Journal of Computational Mathematics, Vol.20, No.6, 2002, 583–590.

NONLINEAR STABILITY OF NATURAL RUNGE-KUTTA METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS*1)

Cheng-jian Zhang
(*Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, Hubei, P. R. China*)

Abstract

This paper first presents the stability analysis of theoretical solutions for a class of nonlinear neutral delay-differential equations (NDDEs). Then the numerical analogous results, of the natural Runge-Kutta (NRK) methods for the same class of nonlinear NDDEs, are given. In particular, it is shown that the (k,l)-algebraic stability of an RK method for ODEs implies the generalized asymptotic stability and global stability of the induced NRK method.

Key words: Nonlinear stability, Neutral delay differential equations, Natural Runge-Kutta methods.

1. Introduction

In the last several decades, there has been a growing interest in the numerical stability for DDEs(cf. [1-14]). In 1988, A.Bellen, Z. Jackiewicz and M.Zennaro[7] first extend the researches to the scalar linear NDDEs. Latterly, a lot of works for the systems of linear NDDEs were presented(cf.[8-12]). However, there are much difficulties to assess the numerical stability of nonlinear NDDEs. In view of this, T.Koto [13] adapted NRK methods (cf.[14]) to a class of nonlinear NDDEs in real space $\mathbf{R}^d$, and studied their asymptotic stability with a discrete analogue of the Liapunov functional.

In this paper, by an alternative approach, we futher deal with the stability of theoretical and numerical solutions for a class of nonlinear NDDEs in complex space $\mathbf{C}^d$. Particularly, it is shown that a NRK method induced by a $(k,l)-$algebraically stable RK methods for ODEs, under suitable conditions,preserves the analogous stability of the original equations.

2. Test Problem and Its Stability

For giving subsequent analysis, we first set some notational conventions. Let $\langle\bullet,\bullet\rangle, \|\bullet\|$ denote the inner product and the induced norm in space $\mathbf{C}^d$, respectively. Correspondingly, the inner product and the induced norm in space $(\mathbf{C}^d)^l$ are defined as follows:

$$\langle U,V\rangle = \sum_{i=1}^{l}\langle u_i, v_i\rangle, \quad \|U\|^2 = \langle U,U\rangle,$$

where $U=(u_1,u_2,\ldots,u_l), U=(v_1,v_2,\ldots,v_l)\in(\mathbf{C}^d)^l$ and $u_i,v_i\in\mathbf{C}^d (i=1,2,\ldots,l)$. Moreover, it is always assumed that each matrix norms, arising in the following, is subject to the corresponding vector norm.

* Received.
1) This project is supported by NSFC (No.69974018) and China Postdoctoral Science Foundation.

Consider the following systems of nonlinear NDDEs

$$\begin{cases} \frac{d}{dt}[y(t) - Ny(t-\tau)] = f(t, y(t), y(t-\tau)), \quad t \geq 0, \\ y(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \frac{d}{dt}[z(t) - Nz(t-\tau)] = f(t, z(t), z(t-\tau)), t \geq 0, \\ z(t) = \psi(t), -\tau \leq t \leq 0, \end{cases} \tag{2.2}$$

where $\tau > 0$ is constant delay, $N \in \mathbf{C}^{d \times d}$ stand for a constant matrix with $\|N\| < 1$, $\phi, \psi: [-\tau, 0] \to \mathbf{C}^d$ are continuous functions, and $f: [0, +\infty) \times \mathbf{C}^d \times \mathbf{C}^d \to \mathbf{C}^d$ is a assigned mapping subject to

$$\begin{aligned} & \mathrm{Re}\langle (x_1 - x_2) - N(y_1 - y_2), f(t, x_1, y_1) - f(t, x_2, y_2) \rangle \\ \leq \; & \alpha \|x_1 - x_2\|^2 + \beta \|y_1 - y_2\|^2, \quad t \geq 0, \;\; x_1, x_2, y_1, y_2 \in \mathbf{C}^d, \end{aligned} \tag{2.3}$$

in which α, β are real constants.

The problems of the form (2.1) can be found in the systems with lossless transmission lines (cf.[16]). In the following,all the problems (2.1) with (2.3) will be referred as *the class* $R_{\alpha,\beta}$. For instance, a complex d-dimensional linear system

$$\begin{cases} \frac{d}{dt}[y(t) - Ny(t-\tau)] = Ly(t) + My(t-\tau), \quad t \geq 0, \\ y(t) = \phi(t), \quad -\tau \leq t \leq 0, \end{cases}$$

belongs to the class $R_{\alpha,\beta}$ whenever the matrix

$$G = \frac{1}{2} \begin{pmatrix} L + L^* - 2\alpha I & M - L^*N \\ M^* - N^*L & -N^*M - M^*N - 2\beta I \end{pmatrix}$$

is negative definite, where I denote a d-dimensional identity matrix and $*$ is the conjugate transpose symbol of the matrices, since

$$\begin{aligned} & Re\langle (x_1 - x_2) - N(y_1 - y_2), L(x_1 - x_2) + M(y_1 - y_2) \rangle \\ & -\alpha \|x_1 - x_2\|^2 - \beta \|y_1 - y_2\|^2 \\ = \; & \langle \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix}, G \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \rangle, \quad \forall x_1, x_2, y_1, y_2 \in \mathbf{C}^d. \end{aligned}$$

For the problems of the class $R_{\alpha,\beta}$, we obtain the following stability results.

Theorem 2.1. *Suppose problems (2,1),(2.2) belong to the class* $R_{\alpha,\beta}$ *with*

$$\alpha \leq 0, \quad \beta \leq \alpha \|N\|^2. \tag{2.4}$$

Then we have

(a) $\|y(t) - z(t)\| \leq \frac{2}{1 - \|N\|} \max\limits_{-\tau \leq \theta \leq 0} \|\phi(\theta) - \psi(\theta)\|, \;\; t \geq 0,$

(b) *for* $\alpha < 0, \;\; \lim\limits_{t \to +\infty} \|y(t) - z(t) - N(y(t-\tau) - z(t-\tau))\| = 0.$

Proof. Let

$$u(t) = y(t) - z(t), \quad v(t) = \|u(t) - Nu(t-\tau)\|^2,$$
$$F(t) = f(t, y(t), y(t-\tau)) - f(t, z(t), z(t-\tau)).$$

Then by (2.3)

$$\begin{aligned} v'(t) \; & = 2Re\langle u(t) - Nu(t-\tau), F(t) \rangle \\ & \leq 2[\alpha \|u(t)\|^2 + \beta \|u(t-\tau)\|^2], \quad t \geq 0, \end{aligned}$$

which gives

$$[\exp(-\alpha\xi)v(\xi)]' \leq \exp(-\alpha\xi)[-\alpha v(\xi) + 2\alpha\|u(\xi)\|^2 + 2\beta\|u(\xi-\tau)\|^2], \;\; \xi \geq 0. \tag{2.5}$$

On the other hand, condition (2.4) leads to

$$\begin{aligned}
& \exp(-\alpha\xi)[-\alpha v(\xi) + 2\alpha\|u(\xi)\|^2 + 2\beta\|u(\xi-\tau)\|^2] \\
= \;& \exp(-\alpha\xi)[-\alpha(\|u(\xi)\|^2 - 2Re\langle u(\xi), Nu(\xi-\tau)\rangle + \|Nu(\xi-\tau)\|^2) \\
& \quad +2\alpha\|u(\xi)\|^2 + 2\beta\|u(\xi-\tau)\|^2] \\
\leq \;& \exp(-\alpha\xi)[-\alpha(\|u(\xi)\| + \|Nu(\xi-\tau)\|)^2 + 2\alpha\|u(\xi)\|^2 + 2\beta\|u(\xi-\tau)\|^2] \\
\leq \;& \exp(-\alpha\xi)[-2\alpha(\|u(\xi)\|^2 + \|N\|^2\|u(\xi-\tau)\|^2) + 2\alpha\|u(\xi)\|^2 + 2\beta\|u(\xi-\tau)\|^2] \\
= \;& 2\exp(-\alpha\xi)(\beta - \alpha\|N\|^2)\|u(\xi-\tau)\|^2 \leq 0, \quad \xi \geq 0.
\end{aligned} \tag{2.6}$$

Combining (2.5) with (2.6) yields

$$[\exp(-\alpha\xi)v(\xi)]' \leq 0, \;\; \xi \geq 0. \tag{2.7}$$

Thus

$$\int_0^t [\exp(-\alpha\xi)v(\xi)]' d\xi \leq 0, \;\; t \geq 0,$$

which implies

$$\sqrt{v(t)} \leq \exp(\frac{\alpha t}{2})\sqrt{v(0)}, \;\; t \geq 0, \tag{2.8}$$

With (2.8), we conclude immdiately that (b) holds and

$$\|u(t)\| \leq \|N\|\|u(t-\tau)\| + \sqrt{v(0)}, \;\; t \geq 0. \tag{2.9}$$

In view of the fact that there exists a positive integer q for each $t \geq 0$ such that $(q-1)\tau \leq t < q\tau$, we can get by an induction to (2.9)

$$\|u(t)\| \leq \|N\|^q\|u(t-q\tau)\| + (1 + \sum_{i=1}^{q-1}\|N\|^i)\sqrt{v(0)}, \;\; t \geq 0. \tag{2.10}$$

Therefore, (a) is true in accordance with both (2.10) and $\|N\| < 1$.

Terms (a), (b) characterize the global stability and the generalized asymptotic stability of the problems (2.1) of the class $R_{\alpha,\beta}$, respectively. In particular, when set $N = 0$ in (2.1) and (2.2), with a slight modification to the proof of Theorem 2.1, we obtain at once

Theorem 2.2. *Suppose problems (2,1),(2.2), when $N = 0$, belong to the class $R_{\alpha,\beta}$ with*

$$\alpha \leq 0, \quad \beta \leq 0. \tag{2.11}$$

Then we have

($\hat{a}$) $\|y(t) - z(t)\| \leq \max_{-\tau \leq \theta \leq 0} \|\phi(\theta) - \psi(\theta)\|, \;\; t \geq 0,$

($\hat{b}$) *for* $\alpha < 0, \;\; \lim_{t \to +\infty} \|y(t) - z(t)\| = 0.$

3. Stability of The Methods

Before proceeding with the numerical stability analysis for NDDEs, we first make a brief review of the related concepts and results on RK methods. A s-stage RK method for ODEs can be expressed as

$$\begin{array}{c|c} \mathrm{c} & \mathrm{A} \\ \hline & b^T \end{array}, \tag{3.1}$$

where $A=(a_{ij})\in\mathbf{R}^{s\times s}$, $b=(b_1,b_2,\ldots,b_s)^T$ and $c=(c_1,c_2,\ldots,c_s)^T\in\mathbf{R}^s$ with $c_i\in[0,1](i=1,2,\ldots,s)$.

Method (3.1) is called $(k,l)-$ *algebrically stable*, if there exist real constants k,l and a nonnegative diagonal matrix $D=diag(d_1,d_2,\ldots,d_s)$ such that matrix

$$G(k,l)=\begin{bmatrix} k-1-2le^TDe & e^TD-b^T-2le^TDA \\ De-b-2lA^TDe & DA+A^TD-bb^T-2lA^TDA \end{bmatrix}$$

is nonnegtive definite, where $e=(1,1,\ldots,1)^T$ (cf.[15]). In particular,the $(1,0)$-algebraically stable RK method is called *algebraically stable*.

M.Zennaro [14] pointed out that every RK methods (3.1) has a natural continuous extension(NCE). Adapting this NCE to the problems (2.1), T.Koto [13] gotten the following NRK methods

$$\begin{cases} \dot{Y}_i^{(n)}=f(T_i^n,Y_i^{(n)},Y_i^{(n-m)})+N\sum\limits_{j=1}^{s}b_j'(c_i)\dot{Y}_j^{(n-m)}, \quad i=1,2,\ldots,s, \\ Y_i^{(n)}=y_n+h\sum\limits_{j=1}^{s}a_{ij}\dot{Y}_j^{(n)}, \quad i=1,2,\ldots,s, \\ y_{n+1}=y_n+h\sum\limits_{j=1}^{s}b_j\dot{Y}_j^{(n)}, \end{cases} \tag{3.2}$$

where $h=\frac{\tau}{m}$, m is certain positive integer, $t_n=nh, T_i^{(n)}=t_n+c_ih; \dot{Y}_i, Y_i^{(n)}$ y_n are approximations to $y'(T_i^{(n)}), y(T_i^{(n)})$ and $y(t_n)$,respectively;particularly, when a mesh point t_n or off-point $T_i^{(n)}$ is in the initial interval $[-\tau,0]$,we always set

$$y_n=\phi(t_n), Y_i^{(n)}=\phi(T_i^{(n)}), \dot{Y}_i=\phi'(T_i^{(n)}).$$

Moreover, in (3.2), the coefficients a_{ij}, b_i and the polynomials $b_i(\zeta)$ are assumed to conform with

$$\begin{cases} b_i(0)=0, \quad b_i(1)=b_i, \quad 1\le i\le s, \\ \sum\limits_{i=1}^{s}b_i(\zeta)c_i^{l-1}=\frac{\zeta^l}{l}, \quad 1\le i\le s, \quad 1\le l\le \max\limits_{1\le i\le s}\{deg(b_i(\zeta))\}, \\ a_{ij}=b_j(c_i), \quad 1\le i,j\le s. \end{cases} \tag{3.3}$$

By (3.3),together with a direct computation, we can get

$$b^TB=b^T, \quad AB=A, \quad \text{where } B=(b_j'(c_i))\in\mathbf{R}^{s\times s}. \tag{3.4}$$

For distinction, the corresponding approximations, produced by applying method (3.2) to (2.2), will be written as $\dot{Z}_i, Z_i^{(n)}$ and z_n, respectively. Furthermore, we also introduced the following notational conventions:

$$u_n=y_n-z_n,\ U_i^{(n)}=Y_i^{(n)}-Z_i^{(n)},\ \dot{U}_i^{(n)}=\dot{Y}_i^{(n)}-\dot{Z}_i^{(n)}$$
$$Q_i^{(n)}=f(T_i^{(n)},Y_i^{(n)},Y_i^{(n-m)})-f(T_i^{(n)},Z_i^{(n)},Z_i^{(n-m)}).$$

$$U^{(n)} = \begin{pmatrix} U_1^{(n)} \\ U_2^{(n)} \\ \vdots \\ U_s^{(n)} \end{pmatrix}, \ \dot{U}^{(n)} = \begin{pmatrix} \dot{U}_1^{(n)} \\ \dot{U}_2^{(n)} \\ \vdots \\ \dot{U}_s^{(n)} \end{pmatrix}, \ Q^{(n)} = \begin{pmatrix} Q_1^{(n)} \\ Q_2^{(n)} \\ \vdots \\ Q_s^{(n)} \end{pmatrix},$$

With (3.2) and the above notational conventions, we have

$$\begin{cases} \dot{U}^{(n)} = Q^{(n)} + (B \otimes N)\dot{U}^{(n-m)}, \\ U^{(n)} = e \otimes u_n + h(A \otimes I)\dot{U}^{(n)}, \\ u_{n+1} = u_n + h(b^T \otimes I)\dot{U}^{(n)}, \end{cases} \tag{3.5}$$

where $\otimes$ denotes the Kronecker product between the two matrices.

Theorem 3.1. *Suppose an RK method (3.1) for ODEs is (k, l) -algebraically stable. Then, the corresponding NRK method (3.2), for the problems of the class $R_{\alpha,\beta}$ with*

$$\alpha \le 0, \ \ \beta \le \alpha \|N\|^2 \ \ and \ \ h\alpha \le 2l, \tag{3.6}$$

satisfies

(I) for $0 \le k \le 1, \quad \|y_n - z_n\| \le \frac{2}{1-\|N\|} \max_{-\tau \le \theta \le 0} \|\phi(\theta) - \psi(\theta)\|,$

(II) for $0 \le k < 1, \quad \lim_{n \to \infty} \|y_n - z_n - N(y_{n-m} - z_{n-m})\| = 0.$

Proof. By (3.4),(3.5) we have

$$\begin{aligned} & u_{n+1} - N u_{n+1-m} \\ = \ & u_n - N u_{n-m} + h(b^T \otimes I)\dot{U}^{(n)} - h(b^T \otimes N)\dot{U}^{(n-m)} \\ = \ & u_n - N u_{n-m} + h(b^T \otimes I)\dot{U}^{(n)} - h(b^T B \otimes N)\dot{U}^{(n-m)} \\ = \ & u_n - N u_{n-m} + h(b^T \otimes I)[\dot{U}^{(n)} - h(B \otimes N)\dot{U}^{(n-m)}] \\ = \ & u_n - N u_{n-m} + h(b^T \otimes I)Q^{(n)}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} & U^{(n)} - (I \otimes N)U^{(n-m)} \\ = \ & e \otimes (u_n - N u_{n-m}) + h(A \otimes I)\dot{U}^{(n)} - h(A \otimes N)\dot{U}^{(n-m)} \\ = \ & e \otimes (u_n - N u_{n-m}) + h(A \otimes I)\dot{U}^{(n)} - h(AB \otimes N)\dot{U}^{(n-m)} \\ = \ & e \otimes (u_n - N u_{n-m}) + h(A \otimes I)[\dot{U}^{(n)} - (B \otimes N)\dot{U}^{(n-m)}] \\ = \ & e \otimes (u_n - N u_{n-m}) + h(A \otimes I)Q^{(n)}. \end{aligned} \tag{3.8}$$

It follows from (3.7),(3.8) and the $(k, l)-$algebraic stability that

$$\begin{aligned} & \|u_{n+1} - N u_{n+1-m}\|^2 - k\|u_n - N u_{n-m}\|^2 \\ & \quad -2 \sum_{i=1}^{s} d_i Re\langle U_i^{(n)} - N U_i^{(n-m)}, hQ_i^{(n)} - l(U_i^{(n)} - N U_i^{(n-m)})\rangle \\ = \ & \|u_{n+1} - N u_{n+1-m}\|^2 - k\|u_n - N u_{n-m}\|^2 \\ & \quad -2Re\langle \dot{U}^{(n)} - (I \otimes N)\dot{U}^{(n-m)}, (D \otimes I)[hQ^{(n)} - l(\dot{U}^{(n)} - (I \otimes N)\dot{U}^{(n-m)})]\rangle \\ = \ & \|u_n - N u_{n-m} + h(b^T \otimes I)Q^{(n)}\|^2 - k\|u_n - N u_{n-m}\|^2 - 2Re\langle e \otimes (u_n - N u_{n-m}) \\ & \quad + h(A \otimes I)Q^{(n)}, (D \otimes I)[hQ^{(n)} - l(e \otimes (u_n - N u_{n-m}) + h(A \otimes I)Q^{(n)})]\rangle \\ = \ & -\langle \begin{pmatrix} u_n - N u_{n-m} \\ hQ^{(n)} \end{pmatrix}, (G(k,l) \otimes I) \begin{pmatrix} u_n - N u_{n-m} \\ hQ^{(n)} \end{pmatrix} \rangle \le 0. \end{aligned} \tag{3.9}$$

A combination of (3.9),(2.3) and (3.6) yields

$$
\begin{aligned}
&\|u_{n+1} - Nu_{n+1-m}\|^2 \\
\leq\ & k\|u_n - Nu_{n-m}\|^2 + 2\sum_{i=1}^{s} d_i Re\langle U_i^{(n)} - NU_i^{(n-m)}, hQ_i^{(n)} - l(U_i^{(n)} - NU_i^{(n-m)})\rangle \\
\leq\ & k\|u_n - Nu_{n-m}\|^2 + 2h\alpha\sum_{i=1}^{s} d_i\|U_i^{(n)}\|^2 + 2h\beta\sum_{i=1}^{s} d_i\|U_i^{(n-m)}\|^2 \\
& -2l\sum_{i=1}^{s} d_i\|U_i^{(n)} - NU_i^{(n)}\|^2 \\
\leq\ & k\|u_n - Nu_{n-m}\|^2 + 2h\alpha\sum_{i=1}^{s} d_i(\|U_i^{(n)}\|^2 + \|N\|^2\|U_i^{(n-m)}\|^2) \\
& -2l\sum_{i=1}^{s} d_i\|U_i^{(n)} - NU_i^{(n-m)}\|^2 \\
\leq\ & k\|u_n - Nu_{n-m}\|^2 + (h\alpha - 2l)\sum_{i=1}^{s} d_i\|U_i^{(n)} - NU_i^{(n-m)}\|^2 \\
\leq\ & k\|u_n - Nu_{n-m}\|^2,
\end{aligned}
$$

i.e.

$$\|u_n - Nu_{n-m}\| \leq \sqrt{k}\|u_{n-1} - Nu_{n-1-m}\|. \tag{3.10}$$

Hence, it follows by an induction to (3.10) that

$$\|u_n - Nu_{n-m}\| \leq (\sqrt{k})^n\|\phi(0) - N\phi(-\tau)\|, \tag{3.11}$$

which implies (II).Further, when $0 \leq k \leq 1$, by (3.11)

$$\|u_n\| \leq \|N\|\|u_{n-m}\| + \|\phi(0) - N\phi(-\tau)\|. \tag{3.12}$$

Therefore, induction of (3.12) results in

$$\|u_n\| \leq \|N\|^q\|u_{n-qm}\| + (1 + \sum_{i=1}^{q-1}\|N\|^i)\|\phi(0) - N\phi(-\tau)\|, \tag{3.13}$$

where q is a positive integer depended only on n and satisfies $(q-1)m \leq n < qm$.Note that $u_{n-qm} = \phi(t_{n-qm})$ and $\|N\| < 1$. So, (I) is proved by (3.13).

Theorem 3.1 shows that the NRK methods preserve the stability properties of the original equations. In particular, when $N = 0$, by a slight modification to the proof of Theorem 3.1 we can conclude

Theorem 3.2. *Suppose an RK method (3.1) for ODEs is (k, l)- algebraically stable.Then, the corresponding NRK method (3.2), for the problems of the class $R_{\alpha,\beta}$ with*

$$N = 0,\ \ \alpha \leq 0,\ \ \beta \leq 0\ \ \textit{and}\ \ h\alpha \leq l,$$

satisfies

(i) for $0 \leq k \leq 1, \quad \|y_n - z_n\| \leq \max_{-\tau\leq\theta\leq 0}\|\phi(\theta) - \psi(\theta)\|,$

(ii) for $0 \leq k < 1, \quad \lim_{n\to\infty}\|y_n - z_n\| = 0.$

Furthermore, using Theorem 3.1,3.2, respectively, we can directly obtain

Corollary 3.1. *Suppose an RK method (3.1) for ODEs is algebraically stable. Then, the corresponding NRK method (3.2), for the problems of the class $R_{\alpha,\beta}$ with $\alpha \leq 0$ and*

$\beta \le \alpha\|N\|^2$, *satisfies*

$$\|y_n - z_n\| \le \frac{2}{1-\|N\|} \max_{-\tau\le\theta\le 0} \|\phi(\theta) - \psi(\theta)\|.$$

Corollary 3.2. *Suppose an RK method (3.1) for ODEs is algebraically stable. Then, the corresponding NRK method (3.2), for the problems of the class $R_{\alpha,\beta}$ with*

$$N = 0, \ \ \alpha \le 0 \ \ and \ \ \beta \le 0,$$

satisfies

$$\|y_n - z_n\| \le \max_{-\tau\le\theta\le 0} \|\phi(\theta) - \psi(\theta)\|.$$

From the presented analysis, it is readily find that all the statements can be extended to the multidelay NDDEs of the form

$$\begin{cases} \frac{d}{dt}[y(t) - \sum_{i=1}^{p} N_i y(t-\tau_i)] = f(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_p)), \ \ t \ge 0, \\ y(t) = \phi(t), \quad -\max_{1\le i\le p}\{\tau_i\} \le t \le 0, \ \ \tau_i > 0 (i = 1, 2, \ldots, p), \end{cases}$$

with

$$\begin{aligned} & Re\langle (x_1 - x_2) - \sum_{i=1}^{p} N_i(y_i - z_i), f(t, x_1, y_1, y_2, \ldots, y_p) - f(t, x_2, z_1, z_2, \ldots, z_p)\rangle \\ \le \ \ & \alpha\|x_1 - x_2\|^2 + \sum_{i=1}^{p} \beta_i \|y_i - z_i\|^2, \quad t \ge 0, \ \ x_1, x_2, y_i, z_i \in \mathbf{C}^d (i = 1, 2, \ldots, p). \end{aligned}$$

References

*setckptjcm*1393*.bbl*

References

[1] V.K.Barwell, Special stability problem for functional differential equations, *BIT* , 15(1975), 130-135.

[2] T.A.Bickart, P-stable and P[α, β]-stable integration/interpolation methods in the solutions of retarded differential-deference equations, *BIT*, 22(1982), 464-476.

[3] M.Z.Liu and M.N.Spijker, The stability of the θ-methods in the numerical solutions of delay differential equations, *IMA J.Numer.Anal.*, 10(1990), 807-814.

[4] K.J.in't Hout, Stability analysis of Runge-Kutta methods for systems of delay differential equations, *IMA J.Numer.Anal.*, 17(1997), 17-27.

[5] L.Torelli, Stability of numerical methods for delay differential equations, *J.Comput. Appl. Math.*, 25(1989), 15-26.

[6] M.Zennaro, Asympototic stability analysis of Runge-Kutta methods for delay differential equations, *Numer.Math.*, 77(1997), 549-563.

[7] C.M.Huang, H.Y.Fu, S.F.Li and G.N.Chen, Stability analysis of Runge-Kutta methods for nonlinear delay differential equations, *BIT*, 39(1999), 270-280.

[8] J.X.Kang, J.X.Xiang and H.J.Tian, The asympototic stability of one-parameter methods for neutral differential equations, *BIT*, 34(1994), 400-408.

[9] T.Koto, A stability property of A-stable collocation-based Runge-Kutta methods neutral delay differential equations, *BIT*, 35(1996), 855-859.

[10] G.D.Hu and T.Mitsui, Stability analysis of numerical methods for systems of neutral delay-differential equations, *BIT*, 35(1995), 504-515.

[11] C.J.Zhang and S.Z.Zhou, The asympototic stability of the theoretical and numerical solutions for systems of neutral mutidelay-differential equations, *Science in China*, 41(1998), 1151-1157.

[12] C.J.Zhang and S.Z.Zhou, Stability analysis of LMMs for systems of neutral mutidelay-differential equations, *Comput. and Math. with Appl.*, 3/4(1999), 113-117.

[13] T.Koto, The stability of natural Runge-Kutta methods for nonlinear neutral delay differential equations, *J.Indust.Appl.Math.*, 14(1997), 111-123.

[14] M.Zennaro, Natural continuous extesions of Runge-Kutta methods, *Math. Comput.*, 46(1986), 119-133.

[15] K.Burrage and J.C.Butcher, Nonlinear stability of a general class of differential equtions methods, *BIT*, 20(1980), 185-203.

[16] J.K.Hale and S.M.V.Lunel, Introduction to functional differential equations, Springer-Verlag, Berlin, 1993.