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ON THE *hp* FINITE ELEMENT METHOD FOR THE ONE DIMENSIONAL SINGULARLY PERTURBED CONVECTION-DIFFUSION PROBLEMS*[1)]

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Abstract

In this work, a singularly perturbed two-point boundary value problem of convection-diffusion type is considered. An *hp* version finite element method on a strongly graded piecewise uniform mesh of Shishkin type is used to solve the model problem. With the analytic assumption of the input data, it is shown that the method converges exponentially and the convergence is uniformly valid with respect to the singular perturbation parameter.

Key words: *hp*-version finite element methods, convection-diffusion, singularly perturbed, exponential rate of convergence.

1. Introduction

In practice, we often encounter differential equations with small (or large) parameters. When these parameters go to extremal, the equations are usually *singularly perturbed*. One typical behavior of the singular perturbation is the so-called *boundary layers*. The existence of the boundary layers causes difficulty in numerically solving these problems. The conventional methods fail to converge since the convergence deteriorated at the limits of the small (large) parameters. A successful numerical algorithm should converge uniformly with respect to singular perturbation parameters. There is a rich literature on numerical methods for problems with boundary layers. The reader is referred to recent books of Miller et al. [13], Morton [14], Roos et al. [15], and references therein.

Concerning singularly perturbed problems, it is common knowledge that solving convection-diffusion equations is usually harder than solving reaction-diffusion equations. The main difficulty with generalizing the theoretical analysis for reaction-diffusion problems to convection-diffusion problems is that the bilinear forms of the latter are not uniformly continuous with respect to the singular perturbation parameter ϵ. To be more precise, for the reaction-diffusion problem, there exists a constant C independent of ϵ, so that the inequality

$$|B_\epsilon(u,v)| \le C\|u\|_\epsilon \|v\|_\epsilon \tag{1.1}$$

holds for an energy norm $\|\cdot\|_\epsilon$. Here $B_\epsilon(\cdot,\cdot)$ is the bilinear form of the variational formulation. However, this property is not valid for the convection-diffusion problem. The lack of the stability property (1.1) prevents us from following the standard analysis. In order to overcome this difficulty, many methods are suggested in the literature, among which, the most popular are streamline-diffusion technique and the Petrov-Galerkin method (see, e.g., [6, 7, 8, 10, 15]). However, the Petrov-Galerkin method is difficult to be generalized to multidimensional settings and the streamline diffusion method alone is not able to resolve the boundary layer. If the

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boundary layer is concerned, a practical procedure would be to start with the streamline diffusion method, adapting the mesh by an *a posteriori* error estimate, and eventually resolving the boundary layer. An important question naturally arises: what is the quality of the numerical approximation when the mesh is consistent with the boundary layer?

In this work, a singularly perturbed two-point boundary value problem of convection-diffusion type is considered. A special hp finite element method that uses piecewise uniform meshes, a uniform mesh outside the boundary layer region and a much smaller uniform mesh in the boundary layer region, is applied. The convergence analysis avoids the use of (1.1) by adopting a different framework from the traditional one. Furthermore, the analysis is carried out on the element level which allows the use of some fundamental results form approximation theory to tracking the exact dependence on p and h, thereby to compensate the boundary layer influence. The main result of this paper is to establish, under the analytic assumption of the input data, an exponential convergent rate for the energy norm, a rate which is uniformly valid with respect to the singular perturbation parameter ϵ for the proposed method.

In an independent work done recently by Melenk and Schwab [12], the authors obtained exponential convergence for both hp and hp streamline diffusion finite element methods. However, the current approach is simpler which provides explicit dependence of the convergent rate on the regularity constants. Furthermore, the proof here is elementary and self-contained.

Since the publication of the first theoretical paper [2] on the p-version finite element method, many works have been done on the p and hp methods. For the general information, the reader is referred to [1, 5, 16, 18] and references therein.

2. Main Results

Consider the following steady state one-dimensional convection-diffusion model problem.

$$(L_\epsilon u)(x) = -\epsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \quad \text{in} \quad \gamma = (0,1), \quad u(0) = u(1) = 0 \tag{2.1}$$

with

$$a(x) \geq \alpha > 0, \quad b(x) - \frac{a'(x)}{2} > 0 \quad \forall x \in \bar{\gamma}. \tag{2.2}$$

It has been shown in [13, Chapter 9] that there is no essential loss of generality in assuming the above rather than

$$a(x) > \alpha > 0, \quad b(x) \geq \beta, \quad \forall x \in \bar{\gamma}. \tag{2.3}$$

Numerical difficulty arises when the diffusion parameter ϵ is small. In this case, the model problem is singularly perturbed. In order to design a good numerical algorithm, it is necessary to understand the boundary layer behavior of the problem. This understanding involves regularity analysis based on the input data. In this work, we utilize the regularity result in [10] for (2.1) under the analytic assumption on the input data. For small ϵ, the solution u can be decomposed into

$$u = \sum_{j=0}^{m} \epsilon^j u_j + u_\epsilon + r_m = w_m + u_\epsilon + r_m \tag{2.4}$$

where u_j, u_ϵ, and r_m are determined by the following initial value problems and boundary value problems:

$$\begin{aligned} a(x)u_0'(x) + b(x)u_0(x) &= f(x), \quad u_0(0) = 0; \\ a(x)u_{j+1}'(x) + b(x)u_{j+1}(x) &= u_j''(x), \quad u_{j+1}(0) = 0, \quad j = 0, 1, \ldots, m-1; \\ L_\epsilon u_\epsilon &= 0, \quad u_\epsilon(0) = 0, \quad u_\epsilon(1) = -w_m(1); \\ L_\epsilon r_m &= \epsilon^{m+1} u_m'', \quad r_m(0) = 0 = r_m(1). \end{aligned}$$

Under the essential condition (2.3) and the analytic assumption of the input data a, b, and f in the following form,

$$\|a^{(k)}\|_{L^\infty(\gamma)} \le C_a\gamma_a^k, \quad \|b^{(k)}\|_{L^\infty(\gamma)} \le C_b\gamma_b^k, \quad \|f^{(k)}\|_{L^\infty(\gamma)} \le C_f\gamma_f^k,$$

for some constants $C_a, C_b, C_f, \gamma_a, \gamma_b, \gamma_f > 0$ and any non-negative integer k, the following regularity results have been established

$$\|u^{(k)}\|_{L^\infty(\gamma)} \le K\gamma^k \max(k, \epsilon^{-1})^k \quad \forall k, \tag{2.5}$$

$$|u_\epsilon^{(k)}(x)| \le K\gamma^k \max(k, \epsilon^{-1})^k e^{-\alpha(1-x)/\epsilon} \quad \forall k, \quad x \in \gamma. \tag{2.6}$$

If $0 < \epsilon m\gamma \le 1$, then

$$\|w_m^{(k)}\|_{L^\infty(\gamma)} \le K\gamma^k k! \quad \forall k, \tag{2.7}$$

$$\|r_m^{(k)}\|_{L^\infty(\gamma)} \le K\epsilon^{1-k}(\epsilon m\gamma)^m \quad k = 0, 1, 2. \tag{2.8}$$

A formal proof of these results can be found in [10]. In order to avoid going into too much technical detail, we simply assume the regularity (2.5) – (2.8) in this work.

By defining bilinear form

$$B_\epsilon(u, v) = \epsilon(u', v') + (au', v) + (bu, v) \quad \text{with} \quad (w, z) = \int_0^1 w(x)z(x)dx,$$

under the homogeneous Dirichlet boundary condition, we can show that

$$(av', v) = -\frac{1}{2}(a'v, v).$$

Therefore,

$$B_\epsilon(v, v) = \epsilon(v', v') + ((b - \frac{1}{2}a')v, v) \ge \epsilon(v', v') + c_0(v, v) = \|v\|_\epsilon^2 \tag{2.9}$$

where

$$c_0 = \min_{x\in\bar{\gamma}}(b(x) - \frac{1}{2}a'(x)).$$

Here we have defined an energy norm $\|\cdot\|_\epsilon$.

The weak form of (2.1) is the following variational problem: Find $u \in H_0^1(\gamma)$ such that

$$B_\epsilon(u, v) = (f, v) \quad \forall v \in H_0^1(\gamma).$$

Next we define the finite element space. Let

$$\tau = \min(\frac{\epsilon(p+1)}{\alpha}, \frac{1}{2}),$$

and divide both intervals $(0, 1-\tau)$ and $(1-\tau, 1)$ into N equal subintervals. Hence there are two different mesh-size $\bar{h} = (1-\tau)/N < 1/N$ and $\underline{h} = \tau/N$. On this piecewise uniform mesh, which we denote as $\mathcal{T}_N^{p,\epsilon}$, we define the standard C^0 finite element space $V_N^{p,\epsilon}$ that contains piecewise polynomials of degree p. The total number of degrees of freedom is then $\dim V_N^{p,\epsilon} = 2pN - 1$. The finite element method is to find $u_{hp} \in V_N^{p,\epsilon}$ such that

$$B_\epsilon(u_{hp}, v) = (f, v) \quad \forall v \in V_N^{p,\epsilon}.$$

Remark 2.1. With minor modification, the analysis can be generalized to piecewise quasi-uniform meshes with $\bar{h} = O(1/N)$ and $\underline{h} = O(\tau/N)$. However, we stay with the piecewise uniform mesh for simplicity. Also, we will assume that $\frac{\epsilon(p+1)}{\alpha} \leq \frac{1}{2}$ since otherwise, it would be the uniform mesh and we are back to the standard case.

Remark 2.2. A strongly graded mesh with one element of length $\kappa\epsilon p$ at boundary layer was first introduced by Schwab and Suri in [17] for a singularly perturbed one-dimensional reaction-diffusion problem with a constant coefficient and later used by Melenk and Schwab in [10] and [12] for the singularly perturbed convection-diffusion problem (2.1).

The analysis will be separated into the asymptotic phase $\epsilon(p+1) \geq 1$ and the pre-asymptotic phase $\epsilon(p+1) < 1$. Indeed, in the asymptotic phase, the boundary layer is compensated by the polynomial degree p, and the uniform or quasi-uniform meshes suffice to warrant an exponential convergence. It is in the pre-asymptotic phase that the mesh refinement in the boundary layer region is necessary.

In order to prove the exponential convergence of the finite element approximation, we shall prove that there exists $I_\epsilon u \in V_N^{p,\epsilon}$ such that

$$|B_\epsilon(u - I_\epsilon u, v)| \leq Cpe^{-\sigma p}\|v\|_\epsilon \quad \forall v \in V_N^{p,\epsilon}, \tag{2.10}$$

or

$$|B_\epsilon(u - I_\epsilon u, v)| \leq C(p + \sqrt{N})e^{-\sigma p}\|v\|_\epsilon \quad \forall v \in V_N^{p,\epsilon}; \tag{2.11}$$

and

$$\|u - I_\epsilon u\|_\epsilon \leq Cpe^{-\sigma p}, \tag{2.12}$$

where constant C is independent of ϵ, p, and N.

Theorem 2.1. Assume that $\epsilon(p+1) \geq 1$. Let u be the solution of (2.1) that satisfies (2.5) and let $I_\epsilon u = u_I$ be the $p+1$-point Gauss-Lobatto interpolation of u on a quasi-uniform mesh with the maximum mesh-size h satisfying $e\gamma h/4 < 1$. Then there exists a constant C independent of h, p, and ϵ, such that both (2.10) and (2.12) are valid with $\sigma = -\ln(e\gamma h/4)$.

Theorem 2.2. Assume that $\epsilon(p+1) < 1$. Let u be the solution of (2.1) that satisfies (2.5) – (2.8), and let

$$\lambda = \frac{e^{1+\frac{1}{N}}\gamma}{4\alpha N} < \frac{1}{2}, \quad \frac{\gamma h}{4} < 1, \quad \mu\gamma < 1 \quad (\mu = \frac{m}{p}).$$

Then there exists $I_\epsilon u \in V_N^{p,\epsilon}$, a special interpolation of u on the piecewise uniform mesh $\mathcal{T}_N^{p,\epsilon}$ with the maximum mesh size h ($< 1/N$), and a constant C independent of N, p, and ϵ, such that, (a) (2.10) and (2.12) are valid when $\epsilon N \geq e\gamma(1+\alpha^{-1})/2$; (b) (2.11) and (2.12) are valid when $\epsilon N < e\gamma(1+\alpha^{-1})/2$; with $\sigma = \min(\ln 2, -\ln(\gamma h/4), -\mu\ln(\mu\gamma))$.

The proof of Theorem 2.1 and Theorem 2.2 will be postponed to the next section. Now we state and prove the main result.

Theorem 2.3. Let u be the solution of the model problem (2.1) that satisfies the regularity assumptions (2.5) – (2.8), and let u_{hp} be the C^0 finite element (of order p) approximation of u on a mesh either in Theorem 2.1 or Theorem 2.2. Then there exists $\sigma > 0$ as in Theorem 2.1 or Theorem 2.2, respectively, such that

$$\|u - u_{hp}\|_\epsilon \leq Cpe^{-\sigma p}, \tag{2.13}$$

or

$$\|u - u_{hp}\|_\epsilon \leq C(p + \sqrt{N})e^{-\sigma p}, \quad \text{when} \quad \epsilon(p+1) < 1, \ \epsilon N < \frac{e\gamma}{2}(1+\alpha^{-1}). \tag{2.14}$$

where C is a constant independent of N, p and ϵ.

Proof. Let $I_\epsilon u$ be the special interpolant as in Theorem 2.1 or Theorem 2.2, respectively. By (2.9) and (2.10),

$$\|u_{hp} - I_\epsilon u\|_\epsilon^2 \leq B_\epsilon(u_{hp} - I_\epsilon u, u_{hp} - I_\epsilon u) = B_\epsilon(u - I_\epsilon u, u_{hp} - I_\epsilon u) \leq Cpe^{-\sigma p}\|u_{hp} - I_\epsilon u\|_\epsilon.$$

Canceling $\|u_{hp} - I_\epsilon u\|_\epsilon$ on both ends, we have

$$\|u_{hp} - I_\epsilon u\|_\epsilon \le Cpe^{-\sigma p}. \tag{2.15}$$

Combining (2.15) with the triangle inequality and (2.12) yields

$$\|u - u_{hp}\|_\epsilon \le \|u - I_\epsilon u\|_\epsilon + \|u_{hp} - I_\epsilon u\|_\epsilon \le Cpe^{-\sigma p}.$$

The case when (2.11) is used, instead of (2.10), can be proved similarly. ■

Remark 2.3. Generalization of the method to a higher dimension is feasible. However, the regularity analysis can be very complicated. As far as the regularity is concerned, this author is not aware of any complete regularity analysis for the two-dimensional convection-diffusion equations in polygonal domains, although there are some recent works on reaction-diffusion equations (see, for example, [9] and [11]).

3. Analysis

The proof of Theorem 2.1 and Theorem 2.2 is based on lemmas in this section. Standard notations are used here. For example, h_i is the length of the element $\gamma_i = (x_{i-1}, x_i)$ and $\|\cdot\|$ denotes the L_2-norm. An index will be given to indicate an inner product or a norm on a sub-domain such as

$$(v, z)_{\gamma_i} = \int_{\gamma_i} v(x)z(x)dx, \quad \|v\|^2_{(0,\tau)} = \int_0^\tau v^2(x)dx.$$

Throughout this section, we use the sub-index I to denote the piecewise Lagrange interpolation at the $p+1$ Gauss-Lobatto points on each element. To be more precise, let $x_{i-1} = t_1^{(i)} < t_2^{(i)} < \cdots < t_{p+1}^{(i)} = x_i$ be the Gauss-Lobatto points on $[x_{i-1}, x_i]$, i.e., $t_2^{(i)} < t_3^{(i)} < \cdots < t_p^{(i)}$ are zeros of the derivative of the Legendre polynomial of degree p on $[x_{i-1}, x_i]$. Then we have

$$\begin{aligned} u_I(x) &= u(t_1^{(i)}) + (x - t_1^{(i)})u[t_1^{(i)}, t_2^{(i)}] + (x - t_1^{(i)})(x - t_2^{(i)})u[t_1^{(i)}, t_2^{(i)}, t_3^{(i)}] \\ &\quad + \cdots + \psi_p(x)u[t_1^{(i)}, t_2^{(i)}, \cdots, t_{p+1}^{(i)}], \end{aligned}$$

$$(u - u_I)(x) = \psi_{p+1}(x)u[t_1^{(i)}, t_2^{(i)}, \cdots, t_{p+1}^{(i)}, x] \tag{3.1}$$

where $u[t_1^{(i)}, t_2^{(i)}, \cdots, t_k^{(i)}]$ is the kth-order Newton divided difference of u and

$$\psi_k(x) = (x - t_1^{(i)})(x - t_2^{(i)}) \cdots (x - t_k^{(i)}).$$

The residue term (3.1) can be estimated by

$$|u[t_1^{(i)}, t_2^{(i)}, \cdots, t_k^{(i)}, x]| \le \frac{1}{k!}\|u^{(k)}\|_{L^\infty(\gamma_i)}, \tag{3.2}$$

$$|\psi_{p+1}(x)| \le \frac{cp^{1/2}}{2^p}\Big(\frac{h_i}{2}\Big)^{p+1} = 2cp^{1/2}\Big(\frac{h_i}{4}\Big)^{p+1}, \tag{3.3}$$

and

$$(u[t_1^{(i)}, t_2^{(i)}, \cdots, t_k^{(i)}, x])' = u[t_1^{(i)}, t_2^{(i)}, \cdots, t_k^{(i)}, x, x] \tag{3.4}$$

$$|\psi'_{p+1}(x)| \le cp^{3/2}\Big(\frac{h_i}{4}\Big)^p, \tag{3.5}$$

for any $x \in (x_{i-1}, x_i)$ with the constant $c \approx \sqrt{\pi}$. The bound (3.2) and the identity (3.4) are from the standard interpolation theory and can be found in most of numerical analysis textbooks. The proof of the estimates (3.3) and (3.5) are provided in the appendix.

In the asymptotic phase when $\epsilon(p+1)\geq 1$. By the regularity result (2.5), we have

$$\|u^{(k)}\|_{L^\infty(\gamma)} \leq K\gamma^k k^k < \frac{K}{\sqrt{2\pi}}(e\gamma)^k k! k^{-1/2}, \quad k\geq p+1. \tag{3.6}$$

Here we have used the Stirling's formula (0.9).

In the analysis we also need an inverse inequality of the hp-version:

$$\Big(\int_{\gamma_i} |v'(x)|^s dx\Big)^{1/s} \leq C\frac{p^2}{h}\Big(\int_{\gamma_i} |v(x)|^s dx\Big)^{1/s} \quad \forall v\in V_N^{p,\epsilon}, \ \ s\in(0,\infty), \tag{3.7}$$

with a constant C independent of p and h. Here the factor p^2 is from the Markov inequality [3, p.402, Theorem A.4.14].

Remark 3.1. By the definition, we can verify that

$$|B_\epsilon(w,v)| \leq C(\|w\|_\epsilon\|v\|_\epsilon + |(w',v)|),$$

or

$$|B_\epsilon(w,v)| \leq C(\|w\|_\epsilon\|v\|_\epsilon + |(w,v')|),$$

for a constant C that depends only on a and b. Therefore, in order to prove (2.10) or (2.11), we only need to establish (2.12) and estimate $|(u'-u_I',v)|$ for $\epsilon(p+1)\geq 1$.

Lemma 3.1. Assume that $\epsilon(p+1)\geq 1$. Let u be the solution of (2.1) that satisfies (2.5) and let u_I be the $p+1$-point Gauss-Lobatto interpolation of u on a quasi-uniform mesh with the maximum mesh-size h satisfying $e\gamma h/4<1$. Then there exist a constant C independent of h, p, and ϵ, such that

$$\|u-u_I\|_\epsilon \leq Ce^{-\sigma p}, \quad |(u'-u_I',v)| \leq Cpe^{-\sigma p}\|v\|,$$

where $\sigma = -\ln(e\gamma h/4)$.

Proof. Using (3.1) – (3.6) and regularity (2.5), we have

$$\begin{aligned}
&(u'-u_I', u'-u_I')_{\gamma_i}\\
= \ & \int_{x_{i-1}}^{x_i} (\psi_{p+1}'(x)u[t_1^{(i)},t_2^{(i)},\cdots,t_{p+1}^{(i)},x] + \psi_{p+1}(x)u[t_1^{(i)},t_2^{(i)},\cdots,t_{p+1}^{(i)},x,x])^2 dx\\
\leq \ & 2\int_{x_{i-1}}^{x_i} (\psi_{p+1}'(x)^2 u[t_1^{(i)},t_2^{(i)},\cdots,t_{p+1}^{(i)},x]^2 + \psi_{p+1}(x)^2 u[t_1^{(i)},t_2^{(i)},\cdots,t_{p+1}^{(i)},x,x]^2) dx\\
\leq \ & \frac{1}{\pi}\Big(c^2p^2(K\gamma e)^2\Big(\frac{e\gamma h_i}{4}\Big)^{2p} + 4c^2(K\gamma e)^2\Big(\frac{e\gamma h_i}{4}\Big)^{2(p+1)}\Big)h_i.
\end{aligned} \tag{3.8}$$

Summing up all elements, we have

$$\|u'-u_I'\|^2 \leq C^2p^2\Big(\frac{e\gamma h}{4}\Big)^{2p} = C^2p^2e^{-2\sigma p}; \tag{3.9}$$

and consequently,

$$|(u'-u_I',v)| \leq \|u'-u_I'\|\|v\| \leq Cpe^{-\sigma p}\|v\|. \tag{3.10}$$

On the other hand,

$$(u-u_I, u-u_I)_{\gamma_i} = \int_{x_{i-1}}^{x_i} \psi_{p+1}(x)^2 u[t_1^{(i)},t_2^{(i)},\cdots,t_{p+1}^{(i)},x]^2 dx \leq \frac{(2cK)^2}{2\pi}\Big(\frac{e\gamma h_i}{4}\Big)^{2(p+1)}h_i.$$

Summing up all elements, we have

$$\|u-u_I\|^2 \le C^2\Big(\frac{e\gamma h}{4}\Big)^{2(p+1)} = C^2 e^{-2\sigma p}. \tag{3.11}$$

Note that $(p+1)^{-1} \le \epsilon$, the conclusion follows by collecting (3.9) – (3.11). ■

Our main effort is devoted to the pre-asymptotic phase $\epsilon(p+1) < 1$ in which case we define a special interpolation in $V_N^{p,\epsilon}$,

$$I_\epsilon u = w_{m,I} + u_{\epsilon,I_\epsilon} + r_{m,l}$$

with $w_{m,I}$, the piecewise Lagrange interpolation of w_m at the $p+1$ Gauss-Lobatto points; $r_{m,l}$, the piecewise linear interpolation of r_m; and $u_{\epsilon,I_\epsilon} = u_{\epsilon,I}$ when $\epsilon N \ge e\gamma(1+\alpha^{-1})/2$ and

$$u_{\epsilon,I_\epsilon} = \begin{cases} u_{\epsilon,I}, & 1-\tau \le x \le 1 \\ l_\tau, & 0 \le x \le 1-\tau \end{cases} \quad \text{when} \quad \epsilon N < \frac{e\gamma}{2}(1+\alpha^{-1}), \tag{3.12}$$

where

$$l_\tau(x) = \begin{cases} u_\epsilon(1-\tau)\dfrac{x-(1-\tau-\bar h)}{\bar h}, & 1-\tau-\bar h \le x \le 1-\tau \\ 0, & 0 \le x \le 1-\tau-\bar h \end{cases}$$

We see that the support of l_τ contains only one element, namely, the first element adjacent to the transition point outside the boundary layer region. Notice that $l_\tau(1-\tau) = u_\epsilon(1-\tau)$ and $l_\tau(1-\tau-\bar h) = 0$. The idea is to make u_{ϵ,I_ϵ} essentially the Gauss-Lobatto interpolation of u_ϵ in the boundary layer region and zero outside. This may result in a discontinuity at the transition point $1-\tau$. By introducing the linear function l_τ, we preserve the continuity of u_{ϵ,I_ϵ}. Note that

$$u_{\epsilon,I_\epsilon}(1-\tau-\bar h) = 0, \quad u_{\epsilon,I_\epsilon}(1-\tau) = u_\epsilon(1-\tau).$$

Using the regularity of u_ϵ, a direct calculation shows that

$$\begin{aligned} \|l_\tau\|^2 &= \bar h u_\epsilon(1-\tau)^2/3 \le C\bar h e^{-2\alpha\tau/\epsilon} = C\bar h e^{-2(p+1)}, && (3.13)\\ \epsilon\|l'_\tau\|^2 &= \epsilon u_\epsilon(1-\tau)^2/\bar h \le C\epsilon N e^{-2\alpha\tau/\epsilon} \le C' e^{-2(p+1)}, \quad \text{if} \quad \epsilon N < \frac{e\gamma}{2}(1+\alpha^{-1}), && (3.14)\\ |(l'_\tau, v)| &\le \|l'_\tau\|\|v\|_{(0,1-\tau)} \le C\sqrt{N}e^{-(p+1)}\|v\|. && (3.15) \end{aligned}$$

Here we have used the inverse inequality (3.7). We see that the existence of l_τ does not influence the exponential convergent rate.

As explained in Remark 3.1, we only need to establish (2.12) and estimate each of

$$|(u_\epsilon - u_{\epsilon,I_\epsilon}, v')|, \quad |(w'_m - w'_{m,I}, v)|, \quad \text{and} \quad |(r'_m - r'_{m,l}, v)|$$

for $\epsilon(p+1) < 1$.

Lemma 3.2. Assume that $\epsilon(p+1) < 1$. Let $u_{\epsilon,I_\epsilon} \in V_N^{p,\epsilon}$ be the special interpolant of u_ϵ defined by (3.12) on the piecewise uniform mesh $\mathcal{T}_N^{p,\epsilon}$ and let N satisfy $\lambda = \dfrac{e^{1+1/N}\gamma}{4\alpha N} < \dfrac{1}{2}$. Then there exists a constant C independent of N, p, and ϵ, such that

$$\|u_\epsilon - u_{\epsilon,I_\epsilon}\|_\epsilon \le Cpe^{-\sigma p};$$

$$|(u_\epsilon - u_{\epsilon,I_\epsilon}, v')| \le \begin{cases} Cpe^{-\sigma p}\|v\|_\epsilon, & \epsilon N \ge \frac{e\gamma}{2}(1+\alpha^{-1}) \\ C(p+\sqrt{N})e^{-\sigma p}\|v\|_\epsilon, & \epsilon N < \frac{e\gamma}{2}(1+\alpha^{-1}) \end{cases}$$

with $\sigma = \ln 2$.

Proof. (a) We first consider $\epsilon N \ge e\gamma(1+\alpha^{-1})/2$ in which case $u_{\epsilon,I_\epsilon} = u_{\epsilon,I}$ and

$$\frac{e\gamma \underline{h}}{4\epsilon} = \frac{e\gamma}{4\epsilon}\frac{\epsilon(p+1)}{\alpha N} < \frac{e\gamma}{4\alpha\epsilon N} < \frac{1}{2};$$

$$\frac{e\gamma \bar{h}}{4\epsilon} < \frac{e\gamma}{4\epsilon N} < \frac{1}{2}.$$

Recalling the regularity (2.6), we use the same residue expression as in (3.8) to estimate

$$\epsilon\|u'_\epsilon - u'_{\epsilon,I}\|^2_{\gamma_i} \le \frac{1}{\epsilon\pi}\Big(c^2p^2(K\gamma e)^2\Big(\frac{e\gamma h_i}{4\epsilon}\Big)^{2p} + 4c^2(K\gamma e)^2\Big(\frac{e\gamma h_i}{4\epsilon}\Big)^{2(p+1)}\Big)e^{-2\alpha(1-x_i)/\epsilon}h_i. \quad (3.16)$$

Summing up all elements we obtain,

$$\epsilon\|u'_\epsilon - u'_{\epsilon,I}\|^2 \le C\epsilon^{-1}p^2 2^{-2p}e^{2\alpha\bar{h}/\epsilon}\sum_{i=1}^{2N} e^{-2\alpha(1-x_{i-1})/\epsilon}h_i \le C'p^2 2^{-2p}. \quad (3.17)$$

Note that

$$\sum_{i=1}^{2N} e^{-2\alpha(1-x_i)/\epsilon}h_i \le e^{2\bar{h}/\epsilon}\sum_{i=1}^{2N} e^{-2\alpha(1-x_{i-1})/\epsilon}h_i \le C\int_0^1 e^{-2\alpha(1-x)/\epsilon}dx \le \frac{C\epsilon}{2\alpha},$$

when $\epsilon N \ge e\gamma(1+\alpha^{-1})/2$.

Similarly, we obtain

$$\|u_\epsilon - u_{\epsilon,I}\|^2_{\gamma_i} \le C\left(\frac{e\gamma h_i}{4\epsilon}\right)^{2(p+1)} e^{-2\alpha(1-x_i)/\epsilon}h_i;$$

$$\|u_\epsilon - u_{\epsilon,I}\|^2 \le C\epsilon 2^{-2(p+1)}, \quad (3.18)$$

$$|(u_\epsilon - u_{\epsilon,I}, v')| \le C\epsilon^{1/2}2^{-(p+1)}\|v'\| \le C2^{-(p+1)}\|v\|_\epsilon. \quad (3.19)$$

Combining (3.17) – (3.19), we have proved the theorem for $\epsilon N \ge e\gamma(1+\alpha^{-1})/2$.

(b) We now consider $\epsilon N < e\gamma(1+\alpha^{-1})/2$.

(b1) When $\gamma_i \subset (1-\tau, 1)$, $u_{\epsilon,I_\epsilon} = u_{\epsilon,I}$, therefore, we can use the same expression (3.16),

$$\epsilon\|u'_\epsilon - u'_{\epsilon,I}\|^2_{\gamma_i} \le C\epsilon^{-1}\left[p^2\left(\frac{e\gamma h_i}{4\epsilon}\right)^{2p} + \left(\frac{e\gamma h_i}{4\epsilon}\right)^{2(p+1)}\right]e^{-2\alpha(1-x_i)/\epsilon}h_i$$

to derive

$$\begin{aligned}\epsilon\|u'_\epsilon - u'_{\epsilon,I}\|^2_{(1-\tau,1)} &\le C\epsilon^{-1}p^2\left(\frac{e\gamma}{4\alpha N}\right)^{2p}e^{2\alpha\underline{h}/\epsilon}\int_{1-\tau}^1 e^{-2\alpha(1-x)/\epsilon}dx \\ &\le C'p^2\left(\frac{e^{1+\frac{1}{N}}\gamma}{4\alpha N}\right)^{2p} \le C'p^2 2^{-2p}.\end{aligned}$$

Similarly,

$$\|u_\epsilon - u_{\epsilon,I}\|^2_{\gamma_i} \le C\left(\frac{e\gamma h_i}{4\epsilon}\right)^{2(p+1)} e^{-2\alpha(1-x_i)/\epsilon}h_i;$$

$$\|u_\epsilon - u_{\epsilon,I}\|^2_{(1-\tau,1)} \le C\epsilon\left(\frac{e^{1+\frac{1}{N}}\gamma}{4\alpha N}\right)^{2(p+1)} \le C\epsilon 2^{-2(p+1)}, \quad (3.20)$$

$$|(u_\epsilon - u_{\epsilon,I}, v')_{(1-\tau,1)}| \le C\epsilon^{1/2}\left(\frac{e^{1+\frac{1}{N}}\gamma}{4\alpha N}\right)^{p+1}\|v'\| \le C2^{-p-1}\|v\|_\epsilon. \tag{3.21}$$

(b2) On $(0,1-\tau)$, $u_{\epsilon,I_\epsilon} = l_\tau$. By the regularity, we have

$$\begin{aligned}
\|u'_\epsilon\|_{(0,1-\tau)} &= \Big(\int_0^{1-\tau} u'_\epsilon(x)^2 dx\Big)^{1/2} \\
&\le K\gamma\epsilon^{-1}\Big(\int_0^{1-\tau} e^{-2\alpha(1-x)/\epsilon}dx\Big)^{1/2} \\
&\le K\gamma\epsilon^{-1}\Big(\frac{\epsilon}{2\alpha}e^{-2\alpha\tau/\epsilon}\Big)^{1/2} = \frac{K\gamma}{\sqrt{2\alpha}}\epsilon^{-1/2}e^{-(p+1)}; \\
\|u_\epsilon\|_{(0,1-\tau)} &= \Big(\int_0^{1-\tau} u_\epsilon(x)^2 dx\Big)^{1/2} \\
&\le K\Big(\int_0^{1-\tau} e^{-2\alpha(1-x)/\epsilon}dx\Big)^{1/2} \le \frac{K}{\sqrt{2\alpha}}\epsilon^{1/2}e^{-(p+1)}.
\end{aligned}$$

Recall the estimates for l_τ, and we have (note that $N \le C\epsilon^{-1}$),

$$\|u'_\epsilon - u'_{\epsilon,I_\epsilon}\|_{(0,1-\tau)} \le \|u'_\epsilon\|_{(0,1-\tau)} + \|l'_\tau\| \le C\epsilon^{-1/2}e^{-(p+1)}; \tag{3.22}$$

$$\|u_\epsilon - u_{\epsilon,I_\epsilon}\|_{(0,1-\tau)} \le \|u_\epsilon\|_{(0,1-\tau)} + \|l_\tau\| \le \frac{C}{N}e^{-(p+1)}, \tag{3.23}$$

$$\begin{aligned}
|(u_\epsilon - u_{\epsilon,I_\epsilon}, v')_{(0,1-\tau)}| &\le |(u_\epsilon, v')_{(0,1-\tau)}| + |(l'_\tau, v)_{(0,1-\tau)}| \\
&\le C\epsilon^{1/2}e^{-(p+1)}\|v'\| + C\sqrt{N}e^{-(p+1)}\|v\| \le C\sqrt{N}e^{-(p+1)}\|v\|_\epsilon.
\end{aligned} \tag{3.24}$$

Conclusion for the case $\epsilon N < e\gamma(1+\alpha^{-1})/2$ follows by combining (3.20) – (3.24). ∎

Lemma 3.3. Assume that $\epsilon(p+1) < 1$. We choose $m = p\mu$ with $\mu\gamma < 1$. Let $w_{m,I}$ be the $p+1$-point Gauss-Lobatto interpolation of w_m, let $r_{m,l}$ be the linear interpolation of r_m, and let the maximum mesh-size h satisfy $\gamma h/4 < 1$. Then there exists a constant C independent of h, p, and ϵ, such that,

$$|(w'_m - w'_{m,I}, v)| \le Cpe^{-\sigma p}\|v\|_\epsilon, \quad |(r'_m - r'_{m,I}, v)| \le Ce^{-\rho p}\|v\|_\epsilon,$$

$$\|w_m - w_{m,I}\|_\epsilon \le Ce^{-\sigma p}, \quad \|r_m - r_{m,I}\|_\epsilon \le C\sqrt{\epsilon}e^{-\rho p},$$

where $\sigma = -\ln(h\gamma/4)$ and $\rho = -\mu\ln(\mu\gamma)$.

Proof. By the choice of μ, we see that $\epsilon m\gamma < 1$ and the regularity (2.7) – (2.8) are valid. Based on (2.7), we can obtain the desired estimate for $w_m - w_{m,I}$ the same way as we did for $u - u_I$ in the proof of Lemma 3.1. Since the term $k!$ instead of k^k appears in the regularity (2.7), the Stirling's formula is not needed, and consequently, the factor e does not appear. Therefore, the condition here is $\gamma h/4 < 1$ instead of $e\gamma h/4 < 1$ in Lemma 3.1.

Using the regularity result (2.8) for r_m with $k = 0, 1$ and the piecewise linear interpolation $r_{m,l}$ for r_m, we have

$$|(r'_m - r'_{m,l}, v)| \le (\|r'_m\| + \|r'_{m,l}\|)\|v\| \le C(\epsilon m\gamma)^m\|v\|.$$

$$\epsilon\|r'_m - r'_{m,l}\|^2 + \|r_m - r_{m,l}\|^2 \le C\epsilon(\epsilon m\gamma)^{2m}.$$

Recall $p\epsilon < 1$, $m = \mu p$ with μ satisfying $\mu\gamma < 1$, and we have,

$$|(r'_m - r'_{m,I}, v)| < C(\mu\gamma)^{\mu p}\|v\| = e^{-\rho p}\|v\|, \quad \|r_m - r_{m,l}\|_\epsilon \le C\epsilon^{1/2}e^{-\rho p},$$

with $\rho = -\mu \ln(\mu\gamma)$. ■

Proof of (2.10) – (2.12).

(a) $\epsilon(p+1) \geq 1$. Choose $I_\epsilon u = u_I$, recall Remark 3.1, and Lemma 3.1 yields (2.12), (2.10).

(b) $\epsilon(p+1) < 1$. By the decomposition of (2.4), we have

$$\|u - I_\epsilon u\|_\epsilon \leq \|w_m - w_{m,I}\|_\epsilon + \|u_\epsilon - u_{\epsilon,I_\epsilon}\|_\epsilon + \|r_m - r_{m,l}\|_\epsilon;$$

$$|((u - I_\epsilon u)', v)| \leq |(w'_m - w'_{m,I}, v)| + |(u_\epsilon - u_{\epsilon,I_\epsilon}, v')| + |(r'_m - r'_{m,l}, v)|.$$

and consequently, (2.12) and (2.10) (or (2.11)) follow from Lemmas 3.2–3.3 (see Remark 3.1). ■

Appendix

By linear mapping from $[-1,1]$ to $[x_{i-1}, x_i]$, the terms $\left(\frac{h_i}{2}\right)^{p+1}$ (for $\psi_{p+1}(x)$) and $\left(\frac{h_i}{2}\right)^{p}$ (for $\psi'_{p+1}(x)$) are generated. In order to obtain the constants $cp^{1/2}/2^p$ (for $\psi_{p+1}(x)$) and $cp^{3/2}/2^p$ (for $\psi'_{p+1}(x)$), we only need to consider the related interpolation on $[-1,1]$.

Let $\{\xi_i\}_{i=0}^p$ be Gauss-Lobatto interpolation points. According to [4, p.57], $\xi_0 = -1$, $\xi_p = 1$, and ξ_i $(i = 1, \ldots, p-1)$ are zeros of $L'_p(\xi)$ where L_p is the Legendre polynomial of degree p. We list some basic formulas of the Legendre polynomials [4, pp.60-62]:

$$((1-\xi^2)L'_p(\xi))' + p(p+1)L_p(\xi) = 0; \tag{0.1}$$

$$L_p(\xi) = \frac{1}{2^p}\sum_{l=0}^{[p/2]}(-1)^l \binom{p}{l}\binom{2p-2l}{p}\xi^{p-2l}; \tag{0.2}$$

$$(2p+1)L_p(\xi) = L'_{p+1}(\xi) - L'_{p-1}(\xi). \tag{0.3}$$

We see that

$$\psi_{p+1}(\xi) = (\xi - \xi_0)(\xi - \xi_1)\cdots(\xi - \xi_p) = c(p)(\xi^2 - 1)L'_p(\xi), \tag{0.4}$$

where

$$c(p) = \frac{2^p}{p\binom{2p}{p}},$$

which is determined by utilizing (0.2) and comparing the coefficients of the leading term ξ^p. We shall estimate

$$|\psi_{p+1}(\xi)|, \quad |\psi'_{p+1}(\xi)|.$$

Define, as in [18, p.38],

$$\phi_{p+1}(\xi) = \sqrt{\frac{2p+1}{2}}\int_{-1}^{\xi} L_p(t)dt, \quad p = 1, 2, \ldots \tag{0.5}$$

¿From (0.3), we are able to verify that

$$\phi_{p+1}(\xi) = \frac{1}{\sqrt{2(2p+1)}}(L_{p+1}(\xi) - L_{p-1}(\xi)). \tag{0.6}$$

Substitute (0.3) into (0.1), integrate the resultant, and we have

$$(1-\xi^2)L'_p(\xi) + \frac{p(p+1)}{2p+1}(L_{p+1}(\xi) - L_{p-1}(\xi)) = 0. \tag{0.7}$$

Recall (0.4), we then have

$$\psi_{p+1}(\xi) = c_1(p)(L_{p+1}(\xi) - L_{p-1}(\xi)) \tag{0.8}$$

where

$$c_1(p) = \frac{2^p(p+1)}{(2p+1)\binom{2p}{p}} = \frac{2^p}{\binom{2p+1}{p}} = \frac{2^{p+1}}{\binom{2p+2}{p+1}}.$$

We use the Stirling's formula

$$\sqrt{2\pi}n^{n+1/2} < n!e^n < \sqrt{2\pi}n^{n+1/2}\Big(1+\frac{1}{4n}\Big) \tag{0.9}$$

to obtain estimates

$$\begin{aligned} c_1(p) = \frac{(p+1)!^2 2^{p+1}}{(2p+2)!} &< \frac{(p+1)^{2p+2}2\pi(p+1)2^{p+1}}{(2p+2)^{2p+2}\sqrt{2\pi(2p+2)}}\Big(1+\frac{1}{4(p+1)}\Big)^2 \\ &= \frac{\sqrt{\pi(p+1)}}{2^{p+1}}\Big(1+\frac{1}{4(p+1)}\Big)^2, \end{aligned}$$

and

$$c_1(p) > \frac{\sqrt{\pi(p+1)}}{2^{p+1}}\Big(1+\frac{1}{8(p+1)}\Big)^{-1}.$$

Using the fact that $|L_k(\xi)| \le 1$ on $[-1,1]$, we then have

$$|\psi_{p+1}(\xi)| \le 2c_1(p) \approx \frac{\pi^{1/2}(p+1)^{1/2}}{2^p}.$$

Note that

$$\min_{\xi_0,\xi_1,\ldots,\xi_p\in[-1,1]}\max_{-1\le\xi\le1}|(\xi-\xi_0)(\xi-\xi_1)\cdots(\xi-\xi_p)| = \max_{-1\le\xi\le1} T_{p+1}(\xi)/2^p = 2^{-p},$$

where T_{p+1} is the Chebyshev polynomial (of degree $p+1$) of the first kind.

Comparing (0.8) with (0.6), we have

$$\psi_{p+1}(\xi) = c_1(p)\sqrt{2(2p+1)}\phi_{p+1}(\xi).$$

Then using (0.5),

$$\psi'_{p+1}(\xi) = c_1(p)\sqrt{2(2p+1)}\sqrt{\frac{2p+1}{2}}L_p(\xi),$$

and hence,

$$|\psi'_{p+1}(\xi)| \le c_1(p)(2p+1) \approx \frac{\pi^{1/2}(p+1)^{3/2}}{2^p}.$$

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