Journal of Computational Mathematics, Vol.20, No.6, 2002, 643–652.

Regular Splitting and Potential Reduction Method for Solving Quadratic Programming Problem with Box Constraints*[1)]

Zi-Luan Wei
(*Institute of Computational Mathematics and Scientific / Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing, 100080*)

Abstract

A regular splitting and potential reduction method is presented for solving a quadratic programming problem with box constraints (QPB) in this paper. A general algorithm is designed to solve the QPB problem and generate a sequence of iterative points. We show that the number of iterations to generate an ϵ-minimum solution or an ϵ-KKT solution by the algorithm is bounded by $O(\frac{n^2}{\epsilon}log\frac{1}{\epsilon}+nlog(1+\sqrt{2n}))$, and the total running time is bounded by $O(n^2(n+logn+log\frac{1}{\epsilon})(\frac{n}{\epsilon}log\frac{1}{\epsilon}+logn))$ arithmetic operations.

Key words: Quadratic programming problem, Regular splitting, Potential reduction algorithm, Complexity analysis.

1. Introduction

In this paper, we consider a specail form of a quadratic programming problem with box constrained variables(QPB) as follows:

$$QPB: \qquad min\ q(x) \quad \text{s.t. } (x,s)\in\Omega$$

where $\Omega=\{(x,s)\in R^n\times R^n : x+s=e,\ x\geq 0,\ s\geq 0\}$ is the feasible region of the problem and s is a slack vector, and Ω^0 denotes the set of interior points of Ω, and $q(x)=\frac{1}{2}x^THx+c^Tx$, and $H\in R^{n\times n}$ is a symmetric matrix, and $c,\ e\in R^n$ are given vectors and all the elements of e are one. Without loss of generality, if the constrained variables of a quadratic programming problem with box constraints are bounded, then the problem can be transformed into the QPB specail form.

This problem arises in several areas of applications, such as problem of differential equations, discrete optimal control with continue time and design engineering, linear least square problem with box constraints or as a sequential subproblem of nonlinear programming methods. Therefore, it has a special importance.

Many different algorithms have been studied for solving this type of problem, such as projection gradient method[1], active-set method[12], matrix splitting methods[2,3,9], and the interior point method[10,11]. If the QPB problem is a convex problem, then it can be solved in polynomial time. If the QPB problem is a nonconvex problem, then it becomes a hard problem–NP complete problem. Some of algorithms can be also used to solve the problem, but it is difficult to obtain a global or local minimal solution[5-8]. On the other hand, searching a local minimum or checking the existence of a KKT point are an NP complete problem for a class of nonconvex optimization problems[7]. Therefore, ϵ-approximate minimizer or ϵ-KKT point was introduced in combinatorial optimization[6,7]. Finding an ϵ-minimizer or ϵ-KKT point is also

* Received.
1) This research supported partially by The National Natural Science Foundation of China–19771079 and 19601035, and State Key Laboratory of Scientific and Engineering Computing. e-mail address: wzl.lsec.cc.ac.cn, Fax 186–10–6254.2285.

hard problem. The complexity of finding an ϵ-approximate minimizer or ϵ-KKT point have been studied by many authors, and some of the results have been used in practice[11]. It would be mentioned that the steepest-descent-type method was used to compute an ϵ-KKT point of the QPB problem, and the complexity of the algorithm was analyzed, and the arithmetic operations of the algorithm was bounded by $O(n^3(\frac{L}{\epsilon})^2)$, where L is a fixed number depending on the problem data[6,7]. Other results are also discussed in [11].

In this paper, we present a regular splitting and potential reduction method for solving the QPB problem. The goal of the paper is to try finding a easy way to solve the problem. The main idea of the algorithm is to introduce a potential function for the original QPB problem and split the matrix H into the sum of two matrices H_1 and H_2 such that $(H_1 - H_2)$ is a symmetric positive definite matrix, and a new minimization problem with Hessian matrix H_1 and an ellipsoid constraint is considered instead of solving the original QPB problem. The potential reduction techniques are used to solve the new problem such that the value of the potential function is reduced by a constant at each iteration. An ϵ-minimum solution and ϵ-KKT solution for QPB problem is defined, respectively. A general algorithm is designed to solve the QPB problem and generates a sequence of iterative points. We show that the number of total iterations to generate an ϵ-minimum solution or an ϵ-KKT solution by the algorithm is bounded by $O(\frac{n^2}{\epsilon}log\frac{1}{\epsilon} + nlog(1+\sqrt{2n}))$, and the total running time is bounded by $O(n^2(n + logn + log\frac{1}{\epsilon})(\frac{n}{\epsilon}log\frac{1}{\epsilon} + logn))$ arithmetic operations.

2. Regular splitting and potential reduction algorithm

The regular splitting and potential reduction algorithm for solving the QPB problem will be described in this section. For the sake of convenience, some of definitions and the basic results are firstly introduced.

Proposition 1. $(x^*, s^*) \in R^n \times R^n$ is a minimum solution of the QPB problem, then there is $(y, z) \in R^n \times R^n$ such that the following relationships hold

$$x^* + s^* = e, \quad x^* \geq 0, \quad s^* \geq 0, \tag{2.1a}$$

$$Hx^* + c + y - z = 0, \quad y \geq 0, \quad z \geq 0, \tag{2.1b}$$

$$y^T x^* = 0, \quad z^T s^* = 0. \tag{2.1c}$$

The formula (2.1) is the first order optimality conditions or KKT condition of the QPB problem. Let $\bar{\Omega} = \{(x, y, z) \in R^n \times R^n \times R^n : Hx + c + y - z = 0, x \geq 0, y \geq 0, z \geq 0\}$. Thus, $\bar{\Omega}$ is the set of dual feasible region of the QPB problem.

Definition 1. $(H_1, H_2) \in R^{n\times n} \times R^{n\times n}$ is said to be a regular splitting of $H \in R^{n\times n}$ if (i) $H = H_1 + H_2$ and (ii)$(H_1 - H_2)$ is a positive definite matrix.

Let l_e and u_e denote the minimal and maximal objective value of the QPB problem on Ω, respectively. Then we can define an ϵ-minimal solution or ϵ-KKT solution of the QPB problem, respectively.

Definition 2. $(x, s) \in \Omega$ is said to be an ϵ-minimum solution of the QPB problem, $\epsilon \in (0, 1)$ if $\frac{q(x) - l_e}{u_e - l_e} \leq \epsilon$. Similarly, $(x, s) \in \Omega$ is said to be an ϵ-KKT solution for the QPB problem if $(x, y, z) \in \bar{\Omega}$, and $\frac{x^T y + s^T z}{u_e - l_e} \leq \epsilon$.

As is well known, the potential reduction algorithm is usually required to start at an analytic center point or an approximate analytic center point of the feasible region for the solved problem. So, it is easy to show that $x^0 = \frac{1}{2}e$ and $s^0 = \frac{1}{2}e$ are the analytic center point of the feasible region Ω, and that there are two ellipsoids V_1 and V_2 such that $\Omega \supset V_1 = \{(x, s) \in \Omega, ||(X^0)^{-1}(x - x^0)||^2 + ||(S^0)^{-1}(s - s^0)||^2 \leq 1\}$, and $\Omega \subset V_2 = \{(x, s) \in \Omega, ||(X^0)^{-1}(x - x^0)||^2 + ||(S^0)^{-1}(s - s^0)||^2 \leq 2n\}$. Where X, S denote the diagonal matrices with elements of x, s, respectively. In other word, Ω is inscribed and outscribed by V_1 and V_2, respectively. Thus, we have the following conclusion.

Proposition 2. $\forall x \in \Omega^0$, then

$$\sum_{i=1}^{n} log(\frac{x_i^0}{x_i}) \geq -nlog(1+\sqrt{2n}), \qquad \sum_{i=1}^{n} log(\frac{s_i^0}{s_i}) \geq -nlog(1+\sqrt{2n}). \tag{2.2}$$

Proof. It follows from the definition of the outscribed ellipsoid V_2 that x^0, x are both in Ω and $\frac{x_i}{x_i^0} \leq 1+\sqrt{2n}$. This gives $\frac{x_i^0}{x_i} \geq \frac{1}{1+\sqrt{2n}}$, which implies the first part of the proposition is true. Similarly, the second part of the proposition is also true.

Proposition 3. If $d \in R^n$ and $||d||_\infty < 1$, then

$$e^T d \geq \sum_{i=1}^{n} log(1+d_i) \geq e^T d - \frac{||d||_\infty^2}{2(1-||d||_\infty)}. \tag{2.3}$$

Now a potential function $F(x,s)$ used to solve the QPB problem is defined as

$$F(x,s) = (n+\rho)log(q(x)-w) - \sum_{i=1}^{n} log x_i s_i \tag{2.4}$$

where $(x,s) \in \Omega^0$, parameter $\rho \geq 0$, and $w < l_e$, here w is fixed number during the iterative process. Assuming that $\forall (x,s) \in \Omega^0$, and $(x^+, s^+) \in \Omega^0$, we have

$$x^+ = x + (x^+ - x) = x + dx, \quad s^+ = s + (s^+ - s) = s + ds. \tag{2.5}$$

Substituting the above expression into the potential function $F(x,s)$, it is easy to derive that

$$\begin{aligned} F(x^+,s^+) - F(x,s) &= (n+\rho)log(\frac{q(x^+)-w}{q(x)-w}) - \sum_{i=1}^{n}(log\frac{x_i^+}{x_i} + log\frac{s_i^+}{s_i}) \\ &=(n+\rho)log(\frac{q(x^+)-w}{q(x)-w}) - \sum_{i=1}^{n}(log(1+x_i^{-1}dx_i) + log(1+s_i^{-1}ds_i)). \end{aligned} \tag{2.6}$$

On the other hand, if set $\Delta = q(x)-w, g = g(x) = Hx + c$, and (dx, ds) is very small, then from the definition of $q(x)$ and (2.5), it is clear that

$$\begin{aligned} log\frac{(q(x^+)-w)}{q(x)-w} &=log\frac{q(x)-w+\frac{1}{2}dx^T H dx + g^T dx}{q(x)-w} \\ &=log(1+\frac{1}{\Delta}(\frac{1}{2}dx^T H dx + g^T dx)) \leq \frac{1}{\Delta}(\frac{1}{2}dx^T H dx + g^T dx). \end{aligned} \tag{2.7}$$

Furthermore, if $||X^{-1}dx||^2 + ||S^{-1}ds||^2 \leq \alpha^2 < 1$, and set $\bar{g} = Hx + c - \frac{\Delta}{n+\rho}X^{-1}e$, then it follows from (2.6)-(2.7) and the proposition 3 that

$$F(x^+,s^+) - F(x,s) \leq \frac{n+\rho}{\Delta}(\frac{1}{2}dx^T H dx + \bar{g}^T dx - \frac{\Delta}{n+\rho}e^T S^{-1} ds) + \frac{\alpha^2}{(1-\alpha)} \tag{2.8}$$

Now assume that (H_1, H_2) is a regular splitting of the matrix H, then the following quadratic programming problem(BQP) with an ellipsoid constraint is introduced in order to achieve a potential reduction.

$$min \quad \varphi(x,s) = \frac{1}{2}dx^T H_1 dx + \bar{g}^T dx - \frac{\Delta}{n+\rho}e^T S^{-1} ds \tag{2.9a}$$

$$\text{s.t.} \quad ||X^{-1}dx||^2 + ||S^{-1}ds||^2 \leq \alpha^2 < 1, \quad dx + ds = 0. \tag{2.9b}$$

Assume that (dx, ds) is an optimal solution of the BQP problem, and λ denotes the Lagrange multiplier associated with the ellipsoid constraint. Let $p_x = -\lambda X^{-1} dx, \quad p_s = -\lambda S^{-1} ds$. Thus, one can see that

$$||p||^2 = ||p_x||^2 + ||p_s||^2 = \lambda^2(||X^{-1}dx||^2 + ||S^{-1}ds||^2) = \lambda^2\alpha^2.$$

where $p^T=(p_x^T,p_s^T)$. Obviously, it follows from the above relation that $||p||=\lambda\alpha, \qquad \lambda=\frac{||p||}{\alpha}$. This gives the conclusion(it will be shown in the next section)

$$F(x^+,s^+)-F(x,s)\le -\frac{n+\rho}{\Delta}\alpha||p||+\frac{\alpha^2}{1-\alpha}=-\alpha(\frac{n+\rho}{\Delta}||p||-\frac{\alpha}{1-\alpha}).$$

Clearly, it is desirable to choose a suitable parameter ρ and α such that the potential function is reduced by a constant at each iteration. A regular splitting and potential reduction algorithm can be constructed from starting point $x^0=s^0=\frac{1}{2}e$ and an iterative sequence (x^k,s^k) is generated by solving the problem BQP, then one can easily show that

$$F(x^k,s^k)-F(x^0,s^0)=(n+\rho)log(\frac{q(x^k)-w}{q(x^0)-w})-\sum_{i=1}^{n}(log\frac{x_i^k}{x_i^0}+log\frac{s_i^k}{s_i^0}).$$

If $\frac{q(x^k)-w}{q(x^0)-w}\le\epsilon$, then one can directly derive that $\frac{q(x^k)-l_e}{u_e-l_e}\le\frac{q(x^k)-l_e}{q(x^0)-l_e}\le\frac{q(x^k)-w}{q(x^0)-w}\le\epsilon$. This means that (x^k,s^k) is an ϵ-minimal solution. Therefore, it is easy to see that if $F(x^k,s^k)-F(x^0,s^0)\le(n+\rho)log\epsilon-2nlog(1+\sqrt{2n})$ holds, then (x^k,s^k) is an ϵ-minimizer for the QPB problem. Thus, it is hopeful that if $\alpha=\frac{1}{4}$ and $\frac{n+\rho}{\Delta_k}||p^k||\ge\frac{3}{4}$ holds at each iteration, then an ϵ-minimizer for the QPB problem can be generated in the number of $O((n+\rho)log\frac{1}{\epsilon}+nlog(1+\sqrt{2n}))$ iterations.

Thus, a regular splitting and potential reduction algorithm can be described as follows:

Algorithm A:

Assume that choose parameters $\rho+n=\frac{4n(2n+\sqrt{n})}{\epsilon}$, $\alpha=\frac{1}{4}$, and $\epsilon\in(0,1)$ and (H_1,H_2) is a regular splitting of matrix H, and that $x^0=s^0=\frac{1}{2}e\in\Omega$ is a given initial vector. Calculate $q(x^0)$ and lower bound value w. Let $k:=0$, then carry out the following steps.

(i) Compute $q(x^k)$. If $\frac{q(x^k)-w}{q(x^0)-w}\le\epsilon$, then terminate and (x^k,s^k) is an ϵ-minimal solution for the QPB subproblem; else, go to next step.

(ii) Solve the BQP subproblem. If $\frac{n+\rho}{\Delta_k}||p^k||\le\frac{3}{4}$, then terminate and (x^k,s^k) is an ϵ-KKT solution for the QPB problem; else, go to the next step.

(iii) Let $x^{k+1}=x^k+dx^k,\ \ s^{k+1}=s^k+ds^k$ and $k:=k+1$, return to the step (i).

It follows from the construction of the algorithm that the main computational work is from solving the BQP subproblem at each iteration.

3. Complexity analysis of the algorithm

This section deals with the complexity analysis of the algorithm A defined in the previous section. For the purpose, several lemmas are described as follows.

Lemma 3.1. Suppose that $(x,s)\in\Omega^0$, $\alpha=\frac{1}{4}$ and (dx,ds) is an optimal solution of the BQP problem, if $\frac{n+\rho}{\Delta}||p||\ge\frac{3}{4}$, then

$$F(x^+,s^+)-F(x,s)\le-\frac{5}{48}. \tag{3.1}$$

Proof. Assume that (dx,ds) is the optimal solution of the BQP problem. Thus, it follows from the 1-st order optimality conditions of the BQP problem that there are λ and $\bar{y}\in R^n$ such that

$$H_1dx+Hx+c-\frac{\Delta}{n+\rho}X^{-1}e-\bar{y}+\lambda X^{-2}dx=0, \tag{3.2a}$$

$$-\frac{\Delta}{n+\rho}S^{-1}e-\bar{y}+\lambda S^{-2}ds=0, \tag{3.2b}$$

$$||X^{-1}dx||^2+||S^{-1}ds||^2\le\alpha^2 \tag{3.2c}$$

$$dx+ds=0, \tag{3.2d}$$

$$\lambda(\alpha^2-||X^{-1}dx||^2-||S^{-1}ds||^2)=0, \qquad \lambda\ge 0, \tag{3.2e}$$

hold. Now multiplying both sides of (3.2a) and (3.2b) by dx^T and ds^T, respectively, and adding the both sides of them together, then from (3.2d)- (3.2e) it gives that

$$\frac{1}{2}dx^T H_1 dx + (Hx + c - \frac{\Delta}{n+\rho}X^{-1}e)^T dx - \frac{\Delta}{n+\rho}e^T S^{-1} ds = -\frac{1}{2}dx^T H_1 dx - \lambda\alpha^2, \qquad (3.3)$$

From (2.9) and (3.3), and the regular splitting of the matrix H, it is easy to show that

$$\begin{aligned} F(x^+, s^+) - F(x,s) \le & \frac{n+\rho}{\Delta}[-\frac{1}{2}dx^T H_1 dx - \lambda\alpha^2 + \frac{1}{2}dx^T H_2 dx] + \frac{\alpha^2}{1-\alpha} \\ =& \frac{n+\rho}{\Delta}[-\lambda\alpha^2 - \frac{1}{2}dx^T(H_1 - H_2)dx] + \frac{\alpha^2}{1-\alpha} \le -\frac{n+\rho}{\Delta}\lambda\alpha^2 + \frac{\alpha^2}{1-\alpha} \\ =& \alpha[-\frac{n+\rho}{\Delta}(\lambda\alpha) + \frac{\alpha}{1-\alpha}] = \alpha(-\frac{n+\rho}{\Delta}||p|| + \frac{\alpha}{1-\alpha}). \end{aligned} \qquad (3.4)$$

since $||p|| = \lambda\alpha$. If $\frac{n+\rho}{\Delta}||p|| \ge \frac{3}{4}$ and $\alpha = \frac{1}{4}$, then it is clear that

$$F(x^+, s^+) - F(x,s) \le \frac{1}{4}(-\frac{3}{4} + \frac{1}{3}) = -\frac{1}{4} \times \frac{5}{12} = -\frac{5}{48},$$

which proves the conclusion of the lemma.

Remark: From the formula (3.4), it is clear that the semi-positive definiteness of the matrix $(H_1 - H_2)$ can also guarantee the value of the potetial function is reduced by a constant at each iteration.

Lemma 3.2. Assume that $A \in R^{m\times n}$ and $r > 0$ is given scale, and let $d(r)$ be the minimizer for the following problem

$$min \quad q(d) = \frac{1}{2}d^T H d + c^T d, \quad s.t. \quad Ad = 0, \quad ||d||^2 \le r^2.$$

Then $q(0) - q(d(r)) \ge (\frac{r}{R})^2 (q(0) - q(d(R)))$ holds for $0 < r \le R$.

Now we consider the following quadratic programming problem

$$\begin{aligned} & min \quad (q(x) - q(x^0)) = \frac{1}{2}(x - x^0)^T H (x - x^0) + c^T(x - x^0) \\ & s.t. \quad (x - x^0) + (s - s^0) = 0, \qquad ||(x - x^0)||^2 + ||(s - s^0)||^2 \le \frac{1}{2}. \end{aligned}$$

Assume that $(\hat{x}, \hat{s})$ is the minimal solution of the above problem, and let

$$w = q(x^0) - 2n(q(x^0) - q(\hat{x})), \qquad (3.5)$$

then $w < l_e$. From the Lemma 3.2, it is straightforward to show that the above inequality is true.

Lemma 3.3. Suppose that $\rho + n = \frac{4n(2n+\sqrt{n})}{\epsilon}$ and w is chosen as in (3.5) and $\alpha = \frac{1}{4}$ in condition (3.2), if $||p^k|| < \frac{\Delta_k}{n+\rho}$, then $x^{k+1} = x^k + dx^k$, $s^{k+1} = s^k + ds^k$ is an ϵ-KKT solution for the QPB problem.

Proof: It follows from the first order optimality conditions of the BQP problem and the definition of $||p^k||$ that

$$\begin{aligned} p_x^k &= X^k(H_1(x^k + dx^k) + H_2 x^k + c - \frac{\Delta_k}{n+\rho}(X^k)^{-1}e - \bar{y}(\lambda_k)) = X^k u(\lambda_k) - \frac{\Delta_k}{n+\rho}e, \\ p_s^k &= S^k(-\frac{\Delta_k}{n+\rho}(S^k)^{-1}e - \bar{y}(\lambda_k)) = -S^k\bar{y}(\lambda_k) - \frac{\Delta_k}{n+\rho}e. \end{aligned}$$

where $u(\lambda_k) = H_1(x^k + dx^k) + H_2 x^k + c - \bar{y}(\lambda_k) = H_1 x^{k+1} + H_2 x^k + c - \bar{y}(\lambda_k)$.

By the assumption $||p^k|| < \frac{\Delta_k}{n+\rho}$, which implies that $\frac{n+\rho}{\Delta_k}||p^k|| < 1$. Namely,

$$\begin{aligned} (\frac{n+\rho}{\Delta_k}||p^k||)^2 &= (\frac{n+\rho}{\Delta_k})^2(||p_x^k||^2 + ||p_s^k||^2) \\ &= ||\frac{n+\rho}{\Delta_k}X^k u(\lambda_k) - e||^2 + || - \frac{n+\rho}{\Delta_k}S^k\bar{y}(\lambda_k) - e||^2 < 1. \end{aligned} \qquad (3.6)$$

Since $X^k = diag(x^k) > 0,\ S^k = diag(s^k) > 0$, from (3.6), it is easy to derive that

$$u(\lambda_k) = H_1(x^k + dx^k) + H_2x^k + c - \bar{y}(\lambda_k) > 0, \quad -\bar{y}(\lambda_k) > 0.$$

If set $y(\lambda_k) = -\bar{y}(\lambda_k)$, then the above formula can be rewritten as

$$u(\lambda_k) = H_1(x^k + dx^k) + H_2x^k + c + y(\lambda_k) > 0, \quad y(\lambda_k) > 0. \tag{3.7}$$

By the above relationship it is easy to verify that

$$\begin{aligned}
(\frac{n+\rho}{\Delta_k}||p^k||)^2 =&||\frac{n+\rho}{\Delta_k}X^k u(\lambda_k) - (\frac{n+\rho}{\Delta_k})\frac{(x^k)^T u(\lambda_k)}{n}e + (\frac{n+\rho}{\Delta_k})\frac{(x^k)^T u(\lambda_k)}{n}e - e||^2 \\
&+ ||\frac{n+\rho}{\Delta_k}S^k y(\lambda_k) - (\frac{n+\rho}{\Delta_k})\frac{(s^k)^T y(\lambda_k)}{n}e + (\frac{n+\rho}{\Delta_k})\frac{(s^k)^T y(\lambda_k)}{n}e - e||^2 \\
=&||\frac{n+\rho}{\Delta_k}X^k u(\lambda_k) - (\frac{n+\rho}{\Delta_k})\frac{(x^k)^T u(\lambda_k)}{n}e||^2 + ||(\frac{n+\rho}{\Delta_k})\frac{(x^k)^T u(\lambda_k)}{n}e - e||^2 \\
&+ ||\frac{n+\rho}{\Delta_k}S^k \bar{y}(\lambda_k) - (\frac{n+\rho}{\Delta_k})\frac{(s^k)^T y(\lambda_k)}{n}e||^2 + ||(\frac{n+\rho}{\Delta_k})\frac{(s^k)^T y(\lambda_k)}{n}e - e||^2 \\
\geq&||\frac{(n+\rho)(x^k)^T u(\lambda_k)}{n\Delta_k}e - e||^2 + ||\frac{(n+\rho)(s^k)^T y(\lambda_k)}{n\Delta_k}e - e||^2 \\
\geq& n[\frac{(n+\rho)(x^k)^T u(\lambda k)}{n\Delta_k} - 1]^2 + n[\frac{(n+\rho)(s^k)^T y(\lambda_k)}{n\Delta_k} - 1]^2.
\end{aligned}$$

By the above formula and (3.7) one can see that

$$[\frac{(n+\rho)(x^k)^T u(\lambda_k)}{n\Delta_k} - 1]^2 + [\frac{(n+\rho)(s^k)^T y(\lambda_k)}{n\Delta_k} - 1]^2 \leq \frac{1}{n} = (\frac{1}{\sqrt{n}})^2.$$

Thus, it is straightforward to derive that

$$\frac{2n-\sqrt{n}}{n+\rho} \leq \frac{(x^k)^T u(\lambda_k) + (s^k)^T y(\lambda_k)}{\Delta_k} \leq \frac{2n+\sqrt{n}}{n+\rho}. \tag{3.8}$$

On the other hand, it follows from the definition of the algorithm A that $x^{k+1} = x^k + dx^k > 0,\ s^{k+1} = s^k + ds^k > 0$. So, one can see that

$$\begin{aligned}
\omega_{k+1} = (x^{k+1})^T u(\lambda_k) + (s^{k+1})^T y(\lambda_k) &= (x^k)^T (X^k)^{-1} X^{k+1} u(\lambda_k) + (s^k)^T (S^k)^{-1} S^{k+1} y(\lambda_k) \\
&\leq ||(X^k)^{-1}X^{k+1}||(x^k)^T u(\lambda_k) + ||(S^k)^{-1}S^{k+1}||(s^k)^T y(\lambda_k) \\
&\leq (1+\alpha)((x^k)^T u(\lambda_k) + (s^k)^T y(\lambda_k)) \leq 2((x^k)^T u(\lambda_k) + (s^k)^T y(\lambda_k)).
\end{aligned}$$

From the above relationship and (3.9), we have

$$\frac{\omega_{k+1}}{\Delta_k} \leq \frac{2((x^k)^T u(\lambda_k) + (s^k)^T y(\lambda_k))}{\Delta_k} \leq \frac{2(2n+\sqrt{n})}{n+\rho} \leq \frac{\epsilon}{2n}.$$

This gives $n + \rho = \frac{4n(2n+\sqrt{n})}{\epsilon}$, which will guarantee that (x^{k+1}, s^{k+1}) is an ϵ-KKT solution.

Obviously, it follows from the definition of ϵ-KKT solution that

$$\frac{\omega_{k+1}}{u_e - l_e} \leq \frac{\omega_{k+1}}{u_e - q(\hat{x})} \leq \frac{\omega_{k+1}}{q(x^0) - q(\hat{x})} or \ \ (\frac{\omega_{k+1}}{q(x^k) - q(\hat{x})}). \tag{3.9}$$

Moreover, by the above formula, the conclusion of Lemma 3.2 and (3.5), one can derive that

$$\frac{\omega_{k+1}}{\Delta_k} = \frac{\omega_{k+1}}{q(x^k) - q(x^0) + 2n(q(x^0) - q(\hat{x}))} \leq \frac{\epsilon}{2n}.$$

Now if $q(x^k) \leq q(x^0)$, then $\frac{\omega_{k+1}}{2n(q(x^0)-q(\hat{x}))} \leq \frac{\omega_{k+1}}{q(x^k)-q(x^0)+2n(q(x^0)-q(\hat{x}))} \leq \frac{\epsilon}{2n}$. else,

$$\begin{aligned}
\frac{\omega_{k+1}}{2n(q(x^k) - q(\hat{x}))} &\leq \frac{\omega_{k+1}}{2n(q(x^k) - q(\hat{x})) - (2n-1)(q(x^k) - q(x^0))} \\
&= \frac{\omega_{k+1}}{q(x^k) - q(x^0) + 2n(q(x^0) - q(\hat{x}))} \leq \frac{\epsilon}{2n}.
\end{aligned}$$

Both of them lead to $\frac{\omega_{k+1}}{u_e - q(\hat{x})} \leq \epsilon$. From the above relationships and the definition of ϵ-KKT solution it is straightforward to show that (x^{k+1}, s^{k+1}) is an ϵ-KKT solution. This proves the lemma.

Theorem 3.4. Suppose that $H \in R^{n\times n}$ is a symmetric matrix, (H_1, H_2) is a regular splitting of the matrix H, and that w and ρ are chosen as in algorithm A, respectively, $\alpha = \frac{1}{4}$,and the sequence (x^k, s^k) is generated by the algorithm A. Then the number of total iterations to generate an ϵ-minimal or an ϵ-KKT solution for the QPB problem is bounded by $O(\frac{n^2}{\epsilon} log\frac{1}{\epsilon} + nlog(1+\sqrt{2n}))$.

Proof. This theorem is an immediate consequence of lemma 3.1 and lemma 3.3.

4. Solving the BQP subproblem

This section describes how to solve the BQP subproblem and analyzes the complexity bound on obtaining an approximate solution. It is clear that computing the optimal solution of the BQP subproblem is equivalent to solve the system of equations (3.2). It may be difficult to solve (3.2) exactly, but an approximate solution can be generated to guarantee the lemma 3.1 and 3.3, and theorem 3.4 hold. First of all, Subtracting (3.2b) from (3.2a), and by (3.2d), it is easy to derive that

$$[(H_1 + \lambda(X^{-2} + S^{-2})]dx = -[Hx + c - \frac{\Delta}{n+\rho}(X^{-1} - S^{-1})e]. \tag{4.1}$$

Since it is trivial when $\lambda = 0$, we assume that $\lambda > 0$ and $H_1 = diag(h_1, h_2, \cdots, h_n)$, and let $\bar{c} = [Hx + c - \frac{\Delta}{n+\rho}(X^{-1} - S^{-1})e]$, $Z = X^{-2} + S^{-2}$. Thus, by (4.1) and (3.2d) one can see that $dx_i = -\frac{\bar{c}_i}{h_i + z_i\lambda}$, $ds_i = \frac{\bar{c}_i}{h_i + z_i\lambda}$. It follows from $\lambda > 0$ and (3.2e) that $||X^{-1}dx||^2 + ||S^{-1}ds||^2 = \alpha^2$. The above relationships imply that

$$\phi(\lambda) = \sqrt{\sum_{i=1}^{n}(x_i^{-2} + s_i^{-2})\frac{\bar{c}_i^2}{(h_{1i} + z_i\lambda)^2}} - \alpha = \sqrt{\sum_{i=1}^{n}\frac{\bar{c}_i^2}{z_i(\frac{h_{1i}}{z_i} + \lambda)^2}} - \alpha = 0. \tag{4.2}$$

Now the main point is how to generate a suitable $\lambda > 0$ such that (4.2) holds.

Lemma 4.1. As the assumptions as in Theorem 3.4, and let $\Delta_k = q(x^k) - w$, then

$$0 \leq \lambda_k \leq 17\Delta_k. \tag{4.3}$$

Proof. It follows from the definition of the function $q(x)$, the algorithm A and (3.3)-(3.4) that

$$\begin{aligned} q(x^+) - q(x) =& \frac{1}{2}dx^T H dx + g^T dx \\ =& -\frac{1}{2}dx^T(H_1 - H_2)dx + \frac{\Delta}{n+\rho}e^T(X^{-1}dx + S^{-1}ds) - \lambda\alpha^2. \end{aligned} \tag{4.4}$$

This gives

$$\begin{aligned} &\lambda\alpha^2 = q(x) - q(x^+) + \frac{\Delta}{n+\rho}e^T(X^{-1}dx + S^{-1}ds) - \frac{1}{2}dx^T(H_1 - H_2)dx \\ &\leq q(x) - q(x^+) + \frac{\Delta}{n+\rho}e^T(X^{-1}dx + S^{-1}ds) \\ &\leq q(x) - q(x^+) + \frac{\Delta}{n+\rho}||e||(||X^{-1}dx|| + ||S^{-1}ds||) \leq q(x) - q(x^+) + \frac{\Delta}{n+\rho}\sqrt{n}\sqrt{2n}\alpha. \end{aligned}$$

From the assumption that $q(x) - q(x^+) \leq q(x) - w$ and $n + \rho = \frac{4n(2n+\sqrt{n})}{\epsilon}$, it is clear that

$$\lambda \leq (q(x) - w)(\frac{1}{\alpha^2} + \frac{\sqrt{2}\epsilon n}{4n(2n+\sqrt{n})\alpha}) \leq (q(x) - w)(\frac{1}{\alpha^2} + \frac{\sqrt{2}\epsilon}{4\alpha(2n+\sqrt{n})}).$$

Since ϵ is very small and $n \geq 2$, and $\alpha = \frac{1}{4}$, set $\Delta = \Delta_k, \lambda = \lambda_k$, then, it is easy to see that $0 \leq \lambda_k \leq 17\Delta_k$, which proves the lemma.

Now we try to estimate the computational work to obtain the solution of the equation (4.2). First of all, if set $a_i = \frac{\bar{c}_i}{\sqrt{z_i}}$, $b_i = \frac{h_{1i}}{z_i}$, and let $A = \sqrt{\sum_{i=1}^n \frac{a_i^2}{(b_i+\lambda)^2}}$. it is not difficult to verify that

$$\frac{d\phi(\lambda)}{d\lambda} = \frac{1}{2}\frac{-2}{A}\sum_{i=1}^n \frac{a_i^2}{(b_i+\lambda)^3} = -\frac{1}{A}\sum_{i=1}^n \frac{a_i^2}{(b_i+\lambda)^3} < 0,$$

Therefore, $\phi(\lambda)$ is a monotonic decreasing function on $[0, \infty)$ and $\lim_{\lambda\to\infty} \psi(\lambda) = -\alpha$. Thus, a bisection method can be used to search the root of the equation (4.2) while $\lambda \in (l, u)$ since l, u can be obtained from (4.3). There is no doubt, there is an unique solution of the equation (3.2) for given (x^k, s^k) at the k-th iteration, namely, $(dx^k, ds^k), \lambda_k, \bar{y}^k$ solve the equation (3.2). Actually, it is not necessary to solve the equation (3.2) exactly, one can try to find an approximate solution λ for a given $\bar{\epsilon} \in (0,1)$, and if λ is chosen as the rightend point of the bisection interval, then it is obvious that

$$0 < \lambda - \lambda_k \leq \bar{\epsilon}. \tag{4.5}$$

Moreover, it is easy to show that such λ can be obtained in $log\frac{(u-l)}{\bar{\epsilon}}$ bisection steps.

Now an algorithm B is described to generate an (x^{k+1}, s^{k+1}) is either an ϵ-KKT solution for the QPB problem, or a solution such that potential function $F(x,s)$ is decreased by a constant.

Algorithm B:

Assume that $\bar{\epsilon} = \frac{\Delta_k}{4(n+\rho)\alpha} = \frac{\Delta_k \epsilon}{4n(2n+\sqrt{n})\alpha}$, and $\lambda^0 = 8.5\Delta_k$. Let ϕ_i, φ_i and p_i denote the values of $\phi(u_i), \varphi(u_i)$ and $p(u_i)$ associated with λ^i, respectively, and l_i, u_i denote the left endpoint and right endpoint of the bisection interval($l_0 = 0, u_0 = 17\Delta_k$), $\delta_i = u_i - l_i$, i:=0, then carry out the following steps.

(i) Compute p_i and δ_i. If $||p_i|| < \frac{\Delta_k}{(n+\rho)}$ and $\delta_i \leq \bar{\epsilon}$, then terminate; else, go to next step.

(ii) If $\varphi_i \leq -\frac{\Delta_k \alpha}{2(n+\rho)}$ and $\delta_i \leq \bar{\epsilon}$, then terminate; else, go to next step.

(iii) If $\phi_i > 0$, set $l_{i+1} = \lambda^i, u_{i+1} = u_i, \lambda^{i+1} = \lambda^i + 0.5(u_{i+1} - l_{i+1})$, else, set $l_{i+1} = l_i, u_i = \lambda^i, \lambda^{i+1} = l_{i+1} + 0.5(u_{i+1} - l_{i+1})$, and $i := i+1$, return to (i).

Lemma 4.2. Assume that (x^k, s^k) is an interior point of $\Omega, n+\rho, \alpha$, and w are chosen as in algorithm A, and that the equation (3.2) is solved by the algorithm B and λ is chosen as the right endpoint of the bisearch interval. If the algorithm B terminates at step (i) or (ii) then an approximate solution $(x^k + dx(\lambda), s^k + ds(\lambda))$ is either an ϵ-KKT solution for the QPB problem or a solution such that the potential function $F(x,s)$ is decreased by a constant.

Proof: Let λ be the right endpoint of the interval generated by the algorithm B, then $\lambda_k \leq \lambda$. Assume that the algorithm B terminares at step (i), and $(dx(\lambda), ds(\lambda))$ is the solution of the equation (3.2) associated with λ, thus, $||(X^{-2} + S^{-2})dx(\lambda)|| \leq \alpha$ since $\lambda_k \leq \lambda$. Therefore, it is easy to see that

$$||p(\lambda)|| = \lambda||(X^{-2} + S^{-2})dx(\lambda)|| \leq \alpha\lambda < \frac{\Delta_k}{(n+\rho)}.$$

Moreover, it is clear that $x^k + dx(\lambda) > 0, s^k + ds(\lambda) > 0$. It follows from lemma 3.3 that $(x^k + dx(\lambda), s^k + ds(\lambda))$ is an ϵ-KKT solution for the QPB problem.

On the other hand, if the algorithm B terminates at step (ii), then we have

$$\varphi_\lambda(x,s) = \varphi_{\lambda_k}(x,s) + (\varphi_\lambda(x,s) - \varphi_{\lambda_k}(x,s)),$$

and it is not difficult to verify that

$$\begin{aligned}\varphi_\lambda(x,s) - \varphi_{\lambda_k}(x,s) &= \frac{1}{2}dx^T H_1 dx + \hat{g}^T dx - \frac{1}{2}(dx^k)^T H_1 dx^k + \hat{g}^T dx^k \\ &= \frac{1}{2}(H_1 dx + \hat{g})^T (dx - dx^k) + \frac{1}{2}(H_1 dx^k + \hat{g})^T (dx - dx^k),\end{aligned}$$

where $\hat{g} = H_1 x + c - \frac{\Delta}{(n+\rho)}(X^{-1} - S^{-1})e$. From the formula (3.2), one can see that

$$\varphi_\lambda(x,s) - \varphi_{\lambda_k}(x,s) = -\frac{\lambda}{2}dx^T(X^{-2}+S^{-2})(dx-dx^k) - \frac{\lambda_k}{2}(dx^k)^T(X^{-2}+S^{-2})(dx-dx^k)$$
$$= -\frac{1}{2}(\lambda-\lambda_k)\alpha^2 - \frac{1}{2}(\lambda-\lambda_k)dx^T(X^{-2}+S^{-2})dx^k \leq \bar{\epsilon}\alpha^2. \quad (4.5)$$

It follows from the step (ii) of the algorithm B that $0 < \lambda - \lambda_k \leq \bar{\epsilon}$ and $\varphi_i \leq -\frac{\Delta_k \alpha}{2(n+\rho)}$. Furthermore, from (3.4) and (4.5) we have

$$\varphi_\lambda(x,s) = \varphi_{\lambda_k}(x,s) + (\varphi_\lambda(x,s) - \varphi_{\lambda_k}(x,s)) \leq -\alpha||p^k|| + \frac{\alpha\Delta_k}{4(n+\rho)} \leq -\frac{\alpha\Delta_k}{2(n+\rho)}$$

Then, it is straightforward to show that $||p^k|| \geq \frac{3\Delta_k}{4(n+\rho)}$. From the definition of the algorithm we have $x^k + dx(\lambda) > 0, s^k + ds(\lambda) > 0$, and $\alpha = \frac{1}{4}$. Thus, we have

$$F(x^k+dx, s^k+ds) - F(x^k,s^k) \leq -\frac{2\alpha}{3}\frac{(n+\rho)}{\Delta_k}||p^k|| + \frac{\alpha^2}{1-\alpha} \leq -\frac{1}{24}.$$

This proves the conclusion of the lemma.

Now it follows from the conclusions of the lemma 4.1 and 4.2 that if $\bar{\epsilon}$ is chosen as in algorithm B to generate an (x^{k+1}, s^{k+1}) is either an ϵ-KKT solution for the QPB problem, or a solution such that the potential function $F(x,s)$ is decreased by a constant. Then the number of total bisection search steps to obtain such solution (x^{k+1}, s^{k+1}) is required by $O(log\frac{1}{\epsilon} + logn)$ at each iteration, and it is only necessary to calculate the value of the function $\phi(\lambda)$ for a given λ at each bisection step. Furthermore, if H_1 is chosen as a diagonal matrix, the cost of each bisection step is $O(n)$ arithmetic operations for computing $\phi(\lambda)$. Thus, in order to obtain an (x^{k+1}, s^{k+1}) such that either the potential function $F(x,s)$ is decreased by a constant or (x^{k+1}, s^{k+1}) is an ϵ-KKT solution for the QPB problem, the total arithmetic operations is bounded by $O(nlogn + nlog\frac{1}{\epsilon})$ for the given (x^k, s^k) and $\bar{\epsilon}$. In addition, by the formula (4.3), the cost of computing $\Delta_k, \bar{c}$ is $O(n^2)$ arithmetic operations at each iteration.

Now one can immediately have following conclusion.

Theorem 4.3. As the assumptions as in lemma 4.2, in order to generate an $(x^{k+1}s^{k+1})$ such that either the potential function $F(x,s)$ is decreased by a constant or (x^{k+1}, s^{k+1}) is an ϵ-KKT solution for the QPB problem. Then the total running time is bounded by $O(n(n+logn+log\frac{1}{\epsilon}))$ arithmetic operations at the k-th iteration.

From the conclusion of the theorem 3.4 and 4.3, one can directly see the following consequence.

Theorem 4.4. As the assumptions as in theorem 3.4, and $\bar{\epsilon}$ is chosen as in lemma 4.2, then the total running time of the algorithm A is bounded by $O(n^2(\frac{n}{\epsilon}log\frac{1}{\epsilon} + logn)(n + logn + log\frac{1}{\epsilon}))$ arithmetic operations.

In some cases, we may previously know the global minimal value l_e of the QPB problem, then it is not necessary to compute the value w. one can choose $n + \rho = \frac{4(2n+\sqrt{n})}{\epsilon}$ and the complexity bound can be reduced by a factor n.

From the algorithm B, one can easily revise the algorithm A and have a new algorithm.

The author would thank referees for many valuable comments and suggestions.

References

[1] J.C. Dunn, On the convergence of projected gradient processes to singular critical point, J. Optim. Theory Appl., 55(1987), 203–216.

[2] Z.Q. Luo and P. Tseng, On the convergence of a matrix splitting algorithm for the symmetric monotone linear complementarity problem, SIAM J. Control and Optimization, 29(1991), 1037–1060.

[3] Z.J. Lu and Z.L. Wei, Decomposition method for quadratic programming problem with box constraints, Mathematica Numerica Sinica, 21(1999), pp. 1-8 (in Chinese).

[4] K.G. Murty and S.N. Kabadi, Some NP-complete problem in quadratic and nonlinear programming, Math. Prog. 39(1987), 117–129.

[5] C.H. Papadimitriou and K. Steiglitz. Combinatorial Optimization: Algorithms and Complexity. Pren-tice-Hall, Englewood Cliffs, NJ, 1982.

[6] P.M.Pardalos and J.B.Rosen. Constrained Global Optimization: Algorithms and Applications. Springer-Verlag, Lecture Notes in Computer Sciences 268, 1987.

[7] S.A. Vavasis. Nonlinear Optimization: Complexity Issue. Oxford Science, New York, NY 1991.

[8] S.A. Vavasis. Polynomial time weak approximation algorithms for quadratic programming. In C.A. Floudes and P.M. Pardalos, editors, Complexity in Numerical Optimization. World Scientific, New Jersey, 1993.

[9] Zi-Luan Wei, Subspace search method for quadratic programming with box constraints, JCM, 18(1999), 307–314.

[10] Y. Ye, On an affine scaling algorithm for nonconvex quadratic programming, Mathematical Programming, 52(1992), 285–300.

[11] Y. Ye, Interior Point Algorithms: Theory and Analysis. John Wiley and Sons, Inc. New York, NY, 1997.

[12] Y.Yuan, Numerical Methods for Nonlinear Programming, Shanghai Scientific and Technical Publishers, 1993 (in Chinese).