

THE ARTIFICIAL BOUNDARY CONDITION FOR EXTERIOR OSEEN EQUATION IN 2-D SPACE*¹⁾

Chun-xiong Zheng Hou-de Han

(Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China)

Abstract

A finite element method for the solution of Oseen equation in exterior domain is proposed. In this method, a circular artificial boundary is introduced to make the computational domain finite. Then, the exact relation between the normal stress and the prescribed velocity field on the artificial boundary can be obtained analytically. This relation can serve as an boundary condition for the boundary value problem defined on the finite domain bounded by the artificial boundary. Numerical experiment is presented to demonstrate the performance of the method.

Key words: Artificial boundary, Exterior domain, Oseen equation.

1. Introduction

PDEs defined on exterior domain can be encountered in many application fields. Typically, the incompressible flow around a body is described by the exterior problem of Navier-Stokes equation. In numerical simulation of this kind of problems, the unboundedness of the domain is a common difficulty. There are many methods to overcome this difficulty and the most popular one is the artificial boundary method. By introducing an artificial boundary which divides the exterior domain into a bounded part and an unbounded part, and setting up a suitable artificial boundary condition, one can reduce the original exterior problem to a boundary value problem defined only on the bounded domain which can then be solved by a suitable numerical method. The readers are referred to [3, 4, 5, 6, 7, 8, 10, 11, 12, 13] for the details of various problems.

In this paper, the analogous method is proposed for the numerical solution of steady Oseen equation. Suppose some objects are moving in \mathcal{R}^2 with a constant speed. By dimensionless procedure, the exterior Oseen equation is formulated as the following:

$$\frac{\partial \vec{u}}{\partial x} = -\nabla p + \frac{1}{\text{Re}} \Delta \vec{u}, \text{ in } \Omega, \quad (1)$$

$$\nabla \cdot \vec{u} = 0, \text{ in } \Omega, \quad (2)$$

$$\vec{u} = (-1, 0), \text{ on } \Gamma, \quad (3)$$

$$\vec{u} \rightarrow 0, \text{ when } r \rightarrow +\infty. \quad (4)$$

where Re is the Reynolds number. Γ is the smooth boundary of the objects and Ω is the exterior domain with boundary Γ .

2. Artificial Boundary Condition

Introduce an artificial boundary $\Gamma_R \equiv \{(r, \theta) | r = R\}$ where R is large enough such that $\Gamma_R \subset \Omega$, then the artificial boundary Γ_R divides Ω into two parts: the unbounded part $\Omega_e \equiv$

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$\{(r, \theta) | r > R\}$ and the bounded part $\Omega_R \equiv \Omega \setminus \bar{\Omega}_e$. On the domain Ω_e we consider the restriction of the solution of problem (1)-(4). From [2], we know in Ω_e the velocity field \vec{u} and the pressure field p has the following expression

$$\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} - \frac{1}{2k} \frac{\partial \chi}{\partial x} + \chi \\ u_y &= \frac{\partial \phi}{\partial y} - \frac{1}{2k} \frac{\partial \chi}{\partial y} \\ p &= -\frac{\partial \phi}{\partial x} \end{aligned}$$

where $k = \frac{Re}{2}$, ϕ and χ are two multi-valued functions satisfying the following equations

$$\begin{aligned} \Delta \phi &= 0, \text{ in } \Omega_e \\ (\Delta - 2k \frac{\partial}{\partial x}) \chi &= 0, \text{ in } \Omega_e. \end{aligned}$$

Furthermore, they have the following expansions (see [2])

$$\begin{aligned} \phi &= \frac{a_0}{2} \log r - \frac{b_0}{2} \theta - \sum_{n=1}^{+\infty} \frac{1}{n} \left(\frac{R}{r}\right)^n (a_n \cos n\theta + b_n \sin n\theta) \\ \chi &= \frac{c_0}{2K_0(kR)} e^{kx} K_0(kr) + \frac{d_0}{K_0(kR)} \int_0^{-\infty} e^{k(x+\xi)} \frac{\partial}{\partial y} K_0(kr_\xi) d\xi \\ &\quad + e^{kx} \sum_{n=1}^{+\infty} \frac{K_n(kr)}{K_n(kR)} (c_n \cos n\theta + d_n \sin n\theta) \end{aligned}$$

where $r_\xi = \sqrt{(x - \xi)^2 + y^2}$. Here and hereafter, K_n and I_n denote the first kind and second kind modified Bessel functions respectively (see [1] for detail). After a computation, on Γ_R we have

$$\begin{aligned} u_r &= \sum_{m=0}^{+\infty} \delta_m \left(\sum_{n=0}^{+\infty} \Phi_{mn}^1 c_n + \frac{a_m}{R} \right) \cos m\theta + \sum_{m=1}^{+\infty} \left(\sum_{n=0}^{+\infty} \Phi_{mn}^2 d_n + \frac{b_m}{R} \right) \sin m\theta \\ u_\theta &= \sum_{m=0}^{+\infty} \delta_m \left(\sum_{n=0}^{+\infty} \Psi_{mn}^2 d_n - \frac{b_m}{R} \right) \cos m\theta + \sum_{m=1}^{+\infty} \left(\sum_{n=0}^{+\infty} \Psi_{mn}^1 c_n + \frac{a_m}{R} \right) \sin m\theta \end{aligned}$$

where

$$\begin{aligned} \Phi_{mn}^1 &= \frac{1}{2} \delta_n \left(I'_{m+n} + I'_{m-n} - \frac{K'_n}{K_n} (I_{m+n} + I_{m-n}) \right), m \geq 0, n \geq 0 \\ \Phi_{mn}^2 &= \frac{1}{2} (I'_{m-n} - I'_{m+n} - \frac{K'_n}{K_n} (I_{m-n} - I_{m+n})), m \geq 0, n > 0 \\ \Phi_{mn}^2 &= \frac{I_{m-1} - I_{m+1}}{2}, m \geq 0, n = 0 \\ \Psi_{mn}^1 &= \frac{1}{2kR} \delta_n ((2n - m)I_{m-n} - (2n + m)I_{m+n}), m \geq 0, n \geq 0 \\ \Psi_{mn}^2 &= -\frac{1}{2kR} ((2n - m)I_{m-n} + (2n + m)I_{m+n}), m \geq 0, n > 0 \\ \Psi_{mn}^2 &= I'_m + \frac{K'_0}{K_0} I_m, m \geq 0, n = 0 \end{aligned}$$

and

$$\delta_n = \begin{cases} \frac{1}{2} & n = 0 \\ 1 & n > 0 \end{cases}$$

The argument of the modified Bessel functions K_n and I_n is kR and we omit it for the sake of simplicity in the above and the following. If the prescribed velocity field \vec{u} has the following expansion on Γ_R

$$u_r = \sum_{n=0}^{+\infty} \delta_n A_n \cos n\theta + \sum_{n=1}^{+\infty} B_n \sin n\theta \quad (5)$$

$$u_\theta = \sum_{n=0}^{+\infty} \delta_n C_n \cos n\theta + \sum_{n=1}^{+\infty} D_n \sin n\theta \quad (6)$$

with

$$A_n = \frac{1}{\pi} \int_0^{2\pi} u_r(R, \theta) \cos n\theta d\theta, \quad n \geq 0$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} u_r(R, \theta) \sin n\theta d\theta, \quad n > 0$$

$$C_n = \frac{1}{\pi} \int_0^{2\pi} u_\theta(R, \theta) \cos n\theta d\theta, \quad n \geq 0$$

$$D_n = \frac{1}{\pi} \int_0^{2\pi} u_\theta(R, \theta) \sin n\theta d\theta, \quad n > 0$$

We have

$$\sum_{n=0}^{+\infty} \Phi_{mn}^1 c_n + \frac{a_m}{R} = A_m, m \geq 0 \quad (7)$$

$$\sum_{n=0}^{+\infty} \Phi_{mn}^2 d_n + \frac{b_m}{R} = B_m, m > 0 \quad (8)$$

$$\sum_{n=0}^{+\infty} \Psi_{mn}^2 d_n - \frac{b_m}{R} = C_m, m \geq 0 \quad (9)$$

$$\sum_{n=0}^{+\infty} \Psi_{mn}^1 c_n + \frac{a_m}{R} = D_m, m > 0 \quad (10)$$

then

$$\sum_{n=0}^{+\infty} (\Phi_{mn}^1 - \Psi_{mn}^1) c_n = A_m - D_m, m > 0 \quad (11)$$

$$\sum_{n=0}^{+\infty} (\Phi_{mn}^2 + \Psi_{mn}^2) d_n = B_m + C_m, m > 0 \quad (12)$$

Infinite matrices $\{\Phi_{mn}^1 - \Psi_{mn}^1, m > 0, n \geq 0\}$ and $\{\Phi_{mn}^2 + \Psi_{mn}^2, m > 0, n \geq 0\}$ define two linear operators from l_2 to l_2 and if denote $\{\Xi_{nm}^1, n \geq 0, m > 0\}$ and $\{\Xi_{nm}^2, n \geq 0, m > 0\}$ the

corresponding infinite matrix expressions of the inverse operators (if exist), we have

$$c_n = \sum_{m=1}^{+\infty} \Xi_{nm}^1 (A_m - D_m), n \geq 0 \tag{13}$$

$$d_n = \sum_{m=1}^{+\infty} \Xi_{nm}^2 (B_m + C_m), n \geq 0 \tag{14}$$

Furthermore, from (7) and (9), we get a_n and b_n .

Now we come to the expression of the normal stress. The stress tensor is

$$\sigma = -pE + \frac{1}{\text{Re}} \left(\nabla \vec{u} + (\nabla \vec{u})^T \right)$$

where E is the identity tensor. Then

$$\begin{aligned} \sigma_{nn} &= -p + \frac{2}{\text{Re}} \frac{\partial u_r}{\partial r} \\ \sigma_{nt} &= \frac{1}{\text{Re}} \left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{1}{r} u_\theta \right) \end{aligned}$$

After a computation, we get

$$\begin{aligned} \sigma_{nn} &= \sum_{n=0}^{+\infty} \delta_n A_n \cos(n+1)\theta - \sum_{n=0}^{+\infty} \delta_n \Phi_n^1 \cos(n+1)\theta \\ &\quad - \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (A_n + nD_n) \cos n\theta + \sum_{n=0}^{+\infty} \delta_n \Psi_n^2 \sin(n+1)\theta \\ &\quad - \sum_{n=0}^{+\infty} \delta_n C_n \sin(n+1)\theta - \frac{1}{kR} \sum_{n=1}^{+\infty} (B_n - nC_n) \sin n\theta \\ &\equiv \sigma_{nn}(\vec{u}|_{\Gamma_R}) \end{aligned} \tag{15}$$

$$\begin{aligned} \sigma_{nt} &= -\frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (nA_n + D_n) \sin n\theta + \frac{1}{2} \sum_{n=0}^{+\infty} \delta_n (\Phi_n^1 + \Psi_n^1) \sin(n+1)\theta \\ &\quad - \frac{1}{2} \sum_{n=0}^{+\infty} \delta_n (\Phi_n^1 - \Psi_n^1) \sin(n-1)\theta + \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (nB_n - C_n) \cos n\theta \\ &\quad + \frac{1}{2} \sum_{n=0}^{+\infty} \delta_n (\Phi_n^2 + \Psi_n^2) \cos(n-1)\theta + \frac{1}{2} \sum_{n=0}^{+\infty} \delta_n (\Psi_n^2 - \Phi_n^2) \cos(n+1)\theta \\ &\equiv \sigma_{nt}(\vec{u}|_{\Gamma_R}) \end{aligned} \tag{16}$$

where

$$\Phi_m^1 = \sum_{n=1}^{+\infty} \Theta_{mn}^1 (A_n - D_n), \quad \Phi_m^2 = \sum_{n=1}^{+\infty} \Theta_{mn}^2 (B_n + C_n), \quad m \geq 0 \tag{17}$$

$$\Psi_m^1 = \sum_{n=1}^{+\infty} \Upsilon_{mn}^1 (A_n - D_n), \quad \Psi_m^2 = \sum_{n=1}^{+\infty} \Upsilon_{mn}^2 (B_n + C_n), \quad m \geq 0 \tag{18}$$

Here

$$\Theta_{mn}^1 = \sum_{k=0}^{+\infty} \Phi_{mk}^1 \Xi_{kn}^1, \quad \Theta_{mn}^2 = \sum_{k=0}^{+\infty} \Phi_{mk}^2 \Xi_{kn}^2, \quad m \geq 0 \quad (19)$$

$$\Upsilon_{mn}^1 = \sum_{k=0}^{+\infty} \Psi_{mk}^1 \Xi_{kn}^1, \quad \Upsilon_{mn}^2 = \sum_{k=0}^{+\infty} \Psi_{mk}^2 \Xi_{kn}^2, \quad m \geq 0 \quad (20)$$

(15)-(16) is the exact artificial boundary condition. Now we can reduce the original problem to the following one only defined in the bounded domain Ω_R :

$$\frac{\partial \vec{u}}{\partial x} = -\nabla p + \frac{1}{\text{Re}} \Delta \vec{u}, \quad \text{in } \Omega_R, \quad (21)$$

$$\nabla \cdot \vec{u} = 0, \quad \text{in } \Omega_R, \quad (22)$$

$$\vec{u} = (-1, 0), \quad \text{on } \Gamma, \quad (23)$$

$$\sigma_{nn} = \sigma_{nn}(\vec{u}|_{\Gamma_R}), \quad \text{on } \Gamma_R \quad (24)$$

$$\sigma_{nt} = \sigma_{nt}(\vec{u}|_{\Gamma_R}), \quad \text{on } \Gamma_R \quad (25)$$

3. Variational Formulation and Its Approximation

Define

$$\begin{aligned} W &= \left\{ \vec{w} = (w_1, w_2) \in (H^1(\Omega_R))^2, \vec{w} \Big|_{\Gamma} = \vec{0} \right\} \\ W_* &= \left\{ \vec{w} = (w_1, w_2) \in (H^1(\Omega_R))^2, \vec{w} \Big|_{\Gamma} = (-1, 0) \right\} \\ Q &= \left\{ q \in L^2(\Omega_R), \int_{\Gamma_R} q ds = 0 \right\} \end{aligned}$$

Now the reduced problem (21)-(25) is equivalent to the following mixed variational problem:

Find $(\vec{u}, p) \in W_* \times Q$, such that

$$\begin{cases} a_0(\vec{w}, \vec{u}) + a_1(\vec{w}, \vec{u}) + a_2(\vec{w}, \vec{u}) + b(\vec{w}, p) = 0, \forall \vec{w} \in W \\ b(\vec{u}, q) = 0, \forall q \in Q \end{cases} \quad (26)$$

where

$$a_0(\vec{w}, \vec{u}) = \frac{2}{\text{Re}} \int_{\Omega_R} \epsilon(\vec{w}) : \epsilon(\vec{u}) d\sigma \quad (27)$$

$$a_1(\vec{w}, \vec{u}) = \int_{\Omega_R} \vec{w} \cdot \frac{\partial \vec{u}}{\partial x} d\sigma \quad (28)$$

$$a_2(\vec{w}, \vec{u}) = - \int_{\Gamma_R} \left(w_r \cdot \sigma_{nn}(\vec{u}|_{\Gamma_R}) + w_\theta \cdot \sigma_{nt}(\vec{u}|_{\Gamma_R}) \right) ds \quad (29)$$

$$b(\vec{u}, q) = - \int_{\Omega_R} q \nabla \cdot \vec{u} d\sigma \quad (30)$$

Here $\epsilon(\vec{w}) \equiv \frac{1}{2} (\nabla \vec{w} + (\nabla \vec{w})^T)$ is the strain tensor.

Variational problem (26) has only the theoretical importance, for in the actual implementation three drawbacks are encountered. Firstly, the components of the velocity expansion (5)-(6) should be truncated to L terms. Secondly, it's impossible to get the exact expressions of the infinite matrices $\{\Xi_{nm}^1, n \geq 0, m > 0\}$ and $\{\Xi_{nm}^2, n \geq 0, m > 0\}$ in (13)-(14). But

what we really need is $\Theta_{mn}^1, \Theta_{mn}^2, \Upsilon_{mn}^1$ and Υ_{mn}^2 for $m \geq 0$ and $0 < n \leq L$ in (19)-(20). When $k > L$, all Ξ_{kn}^1 and Ξ_{kn}^2 decrease very rapidly. If we truncate the infinite matrices to $\{\Phi_{mn}^1 - \Psi_{mn}^1, 0 < m \leq M, 0 \leq n < M\}$ and $\{\Phi_{mn}^2 + \Psi_{mn}^2, 0 < m \leq M, 0 \leq n < M\}$ and get their inverse matrices, Ξ_{kn}^1 and Ξ_{kn}^2 can be fairly approximated. Numerical investigation shows when $Re < 30, R < 4, L < 10, M = 50$ is large enough to get a sufficiently close solution. Futhermore (19)-(20) can be replaced by

$$\Theta_{mn}^1 = \sum_{k=0}^M \Phi_{mk}^1 \Xi_{kn}^1, \quad \Theta_{mn}^2 = \sum_{k=0}^M \Phi_{mk}^2 \Xi_{kn}^2, \quad m \geq 0 \tag{31}$$

$$\Upsilon_{mn}^1 = \sum_{k=0}^M \Psi_{mk}^1 \Xi_{kn}^1, \quad \Upsilon_{mn}^2 = \sum_{k=0}^M \Psi_{mk}^2 \Xi_{kn}^2, \quad m \geq 0 \tag{32}$$

in the aspect of numerical approximation. Thirdly, in the expression of the normal stress, the series should not be infinite. But this can be overcome by truncating the first N terms, the standard procedure in artificial boundary method. After all these procedures, we denote the approximate normal stress as σ_n^* , which has the following expression:

$$\begin{aligned} \sigma_{nm}^* &= \sum_{n=0}^L \delta_n A_n \cos(n+1)\theta - \sum_{n=0}^N \delta_n \Phi_n^{1*} \cos(n+1)\theta \\ &\quad - \frac{1}{kR} \sum_{n=0}^L \delta_n (A_n + nD_n) \cos n\theta + \sum_{n=0}^N \delta_n \Psi_n^{2*} \sin(n+1)\theta \\ &\quad - \sum_{n=0}^L \delta_n C_n \sin(n+1)\theta - \frac{1}{kR} \sum_{n=1}^N (B_n - nC_n) \sin n\theta \\ &\equiv \sigma_{nm}^*(\vec{u}|_{\Gamma_R}) \end{aligned} \tag{33}$$

$$\begin{aligned} \sigma_{nt}^* &= -\frac{1}{kR} \sum_{n=0}^L \delta_n (nA_n + D_n) \sin n\theta + \frac{1}{2} \sum_{n=0}^N \delta_n (\Phi_n^{1*} + \Psi_n^{1*}) \sin(n+1)\theta \\ &\quad - \frac{1}{2} \sum_{n=0}^N \delta_n (\Phi_n^{1*} - \Psi_n^{1*}) \sin(n-1)\theta + \frac{1}{kR} \sum_{n=0}^L \delta_n (nB_n - C_n) \cos n\theta \\ &\quad + \frac{1}{2} \sum_{n=0}^N \delta_n (\Phi_n^{2*} + \Psi_n^{2*}) \cos(n-1)\theta + \frac{1}{2} \sum_{n=0}^N \delta_n (\Psi_n^{2*} - \Phi_n^{2*}) \cos(n+1)\theta \\ &\equiv \sigma_{nt}^*(\vec{u}|_{\Gamma_R}) \end{aligned} \tag{34}$$

where

$$\begin{aligned} \Phi_m^{1*} &= \sum_{n=1}^L \Theta_{mn}^1 (A_n - D_n), & \Phi_m^{2*} &= \sum_{n=1}^L \Theta_{mn}^2 (B_n + C_n), \quad m \geq 0 \\ \Psi_m^{1*} &= \sum_{n=1}^L \Upsilon_{mn}^1 (A_n - D_n), & \Psi_m^{2*} &= \sum_{n=1}^L \Upsilon_{mn}^2 (B_n + C_n), \quad m \geq 0 \end{aligned}$$

Finally, if W_*^h and Q^h are two proper subspaces of W_* and Q respectively, we get the discrete approximate mixed variational problem:

Find $(\vec{u}^h, p^h) \in W_*^h \times Q^h$, such that

$$\begin{cases} a_0(\vec{w}^h, \vec{u}^h) + a_1(\vec{w}^h, \vec{u}^h) + a_2^*(\vec{w}^h, \vec{u}^h) + b(\vec{w}^h, p^h) &= 0, \forall \vec{w}^h \in W^h \\ b(\vec{u}^h, q^h) &= 0, \forall q^h \in Q^h \end{cases} \tag{35}$$

where

$$a_2^*(\vec{w}^h, \vec{u}^h) = - \int_{\Gamma_R} \left(w_r^h \cdot \sigma_{nn}^*(\vec{u}^h|_{\Gamma_R}) + w_\theta^h \cdot \sigma_{nt}^*(\vec{u}^h|_{\Gamma_R}) \right) ds$$

4. Numerical Example

We consider the Oseen flow around a circular cylinder with the radius of 1. The numerical results are compared with the analytical solution gained with the method proposed by S. Tomotika and T. Aoi in [9]. Because of the symmetry of this problem, we can consider the problem only in the upper semi-plane. First, we introduce a circle artificial boundary Γ_R with $R = 2$, then we divide the interval $[1, 2]$ into N_r equal parts. We also divide the $[0, \pi]$ into N_θ equal parts and we get $N_r \times N_\theta$ equal rectangles in the polar system. Divide each quadrangle into two triangles in cartesian system, we get the final mesh and denote it with $N_r \times N_\theta$.

Table 1: The error $\|\vec{u} - \vec{u}^h\|_\infty$ on the grid point for $\text{Re} = 2$, $M = 40$ and $N = 20$

Mesh	$L = 2$	$L = 4$	$L = 6$	$L = 10$
12×6	$9.4326E - 002$	$1.4903E - 002$	$1.3078E - 002$	$1.1717E - 002$
24×12	$9.3642E - 002$	$4.8787E - 003$	$3.8912E - 003$	$3.6008E - 003$
36×12	$9.3530E - 002$	$3.2302E - 003$	$2.3233E - 003$	$2.1225E - 003$

$P_2 - P_1$ mixed finite element is used in the computation. From table 1, we see that both increasing the order of the artificial boundary condition and refining the mesh can decrease the error. After the mesh is sufficiently refined, the error originated from the series truncating is dominating and we should use more series terms in order to get a higher approximation. Figure 1 shows the stream line calculated when $L = 10$ and mesh 36×12 is used. It's seen that there are two standing eddies behind the cylinder. This flow pattern is in very good agreement with that gained in [9].

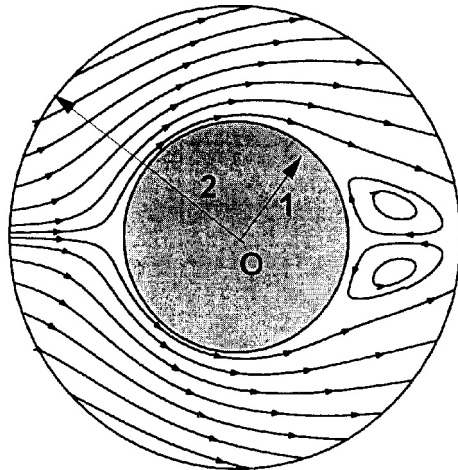


Figure 1: Stream-line plot of Oseen flow with $\text{Re} = 2$ around a circular cylinder with radius of 1. The artificial boundary is a circle with radius of 2.

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