980.