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MULTISCALE ASYMPTOTIC EXPANSION FOR A CLASS OF HYPERBOLIC–PARABOLIC TYPE EQUATION WITH HIGHLY OSCILLATORY COEFFICIENTS*1)

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Abstract

In this paper, we will discuss the asymptotic behaviour for a class of hyperbolic -parabolic type equation with highly oscillatory coefficients arising from the strong-transient heat and mass transfer problems of composite media. A complete multiscale asymptotic expansion and its rigorous verification will be reported.

Key words: Multiscale asymptotic expansion, Hyperbolic-parabolic type equation, Highly oscillatory coefficient.

1. Introduction

Heat transfer theory has systematically been studied based on Fourier's law on theoretical analyses, numerical simulations and engineering applications. However, with rapid development of high-power laser technology, more and more interesting and surprising physical phenomena in strong-transient heat transfer process have been discovered(see, e.g. [13]). To interprete accurately these phenomena, non-Fourier's law was proposed (ref. [5]). It is interesting and important from both theoretical and practical points of view.

It is well known that , in the classical heat transfer theories, there exist some important tools, e.g. wave function expansion method, integral equation method and integral transform method and so on (see, e.g. [4,14] and references therein). Generally speaking, these analytic methods cannot be directly used to solve heat transfer problems of composite media, due to the complicated configurations and rapidly varying coefficients.

Clearly, being a fixed configuration, in principle it is always possible to choose the grid-size h small enough to reflect sufficiently local fluctuation of temperature functions.

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However, this is very often unpractical, since one would obtain linear systems having too many unknown (and actually it may be unfeasible in the higher dimensional case).

To overcome this principal difficulty, homogenization method has previously been analyzed for the above problems in some papers, see [2,6,9,12]. As we know , homogenization method based on average technique reflects only the global macroscopic properties of considering systems, and it is inadequate to describle the local changes of temperture functions. Thus, it is desirable to find out the macro- and meso-scale coupling formulations that can capture the effect of small scales as well as that of loading and constraints, which is an open problem presented by J.L.Lions in [9].

Our goal is to obtain the higher-order multiscale asymptotic expansion for a class of hyperbolic -parabolic type equation with rapidly oscillating coefficients.

The outline of this paper is organized in the following way. In section 2 , the physical model of strong-transient heat and mass transfer problems is introduced. In section 3, the corresponding mathematical equation and its solvablity are considered . In section 4 , we will obtain a multiscale asymptotic expansion for a class of hyperbolic-parabolic type equation with highly oscillatory coefficients in higher dimensional cases. ($n \geq 2$). Finally, we will complete the rigorous verification of the main theoretical result of this paper.

In the following, the Einstein summation convention is used: summation is taken over repeated indices. Throughout, C (with and without a subscript) denotes a generic positive constant, which is independent of ε unless otherwise stated.

2. Physical Model

It is well known that the energy equation of heat transfer problems is the following:

$$-(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}) + \dot{Q} = \rho c \frac{Du}{Dt} \tag{2.1}$$

where the heat flow density vector-valued function $\vec{q} = \{q_x, q_y, q_z\}$, and denotes by $\dot{Q}$ a thermal generation rate, ρ is a density function, c is a specific heat function, u is a temperature function.

If set $\quad \omega_x = \frac{dx}{dt}, \omega_y = \frac{dy}{dt}, \omega_z = \frac{dz}{dt}$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (\omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z}) \tag{2.2}$$

$\frac{Du}{Dt}$ is called the material derivative, which it consists of two terms: the first one $\frac{\partial u}{\partial t}$ reflects local change of temperature, and the second one $(\omega_x \frac{\partial u}{\partial x} + \omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z})$ describes convection heat transfer. In particular, if we consider only the simple heat transfer process, then $\frac{Du}{Dt}$ reduces

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} \tag{2.3}$$

In contrast to classical Fourier's law, the non-Fourier's law needs to be added the delayed time term, i.e.

$$\vec{q} = -\vec{\lambda} \cdot grad\, u - \tau_0 \frac{\partial q}{\partial t} \tag{2.4}$$

where $\vec{\lambda} = \{\lambda_x, \lambda_y, \lambda_z\}$ is the heat conductivity vector-valued function.

Remark 2.1. It should be pointed out that, generally speaking, the delayed time τ_0 is changing in strong-transient heat transfer process, and its value is very small, e.g. $\tau_0 \approx 10^{-9}s$, for helium, and $\tau_0 \approx 10^{-11}s$, for aluminum. In this paper, assume that τ_0 is a constant. If delayed time $\tau_0 \approx 0$, at this moment, the non-Fourier's law reduces classical Fourier's one.

It is not difficult to conclude that the energy equation for strong-transient heat transfer problems is as follows:

$$\rho c\tau_0 \frac{\partial^2 u}{\partial t^2} + \rho c\frac{\partial u}{\partial t} - \Big[\frac{\partial}{\partial x}(\lambda_x \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(\lambda_y \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(\lambda_z \frac{\partial u}{\partial z})\Big] - (\dot{Q} + \frac{\partial \dot{Q}}{\partial t}) = 0 \qquad (2.5)$$

and subject to the appropriate initial conditions and boundary conditions.

3. The Abstract Mathematical Problems

To begin with, let us introduce a small periodic parameter $0 < \varepsilon = \frac{l}{L} \ll 1$, where l and L stand for the sizes of periodic cell and macroscopic domain, respectively, denote by x and $\xi = \varepsilon^{-1}x$ the macro- and meso-scale ,respectively.

Let $\Omega \subset R^n$ be a bounded Lipschitz convex domain, and Ω_0 be union of periodic cells , i.e. $\Omega_0 = \bigcup_{z \in T_\varepsilon} \varepsilon(z+Q)$, $\quad T_\varepsilon = \{z = (z_1, z_2, \cdots z_n) \in Z^n : \varepsilon(z+Q) \subset \Omega\}, z_i \in Z$, and Z be integer set, and $Q = \{\xi : \quad 0 < \xi_j < 1, \quad j = 1, 2, \cdots n\}$, and $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, $\quad \Gamma^* = \partial\Omega_0 \cap \partial\Omega_1$, as shown in Figs.3.1,3.2 , $\frac{1}{2}\varepsilon \le dist(\partial\Omega_1, \partial\Omega) \le 2\varepsilon$.

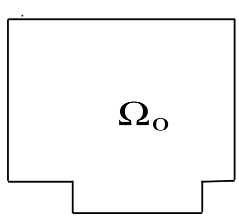

Figure 3.1. subdomain Ω_0

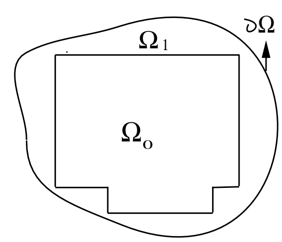

Figure 3.2. boundary layer Ω_1

Consider the following initial–boundary value problem with the Neumann type boundary:

$$\begin{cases} \tau_0 \rho(x, \frac{x}{\varepsilon}) c(x, \frac{x}{\varepsilon}) \dfrac{\partial^2 U^\varepsilon(x,t)}{\partial t^2} + \rho(x, \frac{x}{\varepsilon}) c(x, \frac{x}{\varepsilon}) \dfrac{\partial U^\varepsilon(x,t)}{\partial t} \\ \qquad - \dfrac{\partial}{\partial x_i}(k_{ij}(x, \frac{x}{\varepsilon}) \dfrac{\partial U^\varepsilon(x,t)}{\partial x_j}) = F(x,t), \qquad (x,t) \in \Omega \times (0,T) \\ \sigma_\varepsilon(U^\varepsilon) \equiv \sigma(w^\varepsilon) = \nu_i b_{ij} \dfrac{\partial w^\varepsilon}{\partial x_j} = \psi(x,t), \quad (x,t) \in \partial\Omega \times (0,T) \\ U^\varepsilon(x,t)|_{t=0} = \phi_1(x) \\ \partial_t U^\varepsilon(x,t)|_{t=0} = \phi_2(x) \end{cases} \qquad (3.1)$$

where $\nu = (\nu_1, \nu_2, \cdots \nu_n)$ is the unit outward normal to boundary $\partial\Omega$.

Assume that

$$\rho(x,\frac{x}{\varepsilon})=\begin{cases}\rho(\frac{x}{\varepsilon}) & x\in\overline{\Omega}_0\\ \rho_1 & x\in\Omega_1\end{cases};\qquad c(x,\frac{x}{\varepsilon})=\begin{cases}c(\frac{x}{\varepsilon}) & x\in\overline{\Omega}_0\\ c_1 & x\in\Omega_1\end{cases}\tag{3.2}$$

$$k_{ij}(x,\frac{x}{\varepsilon})=\begin{cases}a_{ij}(\frac{x}{\varepsilon}) & x\in\overline{\Omega}_0\\ b_{ij} & x\in\Omega_1\end{cases};\qquad F(x,t)=\begin{cases}0 & (x,t)\in\overline{\Omega}_0\times(0,T)\\ f(x,t) & (x,t)\in\Omega_1\times(0,T)\end{cases}\tag{3.3}$$

$$\phi_1(x)=\begin{cases}0, & x\in\overline{\Omega}_0\\ \varphi_1(x), & x\in\Omega_1\end{cases};\qquad \phi_2(x)=\begin{cases}0, & x\in\overline{\Omega}_0\\ \varphi_2(x), & x\in\Omega_1\end{cases}\tag{3.4}$$

$$U^\varepsilon(x,t)=\begin{cases}u^\varepsilon(x,t) & (x,t)\in\Omega_0\times(0,T)\\ w^\varepsilon(x,t) & (x,t)\in\overline{\Omega}_1\times(0,T)\end{cases}\tag{3.5}$$

where $\rho_1,\ c_1,$ and $b_{ij},\ i,j=1,\cdots,n$ are constants, and $f(x,t),\ \psi(x,t),\ \varphi_i(x), i=1,2$ are some given functions.

Now let us introduce some Sobolev spaces: $H^m(\Omega)$ is the completion of $C^m(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H^m(\Omega)}=\Big(\sum_{|\alpha|\le m}\|D^\alpha u\|^2_{L^2(\Omega)}\Big)^{1/2}\tag{3.6}$$

and $H_0^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm (3.6)

For $1\le p\le+\infty$, define

$$L^p(0,T;H^m(\Omega))=\Big\{v(\cdot,t):[0,T]\to H^m(\Omega),\quad \Big(\int_0^T\|v(\cdot,t)\|^p_{H^m(\Omega)}dt\Big)^{1/p}<+\infty\Big\}\tag{3.7}$$

$\forall(x,t)\in\Omega\times[0,T]$, in particular $p=2$.

Suppose that

$(\mathbf{A_1})$. Let $\xi=\varepsilon^{-1}x,\ \ \rho(\xi),c(\xi),a_{ij}(\xi)$ be 1-periodic functions in ξ ;

$(\mathbf{A_2})$. $(k_{ij}(x,\frac{x}{\varepsilon}))$ is a symmetric matrix and satisfies the uniform elliptic condition, i.e.

$$\lambda|\eta|^2\le k_{ij}(x,\frac{x}{\varepsilon})\eta_i\eta_j\le\mu|\eta|^2,\quad |\eta|^2=(\eta_1^2+\cdots+\eta_n^2)\tag{3.8}$$

$a.e.\quad x\in\Omega\quad 0<\lambda\le\mu,\quad \forall(\eta_1,\cdots\eta_n)\in R^n$

$(\mathbf{A_3}$). $\rho(x,\frac{x}{\varepsilon}),\ c(x,\frac{x}{\varepsilon}),\quad k_{ij}(x,\frac{x}{\varepsilon})\in L^\infty(\Omega),\quad F,\quad \psi\in L^2(\Omega\times(0,T)),$ $\varphi_i(x)\in L^2(\Omega),\quad i=1,2$;

$(\mathbf{A_4})$. $0<\rho_0\le\rho(x,\frac{x}{\varepsilon})\le\rho_m\quad 0<c_0\le c(x,\frac{x}{\varepsilon})\le c_m,\qquad \forall x\in\Omega$

For convenience, consider the initial–boundary value problem with the homogeneous Neumann type boundary as follows

$$\begin{cases}\tau_0\rho(x,\frac{x}{\varepsilon})c(x,\frac{x}{\varepsilon})\dfrac{\partial^2U^\varepsilon(x,t)}{\partial t^2}+\rho(x,\frac{x}{\varepsilon})c(x,\frac{x}{\varepsilon})\dfrac{\partial U^\varepsilon(x,t)}{\partial t}\\ \qquad -\dfrac{\partial}{\partial x_i}(k_{ij}(x,\frac{x}{\varepsilon})\dfrac{\partial U^\varepsilon(x,t)}{\partial x_j})=F(x,t),\qquad (x,t)\in\Omega\times(0,T)\\ \sigma_\varepsilon(U^\varepsilon)\equiv\sigma(w^\varepsilon)=\nu_i b_{ij}\dfrac{\partial w^\varepsilon}{\partial x_j}=0,\quad (x,t)\in\partial\Omega\times(0,T)\\ U^\varepsilon(x,t)|_{t=0}=\phi_1(x)\\ \partial_tU^\varepsilon(x,t)|_{t=0}=\phi_2(x)\end{cases}\tag{3.9}$$

Theorem 3.1. If the above conditions $(A_2)-(A_4)$ are satisfied, then there exists the following inequality, for $0 \le \tau \le T$

$$\begin{aligned}&\int_\Omega \{(U^\varepsilon(x,t))^2 + (\partial_t U^\varepsilon(x,t))^2 + |\nabla U^\varepsilon(x,t)|^2\}|_{t=\tau} dx \\ &\le C(T)\{\|F(x,t)\|^2_{L^2(\Omega\times(0,T))} + \|\phi_1(x)\|^2_{1,\Omega} + \|\phi_2(x)\|^2_{0,\Omega}\}\end{aligned} \tag{3.10}$$

where $C(T)$ is a constant independent of ε, but does depend on T.

Proof. Multiplying by $\partial_t U^\varepsilon$ in two sides of (3.9), and integrating by parts in $\Omega \times (0,\tau)$, $\quad 0 \le \tau \le T$, $\quad$ one can obtain

$$\begin{aligned}&\tau_0 \int_0^\tau \int_\Omega \rho c \partial_t^2 U^\varepsilon \partial_t U^\varepsilon dxdt + \int_0^\tau \int_\Omega \rho c (\partial_t U^\varepsilon)^2 dxdt \\ &+ \int_0^\tau \int_\Omega \frac{\partial}{\partial x_i}(k_{ij}\frac{\partial U^\varepsilon}{\partial x_j})\partial_t U^\varepsilon dxdt = \int_0^\tau \int_\Omega F(x,t)\partial_t U^\varepsilon dxdt\end{aligned} \tag{3.11}$$

Since $\sigma_\varepsilon(U^\varepsilon) \equiv \nu_i k_{ij}\dfrac{\partial U^\varepsilon}{\partial x_j} = 0, \quad (x,t) \in \partial\Omega \times (0,T)$, hence

$$\begin{aligned}&\tfrac{1}{2}\tau_0 \int_\Omega \rho c(\partial_t U^\varepsilon)^2|_{t=\tau} dx + \tfrac{1}{2}\int_\Omega (k_{ij}\frac{\partial U^\varepsilon}{\partial x_j}\frac{\partial U^\varepsilon}{\partial x_i})|_{t=\tau} dx + \int_0^\tau \int_\Omega \rho c(\partial_t U^\varepsilon)^2 dxdt \\ &= \int_0^\tau \int_\Omega F\partial_t U^\varepsilon dxdt + \tfrac{1}{2}\tau_0 \int_\Omega \rho c(\partial_t U^\varepsilon)|_{t=0} dx + \tfrac{1}{2}\int_\Omega (k_{ij}\frac{\partial U^\varepsilon}{\partial x_j}\frac{\partial U^\varepsilon}{\partial x_i})|_{t=0} dx\end{aligned} \tag{3.12}$$

Given $\int_0^\tau \int_\Omega \rho c(\partial_t U^\varepsilon)^2 dxdt \ge 0$, it implies that

$$\begin{aligned}&\tfrac{1}{2}\rho_0 c_0 \tau_0 \int_\Omega (\partial_t U^\varepsilon)^2|_{t=\tau} dx + \tfrac{1}{2}\lambda \int_\Omega |\nabla U^\varepsilon|^2_{t=\tau} dx \\ &\le \gamma \int_0^\tau \int_\Omega |\partial_t U^\varepsilon|^2 dtdx + \frac{1}{\gamma}\int_0^\tau \int_\Omega |F|^2 dtdx \\ &+ \tfrac{1}{2}\rho_m c_m \tau_0 \int_\Omega |\phi_2|^2 dx + \tfrac{1}{2}\mu \int_\Omega |\nabla \phi_1|^2 dx\end{aligned} \tag{3.13}$$

If choose $\lambda_0 = min(\frac{1}{2}\lambda, \frac{1}{2}\rho_0 c_0 \tau_0) > 0$, and set

$$E(\tau) = \int_0^\tau \int_\Omega (\partial_t U^\varepsilon)^2 dx + \int_0^\tau \int_\Omega |\nabla U^\varepsilon|^2 dtdx \tag{3.14}$$

then we obtain

$$\begin{cases} \dfrac{dE(\tau)}{d\tau} \le \{\dfrac{1}{\lambda_0}\gamma E(\tau) + F_\gamma(\tau)\} \\ E(0) = 0 \end{cases} \tag{3.15}$$

where

$$F_\gamma(\tau) = \frac{1}{\gamma}\int_0^\tau \int_\Omega |F|^2 dtdx + \frac{1}{2}\rho_m c_m \tau_0 \int_\Omega |\phi_2|^2 dx + \frac{1}{2}\mu \int_\Omega |\nabla \phi_1|^2 dx \tag{3.16}$$

It follows from Gronwall's inequality that

$$E(\tau) \le \int_0^\tau e^{\gamma(\tau-t)} F_\gamma(t) dt, \quad \frac{dE(\tau)}{d\tau} \le [(1+\frac{1}{\gamma})(e^{\gamma\tau}-1)]F_\gamma(\tau)$$

If choose $\gamma = \frac{1}{T}$, then

$$\int_\Omega ((\partial_t U^\varepsilon)^2 + |\nabla U^\varepsilon|^2)_{t=\tau} dx \le C(T)\{\|F\|^2_{L^2(\Omega\times(0,T))} + \|\phi_2\|^2_{0,\Omega} + \|\phi_1\|^2_{1,\Omega}\} \tag{3.17}$$

On the other hand

$$\frac{d}{d\tau}\int_\Omega (U^\varepsilon(x,\tau))^2 dx = 2\int_\Omega U^\varepsilon \cdot \frac{\partial U^\varepsilon}{\partial \tau} dx \le \gamma \int_\Omega (U^\varepsilon)^2 dx + \frac{1}{\gamma}\int_\Omega (\frac{\partial U^\varepsilon}{\partial \tau})^2 dx \tag{3.18}$$

Set

$$\Omega(\tau) = \int_\Omega (U^\varepsilon(x,\tau))^2 dx, \quad \begin{cases} \dfrac{d\Omega(\tau)}{d\tau} \le \gamma\Omega(\tau) + G_\gamma(\tau) \\ \Omega(0) = \int\limits_\Omega (U^\varepsilon(x,0))^2 dx \end{cases} \tag{3.19}$$

where $G_\gamma(\tau) = \frac{1}{\gamma}\int\limits_\Omega (\frac{\partial U^\varepsilon(x,\tau)}{\partial \tau})^2 dx$

Using Gronwall's inequality again, we have

$$\Omega(\tau) \le \Omega(0)e^{\gamma\tau} + \int_0^\tau e^{\gamma(\tau-t)} G_\gamma(t) dt \le \Omega(0)e^{\gamma\tau} + \frac{1}{\gamma}e^{\gamma\tau}\int_0^\tau\int_\Omega (\partial_t U^\varepsilon)^2 dx dt \tag{3.20}$$

Choosing $\gamma = \frac{1}{T} > 0$, one can get

$$\begin{aligned} &\int\limits_\Omega (U^\varepsilon(x,\tau))^2 dx \le 3\int\limits_\Omega (U^\varepsilon(x,0))^2 dx + 3\tau \int_0^\tau\int\limits_\Omega (\partial_t U^\varepsilon)^2 dx dt \\ &\le 3\int\limits_\Omega |\phi_1|^2 dx + 3T\int_0^T\int\limits_\Omega (\partial_t U^\varepsilon)^2 dx dt \quad (0 \le \tau \le T) \\ &\le C(T)\{\|F\|^2_{L^2(\Omega\times(0,\tau))} + \|\phi_2\|^2_{0,\Omega} + \|\phi_1\|^2_{1,\Omega}\} \end{aligned} \tag{3.21}$$

Combining (3.17) with (3.21), we can complete the proof of (3.10).

We can prove the following theorem without any difficulty:

Theorem 3.2. Under the assumptions of Theorem 3.1, problem (3.9) has one and only one weak solution $U^\varepsilon(x,t) \in L^2(0,T;H^1(\Omega))$.

4. Multiscale Asymptotic Expansion

For $(x,t) \in \Omega_0 \times (0,T)$, formally set

$$u^\varepsilon(x,t) \cong \sum_{l=0}^{+\infty} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u^0(x,t) \tag{4.1}$$

In contrast to usual expression, here we use the following notation for convenience

$$D^\alpha v = \frac{\partial^l v}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \quad \alpha = \{\alpha_1, \cdots \alpha_l\}, \quad \langle\alpha\rangle = l \tag{4.2}$$

$\alpha_i = 1, 2, \cdots n.$

Substituting (4.1) into (3.9), and taking into account that $\frac{\partial}{\partial x_i} \to \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi_i}$, we obtain the formal equality

$$\begin{aligned} 0 = \sum_{l=0}^{+\infty} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} \rho(\xi)c(\xi)N_{\alpha_1\cdots\alpha_l}(\xi)[\tau_0 D^\alpha \frac{\partial^2 u^0(x,t)}{\partial t^2} \\ +D^\alpha \frac{\partial u^0(x,t)}{\partial t}] - \sum_{l=0}^{+\infty} \varepsilon^{l-2} \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} H_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u^0(x,t) \end{aligned} \tag{4.3}$$

where

$$H_0(\xi) = \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial N_0(\xi)}{\partial \xi_j}) \tag{4.4}$$

$$H_{\alpha_1}(\xi) = \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}) + \frac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi)N_0(\xi)) + a_{\alpha_1 j}(\xi)\frac{\partial N_0(\xi)}{\partial \xi_j} \tag{4.5}$$

For $\langle\alpha\rangle = l \geq 2$

$$\begin{aligned} H_{\alpha_1\cdots\alpha_l}(\xi) &= \frac{\partial}{\partial \xi_i}(a_{ij}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial \xi_j}) + \frac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi)) \\ &+ a_{\alpha_1 j}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_l}(\xi)}{\partial \xi_j} + a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_l}(\xi) \end{aligned} \tag{4.6}$$

If assume that

$$\begin{cases} H_0(\xi) = 0 \\ H_{\alpha_1}(\xi) = 0 \\ H_{\alpha_1\alpha_2}(\xi) = \hat{a}_{\alpha_1\alpha_2} \\ H_{\alpha_1\alpha_2\alpha_3}(\xi) = N_{\alpha_1}(\xi)\hat{a}_{\alpha_2\alpha_3} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ H_{\alpha_1\cdots\alpha_l}(\xi) = N_{\alpha_1\cdots\alpha_{l-2}}(\xi)\hat{a}_{\alpha_{l-1}\alpha_l}, \quad l \geq 2 \end{cases} \tag{4.7}$$

where $\hat{a}_{\alpha_1\alpha_2}$, $N_{\alpha_1\cdots\alpha_j}(\xi)$, $j = 0,1,\cdots,$ $\alpha_j = 1,2,\cdots n$, will be defined below, then holds

$$\begin{aligned} -\sum_{l=2}^{+\infty} \varepsilon^{l-2} \sum_{\alpha_1,\cdots,\alpha_{l-2}=1}^{n} N_{\alpha_1\cdots\alpha_{l-2}}(\xi)\frac{\partial^{l-2}}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_{l-2}}}\{\rho(\xi)c(\xi)\tau_0\frac{\partial^2 u^0(x,t)}{\partial t^2} \\ +\rho(\xi)c(\xi)\frac{\partial u^0(x,t)}{\partial t}) - \sum_{\alpha_{l-1},\alpha_l=1}^{n} \hat{a}_{\alpha_{l-1}\alpha_l}\frac{\partial^2 u^0(x,t)}{\partial x_{\alpha_{l-1}}\partial x_{\alpha_l}}\} = 0 \end{aligned} \tag{4.8}$$

Therefore, if let

$$\tau_0\rho(\xi)c(\xi)\frac{\partial^2 u^0(x,t)}{\partial t^2} + \rho(\xi)c(\xi)\frac{\partial u^0(x,t)}{\partial t} - \frac{\partial}{\partial x_{\alpha_{l-1}}}(\hat{a}_{\alpha_{l-1}\alpha_l}\frac{\partial u^0(x,t)}{\partial x_{\alpha_l}}) = 0 \tag{4.9}$$

then we know that (4.8) is an identity equation.

Using the separation of variables with respect to x, ξ, we can easily verify that (4.9) is equivalent to the following equation

$$\tau_0\langle\rho c\rangle\frac{\partial^2 u^0(x,t)}{\partial t^2} + \langle\rho c\rangle\frac{\partial u^0(x,t)}{\partial t} - \frac{\partial}{\partial x_{\alpha_{l-1}}}(\hat{a}_{\alpha_{l-1}\alpha_l}\frac{\partial u^0(x,t)}{\partial x_{\alpha_l}}) = 0 \tag{4.10}$$

where $\langle\rho c\rangle = \frac{1}{|Q|}\int\limits_Q \rho(\xi)c(\xi)d\xi$

From (4.7), we define the periodic functions $N_0(\xi)$, $N_{\alpha_1\cdots\alpha_j}(\xi)$, $j \geq 1$, $\alpha_j = 1, 2, \cdots n$, in the following ways:

$$N_0(\xi) = 1, \qquad \xi \in Q \tag{4.11}$$

$$\begin{cases} \frac{\partial}{\partial\xi_i}(a_{ij}(\xi)\frac{\partial N_{\alpha_1}(\xi)}{\partial\xi_j}) = -\frac{\partial}{\partial\xi_i}(a_{i\alpha_1}(\xi)) \qquad \text{in } Q \\ N_{\alpha_1}(\xi) = 0 \qquad \text{on } \partial Q \end{cases} \tag{4.12}$$

$$\begin{cases} \frac{\partial}{\partial\xi_i}(a_{ij}(\xi)\frac{\partial N_{\alpha_1\alpha_2}}{\partial\xi_j}) = -\frac{\partial}{\partial\xi_i}(a_{i\alpha_1}(\xi)N_{\alpha_2}(\xi)) \\ \qquad -a_{\alpha_1 j}(\xi)\frac{\partial N_{\alpha_2}(\xi)}{\partial\xi_j} - a_{\alpha_1\alpha_2}(\xi) + \hat{a}_{\alpha_1\alpha_2} \qquad \text{in} \quad Q \\ N_{\alpha_1\alpha_2}(\xi) = 0 \qquad \text{on } \partial Q \end{cases} \tag{4.13}$$

where $\hat{a}_{\alpha_1\alpha_2} = \int_Q (a_{\alpha_1\alpha_2}(\xi) + a_{\alpha_1 k}(\xi)\frac{\partial N_{\alpha_2}(\xi)}{\partial\xi_k})d\xi$

In general, for $\langle\alpha\rangle = l \geq 3$

$$\begin{cases} \frac{\partial}{\partial\xi_i}(a_{ij}(\xi)\frac{\partial N_{\alpha_1\cdots\alpha_l}(\xi)}{\partial\xi_j}) = -\frac{\partial}{\partial\xi_i}(a_{i\alpha_1}(\xi)N_{\alpha_2\cdots\alpha_l}(\xi)) - a_{\alpha_1 j}(\xi)\frac{\partial N_{\alpha_2\cdots\alpha_l}(\xi)}{\partial\xi_j} \\ \qquad -a_{\alpha_1\alpha_2}(\xi)N_{\alpha_3\cdots\alpha_l}(\xi) + N_{\alpha_1\cdots\alpha_{l-2}}(\xi)\hat{a}_{\alpha_{l-1}\alpha_l} \qquad \text{in} \quad Q \\ N_{\alpha_1\cdots\alpha_l}(\xi) = 0 \qquad \text{on} \quad \partial Q \end{cases} \tag{4.14}$$

Remark 4.1. Existence and uniqueness of the solutions $N_{\alpha_1}(\xi), \cdots, N_{\alpha_1\cdots\alpha_l}(\xi)$ for problems (4.12)-(4.14), respectively ,can be easily established by induction with respect to l due to the uniform elliptic condition (A_2) - (A_4), Poincaré-Friedrichs' inequality and Lax-Milgram's lemma. Then they are extended to the whole R^n by the 1-periodicity.

Remark 4.2. It is worthwhile to notice that the periodic functions $N_{\alpha_1}(\xi)$, $\alpha_1 = 1, \cdots n$, defined in this paper, generally speaking, are different from $\tilde{N}_{\alpha_1}(\xi)$ defined in classical homogenization books, (ref. [2,6,9,12]), due to the different boundary conditions on ∂Q. But we can prove that their homogenized matrices are the same in some cases, see Appendix A.

The homogenized initial-boundary value problem corresponding to the original problem (3.1) is the following:

$$\begin{cases} \tau_0\overline{\rho c}\frac{\partial^2\widehat{U}(x,t)}{\partial t^2} + \overline{\rho c}\frac{\partial\widehat{U}(x,t)}{\partial t} - \frac{\partial}{\partial x_i}(\hat{k}_{ij}\frac{\partial\widehat{U}(x,t)}{\partial x_j}) \\ \qquad = F(x,t), \quad (x,t) \in \Omega \times (0,T) \\ \hat{\sigma}(\widehat{U}) = \sigma(\hat{w}) \equiv \nu_i b_{ij}\frac{\partial\hat{w}}{\partial x_j} = 0, \quad (x,t) \in \partial\Omega \times (0,T) \\ \widehat{U}(x,t)|_{t=0} = \phi_1(x) \\ \partial_t\widehat{U}(x,t)|_{t=0} = \phi_2(x) \end{cases} \tag{4.15}$$

$$\hat{k}_{ij} = \begin{cases} \hat{a}_{ij} & x \in \overline{\Omega}_0 \\ b_{ij} & x \in \Omega_1 \end{cases}; \qquad \overline{\rho c} = \begin{cases} \langle\rho c\rangle, & x \in \Omega_0 \\ \rho_1 c_1, & x \in \Omega_1 \end{cases} \tag{4.16}$$

$$\widehat{U}(x,t) = \begin{cases} u^0(x,t) & (x,t) \in \overline{\Omega}_0 \times (0,T) \\ \hat{w}(x,t) & (x,t) \in \Omega_1 \times (0,T) \end{cases} \tag{4.17}$$

Remark 4.3. One can check that $(\hat{k}_{ij})$ is a positive definite matrix, see[2,6,9,12]. Therefore, problem (4.15) has one and only one weak solution.

5. Truncation Error Estimates

For any integer $s \geq 2$, set

$$U_s^\varepsilon(x,t) = \begin{cases} u_s^\varepsilon(x,t) = u^0(x,t) + \sum\limits_{l=1}^{s} \varepsilon^l \sum\limits_{\alpha_1,\cdots,\alpha_l=1}^{n} N_{\alpha_1\cdots\alpha_l}(\xi) D^\alpha u^0(x,t), \\ \qquad (x,t) \in \overline{\Omega}_0 \times (0,T) \\ \hat{w}(x,t), \qquad (x,t) \in \Omega_1 \times (0,T) \end{cases} \tag{5.1}$$

Since $u_s^\varepsilon(x,t)|_{\Gamma^*} = \hat{w}(x,t)|_{\Gamma^*}$, then $U_s^\varepsilon(x,t) \in L^2(0,T;H^1(\Omega)) \bigcap C^0(0,T;L^2(\Omega))$. But, generally speaking, for any fixed $t \in [0,T]$, $[\dfrac{\partial U_s^\varepsilon(x,t)}{\partial n}]|_{\partial\Omega_0 \cap \partial\Omega_1} \neq 0$. To this end, we have to treat it with some regularization operator.

At first, let us introduce a set of open covering $\{\mathcal{V}_l\}_{l=1}^3$ of the bounded closed set $\overline{\Omega} \subset R^n$

$$\begin{array}{l} \mathcal{V}_1 = \{x \in \Omega_0 : \quad dist(x, \partial\Omega_0) > \frac{\delta}{2}\} \\ \mathcal{V}_2 = \{x \in (R^n \setminus \overline{\Omega}_0) : dist(x, \partial\Omega_0) > \frac{\delta}{2}, dist(x, \partial\Omega) < \delta\} \\ \mathcal{V}_3 = \{x \in \Omega : \quad dist(x, \partial\Omega_0) < \delta\} \end{array} \tag{5.2}$$

$\overline{\Omega} \subset \cup_{l=1}^3 \mathcal{V}_l$

Using the partitioning of unity theorem, there exist a set of functions $\{\psi_l(x)\}_{l=1}^3$ such that

(1) $\psi_l(x) \in C_0^\infty(\mathcal{V}_l)$; (2) $\sum\limits_{l=1}^{3} \psi_l(x) \equiv 1, \forall x \in \Omega$

Set $\Omega_0'' = \Omega_0 \setminus \overline{\mathcal{V}}_3$, $\Omega_1'' = \Omega_1 \setminus \overline{\mathcal{V}}_3$, and choose a sufficiently small $\delta > 0$ such that $\delta \leq C \cdot \varepsilon^s, \quad s \geq 2$

Define, for any fixed $t \in [0,T]$

$$\tilde{U}_s^\varepsilon(x,t) = \psi_1(x) \cdot U_s^\varepsilon(x,t) + \psi_2(x) \cdot U_s^\varepsilon(x,t) + J_\delta * (\psi_3(x) \cdot U_s^\varepsilon(x,t)) \tag{5.3}$$

where the regularization $J_\delta * u$ is defined in Section 2.17 of [1].

One can directly verify that $\tilde{U}_s^\varepsilon(x,t) \in L^2(0,T;H^1(\Omega)) \bigcap C^0(0,T;L^2(\Omega))$, and $[\dfrac{\partial \tilde{U}_s^\varepsilon(x,t)}{\partial n}]|_{\partial\Omega_0 \cap \partial\Omega_1} = 0$, for any fixed $t \in [0,T]$.

Theorem 5.1. Let $U^\varepsilon(x,t)$ be the weak solution of problem (3.9), and $U_s^\varepsilon(x,t)$, $\tilde{U}_s^\varepsilon(x,t)$ be the approximate solutions defined by (5.1) and (5.3), respectively . Suppose that $a_{ij}(\frac{x}{\varepsilon}) \in C(\overline{\Omega})$, $\nabla_\xi a_{ij}(\xi) \in L^\infty(\Omega)$, and conditions $(A_2) - (A_4)$ are satisfied. If $u^0 \in L^2(0,T;H^{s+2}(\Omega_0))$, $\partial_t u^0 \in L^2(0,T,H^{s+2}(\Omega_0))$, $0 \leq \tau \leq T$, $\Omega_0 \subset\subset \Omega \subset R^2$, then it holds

$$\int_\Omega \{(U^\varepsilon - U_s^\varepsilon)^2 + [\partial_t(U^\varepsilon - U_s^\varepsilon)]^2 + |\nabla(U^\varepsilon - U_s^\varepsilon)|^2\}|_{t=\tau} dx \leq C(T)\varepsilon^{2(s-1)}, \quad s \geq 2 \tag{5.4}$$

where $C(T)$ is a constant independent of ε, but does depend on T.

Proof. Let

$$\mathcal{L}_\varepsilon = \tau_0 \rho(x, \frac{x}{\varepsilon}) c(x, \frac{x}{\varepsilon}) \frac{\partial^2}{\partial t^2} + \rho(x, \frac{x}{\varepsilon}) c(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i}(k_{ij}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x_j}) \tag{5.5a}$$

$$\widehat{\mathcal{L}} = \tau_0 \overline{\rho c} \frac{\partial^2}{\partial t^2} + \overline{\rho c} \frac{\partial}{\partial t} - \frac{\partial}{\partial x_i}(\hat{k}_{ij} \frac{\partial}{\partial x_j}) \tag{5.5b}$$

If $(x,t) \in \overline{\Omega}_0 \times (0,T)$

$$\begin{aligned}
&\mathcal{L}_\varepsilon(U^\varepsilon(x,t) - U_s^\varepsilon(x,t)) \equiv \mathcal{L}_\varepsilon(u^\varepsilon(x,t) - u_s^\varepsilon(x,t)) \\
&= \varepsilon^{s-1} \sum_{\alpha_1,\cdots,\alpha_{s+1}=1}^{n} a_{\alpha_1 j} \frac{\partial N_{\alpha_2\cdots\alpha_{s+1}}}{\partial \xi_j} D^\alpha u^0(x,t) \\
&+\varepsilon^{s-1} \sum_{\alpha_1,\cdots,\alpha_{s+1}=1}^{n} \frac{\partial}{\partial \xi_i}(a_{i\alpha_1} N_{\alpha_2\cdots\alpha_{s+1}}) D^\alpha u^0(x,t) \\
&+\varepsilon^{s-1} \sum_{\alpha_1,\cdots,\alpha_{s+1}=1}^{n} a_{\alpha_1\alpha_2} N_{\alpha_3\cdots\alpha_{s+1}} D^\alpha u^0(x,t) \\
&+\varepsilon^{s} \sum_{\alpha_1,\cdots,\alpha_{s+2}=1}^{n} a_{\alpha_1\alpha_2} N_{\alpha_3\cdots\alpha_{s+2}} D^\alpha u^0(x,t) = \varepsilon^{s-1} F_0(\varepsilon,x,t)
\end{aligned} \tag{5.6}$$

If $(x,t) \in \Omega_1 \times (0,T)$

$$\mathcal{L}_\varepsilon(U^\varepsilon(x,t) - U_s^\varepsilon(x,t)) \equiv \mathcal{L}_\varepsilon w^\varepsilon - \widehat{\mathcal{L}}\hat{w} = f(x,t) - f(x,t) = 0 \tag{5.7}$$

From (5.3), we can write the unified equation as follows :

$$\begin{cases}
\mathcal{L}_\varepsilon(U^\varepsilon(x,t) - \tilde{U}_s^\varepsilon(x,t)) = \tilde{F}_0(x,t),\ (x,t) \in \Omega \times (0,T) \\
\sigma_\varepsilon(U^\varepsilon - \tilde{U}_s^\varepsilon) = 0,\ (x,t) \in \partial\Omega \times (0,T) \\
(U^\varepsilon(x,t) - \tilde{U}_s^\varepsilon(x,t))|_{t=0} = 0 \\
\partial_t(U^\varepsilon(x,t) - \tilde{U}_s^\varepsilon(x,t))|_{t=0} = 0
\end{cases} \tag{5.8}$$

If $a_{ij}(\frac{x}{\varepsilon}) \in C(\overline{\Omega})$, $\nabla_\xi a_{ij}$, ρ, $c \in L^\infty(R^n)$, $f, \partial_t f \in L^2(0,T;L^2(\Omega))$, then one can verify that(see, e.g. [7,8])

$$u_s^\varepsilon(x,t) \in L^2(0,T;H^2(\Omega_0)) \bigcap C^0(0,T;L^2(\Omega_0))$$

$$\hat{w}(x) \in L^2(0,T;W^{2,p}(\Omega_1)) \bigcap C^0(0,T;L^2(\Omega_1))$$

$$\partial_t(u_s^\varepsilon(x,t)),\ \partial_{tt}(u_s^\varepsilon(x,t)) \in L^2(0,T;L^2(\Omega_0))$$

$$\partial_t(\hat{w}(x)), \partial_{tt}(\hat{w}(x)) \in L^2(0,T;L^p(\Omega_1)), \qquad p \le p_0 < 2$$

where $\Omega_0, \Omega_1, \Omega \subset R^2$ as shown in Figs. 3.1, 3.2.

Using the similar method of [3,11], one can prove that

$$\int_0^\tau \int_\Omega |\tilde{F}_0(x,t)|^2 dxdt \le C\varepsilon^{2(s-1)} \tag{5.9}$$

Let $W_\varepsilon(x,t) = U^\varepsilon(x,t) - \tilde{U}_s^\varepsilon(x,t)$. Multiplying by $\partial_t W_\varepsilon$ on both sides of (5.8), and integrating by parts in $\Omega \times (0,\tau)$, one can obtain

$$\begin{aligned}
&\tfrac{1}{2}\tau_0 \int_0^\tau \int_\Omega \rho(x,\tfrac{x}{\varepsilon}) c(x,\tfrac{x}{\varepsilon}) \frac{\partial}{\partial t}(\partial_t W_\varepsilon)^2 dxdt + \int_0^\tau \int_\Omega \rho(x,\tfrac{x}{\varepsilon}) c(x,\tfrac{x}{\varepsilon}) (\partial_t W_\varepsilon)^2 dxdt \\
&\quad + \int_0^\tau \int_\Omega k_{ij} \frac{\partial W_\varepsilon}{\partial x_j} \frac{\partial}{\partial x_i}(\partial_t W_\varepsilon) dxdt = \int_0^\tau \int_\Omega \tilde{F}_0(\varepsilon,x,t) \partial_t W_\varepsilon dxdt
\end{aligned} \tag{5.10}$$

$$\begin{aligned}&\tfrac{1}{2}\tau_0 \int_0^\tau (\partial_t \int_\Omega \rho(x,\tfrac{x}{\varepsilon})c(x,\tfrac{x}{\varepsilon})(\partial_t W_\varepsilon)^2 dx)dt + \int_0^\tau \int_\Omega \rho(x,\tfrac{x}{\varepsilon})c(x,\tfrac{x}{\varepsilon})(\partial_t W_\varepsilon)^2 dxdt \\ &+\tfrac{1}{2}\int_0^\tau \partial_t(\int_\Omega k_{ij}\frac{\partial W_\varepsilon}{\partial x_j}\frac{\partial W_\varepsilon}{\partial x_i}dx)dt = \int_0^\tau\int_\Omega \tilde{F}_0(\varepsilon,x,t)\partial_t W_\varepsilon dxdt\end{aligned} \tag{5.11}$$

$\rho c(\partial_t W_\varepsilon)^2 \geq 0$ implies that

$$\begin{aligned}&\tfrac{1}{2}\tau_0 \int_\Omega \rho(x,\tfrac{x}{\varepsilon})c(x,\tfrac{x}{\varepsilon})(\partial_t W_\varepsilon)^2|_{t=\tau}dx + \tfrac{1}{2}\int_\Omega \Big(k_{ij}\frac{\partial W_\varepsilon}{\partial x_j}\frac{\partial W_\varepsilon}{\partial x_i}\Big)|_{t=\tau}dx \\ &\leq \Big(\int_0^\tau\int_\Omega |\tilde{F}_0|^2 dxdt\Big)^{1/2}\Big(\int_0^\tau\int_\Omega |\partial_t W_\varepsilon|^2 dxdt\Big)^{1/2} \\ &+\tfrac{1}{2}\tau_0 \int_\Omega \rho(x,\tfrac{x}{\varepsilon})c(x,\tfrac{x}{\varepsilon})(\partial_t W_\varepsilon)^2|_{t=0}dx + \tfrac{1}{2}\int_\Omega \Big(k_{ij}\frac{\partial W_\varepsilon}{\partial x_j}\frac{\partial W_\varepsilon}{\partial x_i}\Big)|_{t=0}dx \\ &\leq \Big(\int_0^\tau\int_\Omega |\tilde{F}_0|^2 dxdt\Big)^{1/2}\Big(\int_0^\tau\int_\Omega |\partial_t W_\varepsilon|^2 dxdt\Big)^{1/2}\end{aligned} \tag{5.12}$$

Choosing a sufficiently small constant $\gamma > 0$ such that $\gamma < \frac{\lambda}{4}$, we have

$$\begin{aligned}&\tfrac{1}{2}\rho_0 c_0 \tau_0 \int_\Omega (\partial_t W_\varepsilon)^2|_{t=\tau}dx + \tfrac{1}{4}\lambda\int_\Omega |\nabla W_\varepsilon|^2_{t=\tau}dx \\ &\leq C\Big\{\gamma\int_0^\tau\int_\Omega |\partial_t W_\varepsilon|^2 dxdt + \frac{1}{\gamma}\int_0^\tau\int_\Omega |\tilde{F}_0|^2 dxdt\}\end{aligned} \tag{5.13}$$

Choose $\lambda_0 = min(\frac{1}{4}\lambda, \frac{1}{2}\rho_0 c_0 \tau_0) > 0$, and set

$$\Theta(\tau) = \int_0^\tau \int_\Omega (\partial_t W_\varepsilon)^2 dxdt + \int_0^\tau\int_\Omega |\nabla W_\varepsilon|^2 dxdt \tag{5.14}$$

$$\begin{cases} \dfrac{d\Theta(\tau)}{d\tau} \leq \dfrac{1}{\lambda_0}[\gamma\Theta(\tau) + \Lambda_\gamma(\tau)] \\ \Theta(0) = 0 \end{cases} \tag{5.15}$$

where

$$\Lambda_\gamma(\tau) = \frac{1}{\gamma}\int_0^\tau\int_\Omega |\tilde{F}_0|^2 dxdt \tag{5.16}$$

It follows from Gronwall's inequality that

$$\Theta(\tau) \leq \frac{1}{\lambda_0}\int_0^\tau e^{\gamma(\tau-t)}\Lambda_\gamma(t)dt$$

$$\frac{d\Theta(\tau)}{d\tau} \leq \frac{1}{\lambda_0}\Big[(1+\frac{1}{\gamma})(e^{\gamma T}-1)\Big]\Lambda_\gamma(\tau)$$

Choosing $\gamma = \dfrac{1}{T}$, we can conclude that

$$\int_\Omega ((\partial_t W_\varepsilon)^2 + |\nabla W_\varepsilon|^2)_{t=\tau}dx \leq C(T)\varepsilon^{2(s-1)} \tag{5.17}$$

Similarly, we can prove that

$$\int_\Omega (W_\varepsilon)^2|_{t=\tau}dx \leq C(T)\varepsilon^{2(s-1)} \tag{5.18}$$

By using Theorem 3.16 of [1] again, it is easy to prove that

$$\int_\Omega \{(U_s^\varepsilon - \tilde{U}_s^\varepsilon)^2 + [\partial_t(U_s^\varepsilon - \tilde{U}_s^\varepsilon)]^2 + |\nabla(U_s^\varepsilon - \tilde{U}_s^\varepsilon)|^2\}|_{t=\tau} dx \leq C(T)\varepsilon^{2s} \tag{5.19}$$

From (5.17), (5.18) and (5.19), we complete the proof of (5.4).

Concluding remark. In this paper, a multiscale asymptotic expansion for a class of hyperbolic-parabolic type equation arising from the strong-transient heat transfer problems of composite media is obtained , which forms the basis of numerical computation. The multiscale numerical method consists of three parts: the first one is to compute the periodic functions $N_{\alpha_1\cdots\alpha_l}(\xi), \quad l \geq 1, \quad \alpha_j = 1, 2\cdots, n$ in the unit cell Q; the second one is to solve the homogenized initial-boundary value problem with piecewise constant coefficients in whole domain Ω by using usual numerical methods; the last one is to compute any derivatives of the homogenized solution $u^0(x,t)$ with respect to x.

Appendix A. The equivalence of two kinds of homogenization methods : In [2], we know that $\tilde{N}_{\alpha_1}(\xi)$ is definded in such a way:

$$\begin{cases} \dfrac{\partial}{\partial \xi_k}(a_{kj}(\xi)\dfrac{\partial \tilde{N}_{\alpha_1}(\xi)}{\partial \xi_j}) = -\dfrac{\partial}{\partial \xi_k}(a_{k\alpha_1}(\xi)), \quad \text{in} \quad Q \\ \tilde{N}_{\alpha_1}(\xi) \quad \text{is 1-periodic in } \xi \\ \int\limits_Q \tilde{N}_{\alpha_1}(\xi) d\xi = 0 \end{cases} \tag{A.1}$$

Define

$$V_{per} = \{v \in H^1(Q): \quad v(\xi) \quad \text{is 1-periodic in } \xi\}$$

Then (A.1) is equivalent to, refer to p.14 of [2]

$$\int\limits_Q a_{kj}(\xi)\frac{\partial \tilde{N}_{\alpha_1}(\xi)}{\partial \xi_j}\frac{\partial \tilde{v}(\xi)}{\partial \xi_k} d\xi = -\int\limits_Q a_{k\alpha_1}(\xi)\frac{\partial \tilde{v}(\xi)}{\partial \xi_k} d\xi$$
$$\tilde{N}_{\alpha_1} \in V_{per}, \quad \int\limits_Q \tilde{N}_{\alpha_1}(\xi) d\xi = 0, \quad \forall \tilde{v} \in V_{per}$$

i.e.

$$\int\limits_Q \Big(a_{k\alpha_1}(\xi) + a_{kj}(\xi)\frac{\partial \tilde{N}_{\alpha_1}(\xi)}{\partial \xi_j}\Big)\frac{\partial \tilde{v}(\xi)}{\partial \xi_k} d\xi = 0, \quad \forall \quad \tilde{v} \in V_{per} \tag{A.2}$$

In practice, if $\tilde{N}_{\alpha_1} \in H^2(Q)$, then one can prove easily that $\int\limits_{\partial Q} \nu_k a_{kj}(\xi)\dfrac{\partial \tilde{N}_{\alpha_1}(\xi)}{\partial \xi_j}$ $\tilde{v}\ (\xi)\ d\xi + \int\limits_{\partial Q} \nu_k a_{k\alpha_1}(\xi)\ \tilde{v}\ (\xi)\ d\xi = 0$, since $a_{kj}(\xi), \tilde{v}(\xi)$ are 1-periodic functions with respect to ξ.

On the other hand, from (4.12), we know

$$\int\limits_Q \Big(a_{k\alpha_1}(\xi) + a_{kj}(\xi)\frac{\partial N_{\alpha_1}(\xi)}{\partial \xi_j}\Big)\frac{\partial v(\xi)}{\partial \xi_k} d\xi = 0, \quad \forall \quad v \in H_0^1(Q) \tag{A.3}$$

In the appendix, assume that $a_{ij}(\xi) = 0, \quad i \neq j$.

For simplicity, we discuss only 2-D problems without loss of generality.

Let $m=(m_1,m_2), q=(q_1,q_2)\in Z^2,\ \xi=(\xi_1,\xi_2)\in(0,1)^2,\ \tilde{v}(\xi)=e^{i2\pi m\cdot\xi},\ (m_1, m_2)\neq(0,0),\ v(\xi)=e^{i2\pi q\cdot\xi}-e^{i2\pi q_2\xi_2}-e^{i2\pi q_1\xi_1}+1,\ q_1\neq 0,\ q_2\neq 0$. One can directly check that $\tilde{v}(\xi)\in V_{per},\ v(\xi)\in H_0^1(Q)$.

Set $\tilde{\Lambda}_{k\alpha_1}(\xi)=\Big(a_{k\alpha_1}(\xi)+a_{kk}(\xi)\dfrac{\partial\tilde{N}_{\alpha_1}(\xi)}{\partial\xi_k}\Big),\quad \Lambda_{k\alpha_1}(\xi)=\Big(a_{k\alpha_1}(\xi)+a_{kk}(\xi)\dfrac{\partial N_{\alpha_1}(\xi)}{\partial\xi_k}\Big)$, $k,\alpha_1=1,2$

Substituting $\tilde{v}(\xi), v(\xi)$ into (A.2),(A.3), respectively, we have

$$\sum_{k=1}^{2} m_k\int_Q \tilde{\Lambda}_{k\alpha_1}(\xi)e^{i2\pi m\cdot\xi}d\xi=0,\quad \forall(m_1,m_2)\neq(0,0)\in Z^2 \tag{A.4}$$

$$\begin{aligned}&\sum_{k=1}^{2} q_k\int_Q \Lambda_{k\alpha_1}(\xi)e^{i2\pi q\cdot\xi}d\xi-q_1\int_Q\Lambda_{1\alpha_1}(\xi)e^{i2\pi q_1\xi_1}d\xi\\ &\quad -q_2\int_Q\Lambda_{2\alpha_2}(\xi)e^{i2\pi q_2\xi_2}d\xi=0,\quad \forall q_1\neq 0, q_2\neq 0\in Z\end{aligned} \tag{A.5}$$

From (A.4) and (A.5), and using the Fourier analysis method, one can prove that

$$\tilde{\Lambda}_{1\alpha_1}(\xi)=\tilde{\phi}_{1\alpha_1}(\xi_2),\tilde{\Lambda}_{2\alpha_1}(\xi)=\tilde{\phi}_{2\alpha_1}(\xi_1),\alpha_1=1,2 \tag{A.6}$$

$$\Lambda_{1\alpha_1}(\xi)=\phi_{1\alpha_1}(\xi_2),\Lambda_{2\alpha_1}(\xi)=\phi_{2\alpha_1}(\xi_1),\alpha_1=1,2 \tag{A.7}$$

Therefore

$$a_{11}\frac{\partial\Big(\tilde{N}_{\alpha_1}(\xi)-N_{\alpha_1}(\xi)\Big)}{\partial\xi_1}=\tilde{\phi}_{1\alpha_1}(\xi_2)-\phi_{1\alpha_1}(\xi_2) \tag{A.8}$$

i.e.

$$\frac{\partial\Big(\tilde{N}_{\alpha_1}(\xi)-N_{\alpha_1}(\xi)\Big)}{\partial\xi_1}=\frac{1}{a_{11}(\xi)}\Big(\tilde{\phi}_{1\alpha_1}(\xi_2)-\phi_{1\alpha_1}(\xi_2)\Big) \tag{A.9}$$

Integrating on both sides of (A.9) with respect to ξ_1 on the interval [0,1], and using the periodic conditions of $\tilde{N}_{\alpha_1}(\xi), N_{\alpha_1}(\xi)$, we have

$$\tilde{\phi}_{1\alpha_1}(\xi_2)=\phi_{1\alpha_1}(\xi_2),\quad \alpha_1=1,2 \tag{A.10}$$

Similarly

$$\tilde{\phi}_{2\alpha_1}(\xi_1)=\phi_{2\alpha_1}(\xi_1),\quad \alpha_1=1,2 \tag{A.11}$$

Therefore

$$\hat{a}_{k\alpha_1}=\int_Q\phi_{k\alpha_1}d\xi=\int_Q\tilde{\phi}_{k\alpha_1}d\xi=\hat{\tilde{a}}_{k\alpha_1},\quad k,\alpha_1=1,2 \tag{A.12}$$

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