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RATE OF CONVERGENCE OF SCHWARZ ALTERNATING METHOD FOR TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEM*[1)]

Jian-wei Hu
(*School of Mathematical Sciences, Nankai University, 300071 Tianjin China*)
(*Laboratory of Computational Physics, IAPCM, 100088 Beijing China*)
Cai-hua Wang
(*School of Mathematical Sciences, Nankai University, 300071 Tianjin China*)
(*Department of Mathematics, Tianjin Normal University, 300073 Tianjin China*)

Abstract

This paper provides a theoretical justification to a overlapping domain decomposition method applied to the solution of time-dependent convection-diffusion problems. The method is based on the partial upwind finite element scheme and the discrete strong maximum principle for steady problem. An error estimate in $L^\infty(0,T;L^\infty(\Omega))$ is obtained and the fact that convergence factor $\rho(\tau,h)\to 0$ exponentially as $\tau,h\to 0$ is also proved under some usual conditions.

Key words: Rate of convergence, Schwarz alternating method, Convection-diffusion problem.

1. Introduction

Schwarz alternating procedure is the earliest method of domain decomposition approach in the context of partial differential equations. It has been paid attention by mathematician and engineer since 1980's although it was proposed in 1869 by H.A.Schwarz. Via maximum priciple it is easy to prove that there exists a convergence factor $\rho\in(0,1)$ such that the error reduce with geometric rate: $\|e^{k+1}\|_\infty\le\rho^k\|e^0\|_\infty$. It is easy to understand that the convergence factor ρ depends on the size of the overlapping domain, but that fact had not been proved untill to the middle of 1980's. For Laplace equations in rectangular domain, with Fourier seriers and Schwarz alternating procedure Evans, Shao, Kang, Chen and Tang[4][11][9] found that the convergence factor depend exponentially on size of overlapping. For second order elliptic partial defferential equations in domain $\Omega\subset\mathbf{R}^d(d\ge 2)$, P.-L.Lions proved that[8][9]: Set Ω has been decomposed two overlapping subdomain $\Omega=\Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2=\Omega_{12}\neq\emptyset$. Let $\gamma_1=\partial\Omega_1\cap\Omega_2$, $\gamma_2=\partial\Omega_2\cap\Omega_1$ and

$$\delta=\mathrm{dist}(\gamma_1,\gamma_2)>0$$

denote the degree of the overlapping. Then there exists a constant $\mu>0$ dependent on the coefficients of equation and subdomain Ω_1 such that the convergence factor of Schwarz alternating procedure $\rho=\exp(-\mu\delta^2)$, i.e. the convergence factor depends exponentially on the degree of overlapping.

Domain decomposition method constitutes an important actively developed approach to approximate realization of implicit schemes for unsteady problems. However, to the author's knowledge, the discussion of convergence factor for discrete approximation of time-dependent

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problems is rare. For one-dimensional diffusion equation with constant coefficients Rui H.-X. proved [10] the convergence factor ρ also depends exponentially on the degree of overlapping and $\rho(\tau, h) \to 0$ exponentially as $\tau, h \to 0$, but the method of proof depended on analytic solution of one-dimensional diffusion problem with constant coefficients and it is not easy to analyse more general equations in the same way.

In this paper, we discuss a domain decomposition method with overlapping applied to the solution of convection-diffusion problems. For the sake of simplicity, we consider two-dimensional problem $(\mathbf{P})$ as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (a\nabla u) + \mathbf{b}\cdot\nabla u + cu = f, & (x,t) \in \Omega \times (0,T]; \\ u(x,0) = u_0(x), & x \in \overline{\Omega}; \\ u(x,t) = g(x,t), & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

where $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$. The coefficents a, $\mathbf{b}$, c and the functions f, u_0, g are smooth enough. Moreover $a(x) \geq a_0 \geq 0$, $c(x) \geq 0$. For simplicity we set $g(x,t) = 0$.

In the next section we shall describe a kind of finite element scheme to approach problem $(\mathbf{P})$ and in the third section we present two kind of Schwarz alternating procedure to solve the numerical approximation of problem $(\mathbf{P})$. In the last section we study the rate of convergence for the above Schwarz alternating procedure and then an error estimate in $L^\infty(0,T;L^\infty(\Omega))$ is obtained, the fact that convergence factor $\rho(\tau,h) \to 0$ exponentially as $\tau, h \to 0$ is also proved under some usual conditions.

2. Partial upwind finite element scheme

In this section we shall describe a kind of finite element scheme to approach problem $(\mathbf{P})$. Special attention is paid particularly to problems with convection dominating over diffusion (i.e. $|\mathbf{b}| >> a$) because of difficulties with an accurate resolution of the so-call boundary layers. For discretization of problem $(\mathbf{P})$, efficient finite difference or finite element schemes are usually based on the use of upwinding or artificial viscosity. But the full upwind scheme, as well as the artificial viscosity scheme, involve additional viscosity, which may cause excessive dullness of numerical solutions. We use an efficient scheme so-called partial upwind finite element in order to recover the shape of the exact solution as sharply as possible.

We denote a scalar product in $L^2(\Omega)$ by $(\cdot,\cdot)$ and define bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ in space $H^1(\Omega)$ as follows

$$a(u,v) = (a\nabla u, \nabla v), \quad b(u,v) = (\mathbf{b}\cdot\nabla u, v).$$

Then the weak form of problem $(\mathbf{P})$ will be that for $t \in (0,T]$ to find $u(t) \in H_0^1(\Omega)$ such that $u(0) = u_0$ and

$$(u'(t),v) + a(u,v) + b(u,v) + (cu,v) = (f,v), \ \forall v \in H_0^1(\Omega) \tag{2.1}$$

For discretization of above problem $(\mathbf{P})$, we use an efficient scheme called partial upwind finite element [3][5]. Let Ω_h be a polygonal approximation of the domain Ω. Consider a regular triangulation[2] $\mathcal{T}_h = \{e\}$ defined over $\bar{\Omega}_h$, where each element e of $\mathcal{T}_h$ is a closed triangle. We denote the internal vertexes by x_i with $i = 1, \cdots, N_p$; the boundary vertexes on $\partial\Omega$ by x_i with $i = N_p + 1, \cdots, M_p$. We put h_e to be the maximum side length and κ_e to be the minimum perpendicular length of triangle e. Set

$$h = \max\{h_e;\ e \in \mathcal{T}_h\}, \quad \kappa = \min\{\kappa_e;\ e \in \mathcal{T}_h\}.$$

We assume that the regular triangulation $\mathcal{T}_h$ is quasi-uniform, i.e. there exist positive constants c_1, c_2 such that

$$c_1 \leq \frac{h_e}{h} \leq 1 < \frac{h_e}{\kappa_e} \leq c_2, \quad \forall e \in \mathcal{T}_h.$$

Now let us construct the dual decomposition $\mathcal{D}_h = \{D_i\}$, where closed polygon D_i is the barycentric domain associated with nodal point x_i:

$$D_i = \bigcup_e \{D_i^e,\ e \in \mathcal{T}_h \text{ such that } x_i \text{ is a vertex of } e\},$$

where

$$D_i^e = \{x,\ x \in e \text{ and } \lambda_i(x) \geq \lambda_j(x) \ \ \forall x_j \in e\ \}$$

and λ_k is the barycentric coordinate with respect to vertex x_k of e. Moreover, we associate the index set

$$\Lambda_i = \{j \neq i;\ x_j \text{ is adjacent to } x_i\ \}.$$

We denote the boundary of D_i by Γ_i and use notation $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$ for $i = 1, \cdots, M_p$ and $j \in \Lambda_i$ and $T(i) = \{e; x_i \in e,\ e \in \mathcal{T}_h\}$. With the characteristic function μ_i of barycentric domain D_i, the mass lumping operator $\wedge : w \in C(\bar{\Omega}) \longrightarrow \widehat{w} \in L^\infty(\Omega)$ is defined by $\widehat{w}(x) = \sum_{i=1}^{M_p} w(x_i)\mu_i(x)$. Let $V_h \subset H_0^1(\Omega)$ be a space of linear conforming finite element. By $I_h : C(\bar{\Omega}) \to V_h$ we denote the operator of the Lagrange interpolation. Let $\tau = T/N_\tau$ be time step and $t_{n+1} = (n+1)\tau$, then Galerkin finite element scheme for problem (**P**) reads:

(i) Set $U^0 = I_h u_0$;

(ii) For $n = 0, 1, \cdots, N_\tau - 1$, to find $U^{n+1} \in V_h$ such that

$$\frac{1}{\tau}(U^{n+1} - U^n, v) + a(U^{n+1}, v) + b(U^{n+1}, v) + (cU^{n+1}, v) = (f^{n+1}, v),\ \forall v \in V_h \tag{2.2}$$

where $f^{n+1} = f(\cdot, t_{n+1})$. However, because of the case with convection dominating over diffusion, it is suitable to modify the convection form $b(U^{n+1}, v)$ as

$$b_h(U^{n+1}, v) = \sum_{i=1}^{N_p} v_i \sum_{j \in \Lambda_i} [(\sigma_{ij} U_i^{n+1} + \sigma_{ji} U_j^{n+1}) - U_i^{n+1}]\beta_{ij}, \tag{2.3}$$

here $U_i^{n+1} = U^{n+1}(x_i)$, $v_i = v(x_i)$ and σ_{ij} , σ_{ji} are partial upwind coefficients which are defined easyly by coefficients of equation a, b and dual decomposition $\mathcal{D}_h$. Let $\mathbf{n}$ denote the unit normal to Γ_i and

$$\beta_{ij} = \int_{\Gamma_{ij}} \mathbf{b} \cdot \mathbf{n} \mathrm{d}s.$$

For details see [6].

We also use mass lumping for each term in (2.2) except for $a(U^{n+1}, v)$, then the partial upwind finite element scheme (**Q**) as follows:

(i) Set $U^0 = I_h u_0$;

(ii) For $n = 0, 1, \cdots, N_\tau - 1$, to find $U^{n+1} \in V_h$ such that

$$\frac{1}{\tau}(\widehat{U}^{n+1} - \widehat{U}^n, \widehat{v}) + a(U^{n+1}, v) + b_h(U^{n+1}, v) + (\widehat{c}\widehat{U}^{n+1}, \widehat{v}) = (\widehat{f}^{n+1}, \widehat{v}),\ \forall v \in V_h \tag{2.4}$$

For simplicity of notation we define bilinear form $L(u, v)$ on $V_h \times V_h$

$$L(u, v) = a(u, v) + b_h(u, v) + (\widehat{c}\widehat{u}, \widehat{v}), \quad \forall u, v \in V_h. \tag{2.5}$$

and linear form $F(v)$

$$F(v) = (\tau \widehat{f}^{n+1} + \widehat{U}^n, \widehat{v}), \ \ \forall v \in V_h.$$

Then it is easy to see that in the scheme (**Q**) (2.4) is equivalent to solve a series of steady problem (**S**): to find $U^{n+1} \in V_h$ such that

$$(\widehat{U}^{n+1}, \widehat{v}) + \tau L(U^{n+1}, v) = F(v), \ \ \forall v \in V_h. \tag{2.6}$$

Let $V_h = \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_{N_p}\}$, here φ_i be basis functions such that $\varphi_i(x_j) = \delta_{ij}$. Futher, we set vectors

$$\mathbf{u} = (U_1^{n+1}, U_2^{n+1}, \cdots, U_{N_p}^{n+1})^T, \quad \mathbf{f} = (F(\varphi_1), F(\varphi_2), \cdots, F(\varphi_{N_p}))^T,$$

and matrices

$$M = (m_{ij}), \quad A = (a_{ij}), \quad B = (b_{ij}), \quad C = (c_{ij}),$$

where

$$m_{ij} = (\widehat{\varphi}_j, \widehat{\varphi}_i), \quad a_{ij} = a(\varphi_j, \varphi_i), \quad b_{ij} = b_h(\varphi_j, \varphi_i), \quad c_{ij} = (\widehat{c}\widehat{\varphi}_j, \widehat{\varphi}_i), \qquad i, j = 1, 2, \cdots, N_p.$$

With equation (2.6) we obtain the linear algebraic system

$$[M + \tau(A + B + C)]\mathbf{u} = \mathbf{f}. \tag{2.7}$$

We assume throughout that the triangulation $\mathcal{T}_h$ is of weakly acute type, that is, all the angles of triangle $e \in \mathcal{T}_h$ are less than or equal to $\pi/2$. More generally and more practically we may assume the triangulation $\mathcal{T}_h$ is a Delaunay triangulation, i.e. any two angles subtended by any given edge add to no more than π. Of course, a weakly acute triangulation is a special Delaunay triangulation [12]. Then it is easy to prove the following lemmas[6].

Lemma 2.1. *The elements of matrices M, B, C can be rewritten in the following way:*

(i)

$$m_{ij} = \begin{cases} |D_i|, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases} \tag{2.8}$$

(ii)

$$b_{ij} = \begin{cases} \sum_{k \in \Lambda_i} (\sigma_{ik} - 1)\beta_{ik}, & \text{if } j = i; \\ (1 - \sigma_{ij})\beta_{ij}, & \text{if } j \in \Lambda_i; \\ 0, & \text{if } j \neq i, \ j \notin \Lambda_i. \end{cases} \tag{2.9}$$

(iii)

$$c_{ij} = \begin{cases} c(x_i)|D_i|, & \text{if } j = i; \\ 0, & \text{if } j \neq i. \end{cases} \tag{2.10}$$

where $|D_i|$ denotes the area of polygon D_i in dual decomposition $\mathcal{D}_h$.

Lemma 2.2. *If the triangulation $\mathcal{T}_h$ is a Delaunay triangulation, then for the matrix $K = A + B + C = (k_{ij})$ following properties hold:*

$$k_{ii} > 0 \quad \forall i; \qquad k_{ij} \leq 0 \quad \forall j \in \Lambda_i; \qquad k_{ij} = 0 \quad \forall j \notin \Lambda_i;$$

$$\sum_{j=1}^{N_p} k_{ij} = c(x_i)|D_i| \geq 0, \qquad i = 1, 2, \cdots, N_p.$$

Moreover, for any two nodes x_i and x_j in triangulation $\mathcal{T}_h$ there exists a path of nodes

$$x_i = x_{j(0)}, \ x_{j(1)}, \ \cdots \ , x_{j(m)} = x_j, \quad (j(n) \in \Lambda_{j(n-1)}, \ n = 1, 2, \cdots, m)$$

such that $k_{j(n-1)\ j(n)} < 0$.

Lemma 2.3. *If the triangulation $\mathcal{T}_h$ is Delaunay and quasi-uniform, then there exists a positive constant c_3 dependent only on $a(x), \mathbf{b}(x), c(x)$ and Ω such that*

$$m_{ii}/k_{ii} \geq c_3 h^2, \quad \forall i. \tag{2.11}$$

Proof. We have[7]

$$\begin{aligned} a_{ii} &= (a\nabla\varphi_i, \nabla\varphi_i) \\ &\le \sum_{e\in T(i)} |D_i^e|(3\|a\|_{L^\infty(\Omega)}/{\kappa_e}^2) \\ &\le |D_i|(3\|a\|_{L^\infty(\Omega)}/\kappa^2) \end{aligned}$$

and it is easy to know[1]

$$\begin{aligned} b_{ii} &= \sum_{k\in\Lambda_i} (\sigma_{ik}-1)\beta_{ik} \\ &\le \sum_{e\in T(i)} 4|D_i^e|\ \|\mathbf{b}\|_{0,e}/\kappa_e \\ &\le |D_i|\ \|\mathbf{b}\|_0/\kappa \end{aligned}$$

here $\|\mathbf{b}\|_{0,e} = \max\{|\mathbf{b}(x)|, x\in e\}$ and $\|\mathbf{b}\|_0 = \max\{\|\mathbf{b}\|_{0,e}, e\in \mathcal{T}_h\}$. Because $\mathcal{T}_h$ is quasi-uniform then $\kappa/h \ge c_1/c_2$,

$$\begin{aligned} k_{ii}/m_{ii} &= (a_{ii}+b_{ii}+c_{ii})/m_{ii} \\ &\le 3\|a\|_{L^\infty(\Omega)}/\kappa^2 + \|\mathbf{b}\|_0/\kappa + \|c\|_{L^\infty(\Omega)} \\ &\le \Big[3\|a\|_{L^\infty(\Omega)}/(\frac{c_1}{c_2})^2 + \|\mathbf{b}\|_0 h/(\frac{c_1}{c_2}) + \|c\|_{L^\infty(\Omega)}h^2\Big]/h^2 \end{aligned}$$

With boundedness of Ω, we take c_3 as follows

$$\frac{1}{c_3} = 3\|a\|_{L^\infty(\Omega)}/(\frac{c_1}{c_2})^2 + \|\mathbf{b}\|_0 \mathrm{diam}(\Omega)/(\frac{c_1}{c_2}) + \|c\|_{L^\infty(\Omega)}(\mathrm{diam}(\Omega))^2$$

then lemma holds.

The solution $U^{n+1}(x)$ in scheme ($\mathbf{Q}$) (or problem ($\mathbf{S}$)) has following strong maximum principle[6]:

Theorem 2.1. *Set the triangulation $\mathcal{T}_h$ is a Delaunay triangulation. If $F(v)\le 0$ (or ≥ 0), $\forall v(x)\ge 0$, then $U^{n+1}(x)$ cannot achieve a non-negative maximum (or non-positive minimum) in the interrior of Ω unless it is constant.*

Moreovre we have error estimate in $L^\infty(\Omega)$ [1][7]:

Theorem 2.2. *If the solution u of problem (P) is sufficiently smooth and the condition of theorem 2.1 holds, then the solution U^{n+1} obtained by scheme ($\mathbf{Q}$) (or problem ($\mathbf{S}$)) satisfies the estimate*

$$\|U^{n+1}-u^{n+1}\|_{L^\infty(\Omega)} \le c_4(h+\tau), \quad \forall n = 0,1,\cdots,N_\tau - 1. \tag{2.12}$$

Where constant c_4 independent of τ and h.

3. Schwarz alternating procedure

We use Schwarz's alternating methods for solving steady problem ($\mathbf{S}$). For the sake of simplicity, only two overlapping subdomain are used. Let $\Omega = \Omega_1\cup\Omega_2$, $\Omega_1\cap\Omega_2 = \Omega_{12} \ne \emptyset$ and $\gamma_1 = \partial\Omega_1\cap\Omega_2$, $\gamma_2 = \partial\Omega_2\cap\Omega_1$.

We assume that $\delta = \mathrm{dist}(\gamma_1,\gamma_2) > 0$ and there exists a cut γ_3 in Ω_{12} such that

$$\delta_1 = \mathrm{dist}(\gamma_1,\gamma_3) > 0, \quad \delta_2 = \mathrm{dist}(\gamma_2,\gamma_3) > 0, \quad \delta_1\approx\delta_2\approx\delta/2$$

and then Ω is decomposed two subdomain again

$$\Omega = \Omega_1'\cup\Omega_2', \quad \overline{\Omega}_1'\cap\overline{\Omega}_2' = \gamma_3.$$

We assume throughout that γ_1, γ_2 and γ_3 are cuts in family of triangulation $\mathcal{T}_h$ and the step length $h < \delta/2$. Set spaces of linear conforming finite element $V_h^i \subset H^1(\Omega_i)$ and $v = 0$ on $\partial\Omega$ $\forall v \in V_h^i$, $V_{0h}^i \subset H_0^1(\Omega_i)$ $(i = 1, 2)$. We give following two schemes of Schwarz alternating procedure:

Scheme 3.1. Let $V^0(x) = U^0(x) = I_h u_0$. For $n = 0, 1, \cdots, N_\tau - 1$ find $V^{n+1} \in V_h$:

1. Set $W_{(0)}^{n+1} = V^n$.

2. For $m = 0, 1, 2, \cdots k$ find $W_{(2m+1)}^{n+1} \in V_h^1$ such that

$$\begin{cases} (\widehat{W}_{(2m+1)}^{n+1},\ \widehat{v}) + \tau L(W_{(2m+1)}^{n+1},\ v) = \widetilde{F}(v), & \forall v \in V_{0h}^1 \\ W_{(2m+1)}^{n+1}(x) = W_{(2m)}^{n+1}(x), & x \in \partial\Omega_1 \end{cases} \tag{3.1}$$

and then find $W_{(2m+2)}^{n+1} \in V_h^2$ such that

$$\begin{cases} (\widehat{W}_{(2m+2)}^{n+1},\ \widehat{v}) + \tau L(W_{(2m+2)}^{n+1},\ v) = \widetilde{F}(v), & \forall v \in V_{0h}^2 \\ W_{(2m+2)}^{n+1}(x) = W_{(2m+1)}^{n+1}(x), & x \in \partial\Omega_2 \end{cases} \tag{3.2}$$

where

$$\widetilde{F}(v) = (\tau \widehat{f}^{n+1} + \widehat{V}^n,\ \widehat{v}).$$

Prolong the field of definition of $W_{(2m+1)}^{n+1}, W_{(2m+2)}^{n+1}$ to whole Ω:

$$W_{(2m+1)}^{n+1}(x) = \begin{cases} W_{(2m+1)}^{n+1}(x), & \text{if } x \in \overline{\Omega}_1 \\ W_{(2m)}^{n+1}(x), & \text{if } x \in \overline{\Omega}\backslash\overline{\Omega}_1 \end{cases} \tag{3.3}$$

$$W_{(2m+2)}^{n+1}(x) = \begin{cases} W_{(2m+2)}^{n+1}(x), & \text{if } x \in \overline{\Omega}_2 \\ W_{(2m+1)}^{n+1}(x), & \text{if } x \in \overline{\Omega}\backslash\overline{\Omega}_2 \end{cases} \tag{3.4}$$

3. Let

$$V^{n+1}(x) = \begin{cases} W_{(2k+1)}^{n+1}(x), & \text{if } x \in \overline{\Omega}_1' \\ W_{(2k+2)}^{n+1}(x), & \text{if } x \in \overline{\Omega}_2' \end{cases} \tag{3.5}$$

be approximation of U^{n+1} for problem (**S**).

It is obvious that the calculation of $W_{(2m+2)}^{n+1}$ must be done after calculation of $W_{(2m+1)}^{n+1}$ in scheme 3.1 and following scheme will be parallel.

Scheme 3.2. Let $V^0(x) = U^0(x) = I_h u_0$. For $n = 0, 1, \cdots, N_\tau - 1$ find $V^{n+1} \in V_h$:

1. Set $W_{(-1)}^{n+1} = W_{(0)}^{n+1} = V^n$.

2. For $m = 0, 1, 2, \cdots k$ find $W_{(2m+1)}^{n+1} \in V_h^1$ just the same as (3.1) and find $W_{(2m+2)}^{n+1} \in V_h^2$ such that

$$\begin{cases} (\widehat{W}_{(2m+2)}^{n+1},\ \widehat{v}) + \tau L(W_{(2m+2)}^{n+1},\ v) = \widetilde{F}(v), & \forall v \in V_{0h}^2 \\ W_{(2m+2)}^{n+1}(x) = W_{(2m-1)}^{n+1}(x), & x \in \partial\Omega_2 \end{cases} \tag{3.2$'$}$$

where

$$\widetilde{F}(v) = (\tau \widehat{f}^{n+1} + \widehat{V}^n,\ \widehat{v}).$$

Prolong the field of definition of $W_{(2m+1)}^{n+1}$ to whole Ω just same as (3.3) and prolongation of $W_{(2m+2)}^{n+1}$ is

$$W_{(2m+2)}^{n+1}(x) = \begin{cases} W_{(2m+2)}^{n+1}(x), & \text{if } x \in \overline{\Omega}_2 \\ W_{(2m-1)}^{n+1}(x), & \text{if } x \in \overline{\Omega}\backslash\overline{\Omega}_2 \end{cases} \tag{3.4$'$}$$

3. Let $V^{n+1}(x)$ be same as (3.5).

4. Analysis of rate of convergence

First we define

$$e^n(x) = V^n(x) - U^n(x), \qquad x \in \Omega$$
$$e^n_{(i)}(x) = W^n_{(i)}(x) - U^n(x), \qquad x \in \Omega$$

Lemma 4.1. *Assume that the condition of theorem 2.2 holds, and* $\|\frac{\partial u}{\partial t}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq c_5$, *then there exists a positive constant* c_6 *independent of* τ *and* h *such that*

$$\|e^{n+1}_{(0)}\|_{L^\infty(\Omega)} \leq c_6(h+\tau) + \|e^n\|_{L^\infty(\Omega)}. \tag{4.1}$$

Proof. We have

$$\begin{aligned} e^{n+1}_{(0)}(x) &= W^{n+1}_{(0)}(x) - U^{n+1}(x) \\ &= V^n(x) - U^n(x) + U^n(x) - U^{n+1}(x). \end{aligned} \tag{4.2}$$

With

$$U^n - U^{n+1} = U^n - u^n + u^n - u^{n+1} + u^{n+1} - U^{n+1},$$

by theorem 2.2 and above assumption it is easy to see that

$$\|U^n - U^{n+1}\|_{L^\infty(\Omega))} \leq 2c_4(h+\tau) + c_5\tau, \tag{4.3}$$

therefore

$$\|e^{n+1}_{(0)}\|_{L^\infty(\Omega)} \leq 2c_4(h+\tau) + c_5\tau + \|e^n\|_{L^\infty(\Omega)}. \tag{4.4}$$

Take $c_6 = 2c_4 + c_5$ then estimation (4.1) follows.

First we make analysis for seheme 3.1. By definition of V^{n+1} the error

$$e^{n+1}(x) = \begin{cases} e^{n+1}_{(2k+1)}(x), & x \in \overline{\Omega}'_1 \\ e^{n+1}_{(2k+2)}(x), & x \in \overline{\Omega}'_2 \end{cases}$$

For $m = 0, 1, \cdots k$, the equations of error can be written as follows:

$$\begin{cases} (\widehat{e}^{n+1}_{(2m+1)}, \widehat{v}) + \tau L(e^{n+1}_{(2m+1)}, v) = (\widehat{e}^n, \widehat{v}), & \forall v \in V^1_{0h} \\ e^{n+1}_{(2m+1)}(x) = e^{n+1}_{(2m)}(x), & x \in \gamma_1 \\ e^{n+1}_{(2m+1)}(x) = 0, & x \in \partial\Omega_1 \backslash \gamma_1 \end{cases} \tag{4.5}$$

$$\begin{cases} (\widehat{e}^{n+1}_{(2m+2)}, \widehat{v}) + \tau L(e^{n+1}_{(2m+2)}, v) = (\widehat{e}^n, \widehat{v}) & \forall v \in V^2_{0h} \\ e^{n+1}_{(2m+2)}(x) = e^{n+1}_{(2m+1)}(x), & x \in \gamma_2 \\ e^{n+1}_{(2m+2)}(x) = 0, & x \in \partial\Omega_2 \backslash \gamma_2 \end{cases} \tag{4.6}$$

For (4.5) we decompose $e^{n+1}_{(2m+1)}(x)$ as

$$e^{n+1}_{(2m+1)}(x) = e_1(x) + e_2(x)$$

where $e_1(x)$ and $e_2(x)$ satisfy respectively:

$$\begin{cases} (\widehat{e}_1, \widehat{v}) + \tau L(e_1, v) = (\widehat{e}^n, \widehat{v}) & \forall v \in V^1_{0h} \\ e_1(x) = 0, & x \in \partial\Omega_1 \end{cases} \tag{4.7}$$

and

$$\begin{cases} (\widehat{e}_2, \widehat{v}) + \tau L(e_2, v) = 0, & \forall v \in V^1_{0h} \\ e_2(x) = e^{n+1}_{(2m)}(x), & x \in \gamma_1 \\ e_2(x) = 0, & x \in \partial\Omega_1 \backslash \gamma_1 \end{cases} \tag{4.8}$$

We estimate $e_1(x)$ first. We have:

Lemma 4.2. *Assume that the condition of theorem 2.1 holds, then $e_1(x)$ in (4.7) has estimation as follows*

$$\max_{x\in\overline{\Omega}_1}|e_1(x)| \le \max_{x\in\overline{\Omega}_1}|e^n(x)|. \tag{4.9}$$

Proof. Set $E_1 = \max_{x\in\overline{\Omega}_1}|e^n(x)|$. By (4.7)

$$(E_1 - \widehat{e}_1(x), \widehat{v}) + \tau L(E_1 - e_1(x), v) = (E_1 - \widehat{e}^n(x), \widehat{v}) + \tau L(E_1, v).$$

For any $v(x) \in V_{0h}^1$ and $v(x) \ge 0$

$$(E_1 - \widehat{e}^n(x), \widehat{v}) \ge 0,$$

and by lemma 2.2 it is easy to know

$$L(E_1, v) = E_1(\widehat{c}(x), \widehat{v}) \ge 0.$$

From two above inequalities and theorem 2.1 we have

$$E_1 - e_1(x) \ge 0 \quad \forall x \in \overline{\Omega}_1.$$

Similarly

$$E_1 + e_1(x) \ge 0 \quad \forall x \in \overline{\Omega}_1.$$

therefore lemma holds.

We are going to estimate $e_2(x)$ of (4.8).

Lemma 4.3. *Assume that the triangulation $\mathcal{T}_h$ is a Delaunay and quasi-uniform, the step length $h < \delta_1/2$, then there exists $0 < \rho(\tau,h) < 1$ such that for $e_2(x)$ in (4.8)*

$$\max_{x\in\overline{\Omega}_1'}|e_2(x)| \le \rho(\tau,h)\max_{x\in\gamma_1}|e_{(2m)}^{n+1}(x)|, \tag{4.10}$$

specially

$$\sup_{x\in\gamma_2}|e_2(x)| \le \rho(\tau,h)\max_{x\in\gamma_1}|e_{(2m)}^{n+1}(x)|. \tag{4.11}$$

where the convergence factor

$$\rho(\tau,h) = \Big(\frac{1}{1+c_3h^2/\tau}\Big)^{\delta/2h-1}. \tag{4.12}$$

depends on δ exponentially.

proof. Let integer $N : N \le \delta_1/h < N+1$. With $\delta_1 =$dist(γ_1,γ_3) it is easy to see that there exist at least $N-1$ non-intersecting node paths $l_1, l_2, \cdots, l_{N-1}$ who consist of edges of $\mathcal{T}_h$ in domain Ω_{12}. Set $l_0 = \gamma_1$ and $l_N = \gamma_3$ and l_k is nearer γ_1 than l_{k+1} $(k = 0, 1, \cdots, N-1)$. For any node $x_i \in l_k$ (4.8) leads

$$m_{ii}e_2(x_i) + \tau k_{ii}e_2(x_i) + \tau\sum_{j\in\Lambda_i}k_{ij}e_2(x_j) = 0 \tag{4.13}$$

With maximum principle we have

$$|e_2(x_j)| \le \max_{x\in l_{k-1}}|e_2(x_j)| \quad \forall j \in \Lambda_i. \tag{4.14}$$

By lemma 2.1 and lemma 2.2,

$$\sum_{j\in\Lambda_i}k_{ij} = a_{ii} + b_{ii} \le k_{ii} \tag{4.15}$$

Summarizing(4.13)–(4.15) we find that

$$|e_2(x_i)| \le \frac{\tau k_{ii}}{\tau k_{ii} + m_{ii}}\max_{x\in l_{k-1}}|e_2(x)|$$

and then with lemma 2.3

$$\max_{x\in l_k}|e_2(x)| \le \frac{1}{1+c_3h^2/\tau}\max_{x\in l_{k-1}}|e_2(x)|$$

For $k=1,2,\cdots,N$ successive estimate we have

$$\max_{x\in\gamma_3}|e_2(x)| \le (\frac{1}{1+c_3h^2/\tau})^N \max_{x\in\gamma_1}|e_2(x)|$$

and by maximum principle also

$$\max_{x\in\overline{\Omega}_1'}|e_2(x)| \le (\frac{1}{1+c_3h^2/\tau})^N \max_{x\in\gamma_1}|e_2(x)|.$$

Let $\rho(\tau,h)=(\frac{1}{1+c_3h^2/\tau})^{\frac{\delta_1}{h}-1}$ and notice $\delta\approx 2\delta_1$ then lemma follows.

Lemma 4.4. *Let $\rho(\tau,h)$ be defined in lemma 4.3, if $\tau=O(h^2)$ then there exists positive constant M independent of τ and h such that*

$$\Big(\frac{1}{1-\rho(\tau,h)}\Big)^n \le M, \qquad (0\le n\le T/\tau). \tag{4.16}$$

Proof. Let

$$f(\tau,h)=\Big(\frac{1}{1-\rho(\tau,h)}\Big)^{T/\tau} \tag{4.17}$$

or equivalent

$$f(\tau,h)=\Big(1+\frac{\rho(\tau,h)}{1-\rho(\tau,h)}\Big)^{\frac{1-\rho(\tau,h)}{\rho(\tau,h)}\cdot\frac{T}{1-\rho(\tau,h)}\cdot\frac{\rho(\tau,h)}{\tau}}$$

By $\tau=O(h^2)$ it is easy to see $\lim\limits_{\tau,h\to 0}\rho(\tau,h)=0$ then

$$\lim_{\tau,h\to 0}f(\tau,h)=\lim_{\tau,h\to 0}e^{\frac{T}{1-\rho(\tau,h)}\cdot\frac{\rho(\tau,h)}{\tau}}$$

and

$$\lim_{\tau,h\to 0}\frac{\rho(\tau,h)}{\tau}=0.$$

Therefore

$$\lim_{\tau,h\to 0}f(\tau,h)=1$$

Then there exists positive constant M independent on τ,h such that

$$\Big(\frac{1}{1-\rho(\tau,h)}\Big)^{T/\tau}\le M$$

that implies the desired result.

With lemma 4.2 and lemma 4.3 we have

$$\max_{x\in\overline{\Omega}_1'}|e^{n+1}_{(2m+1)}(x)| \le \rho(\tau,h)\max_{x\in\gamma_1}|e^{n+1}_{(2m)}(x)|+\|e^n\|_{L^\infty(\Omega)} \tag{4.18}$$

Specially

$$\max_{x\in\gamma_2}|e^{n+1}_{(2m+1)}(x)| \le \rho(\tau,h)\max_{x\in\gamma_1}|e^{n+1}_{(2m)}(x)|+\|e^n\|_{L^\infty(\Omega)} \tag{4.19}$$

Similarly for $e^{n+1}_{(2m+2)}$:

$$\max_{x\in\overline{\Omega}_2'}|e^{n+1}_{(2m+2)}(x)| \le \rho(\tau,h)\max_{x\in\gamma_2}|e^{n+1}_{(2m+1)}(x)|+\|e^n\|_{L^\infty(\Omega)}. \tag{4.20}$$

Specially

$$\max_{x\in\gamma_1}|e^{n+1}_{(2m+2)}(x)| \le \rho(\tau,h)\max_{x\in\gamma_2}|e^{n+1}_{(2m+1)}(x)| + \|e^n\|_{L^\infty(\Omega)} \tag{4.21}$$

Theorem 4.1. *Assume that the triangulation $\mathcal{T}_h$ is a Delaunay and quasi-uniform, time step $\tau = O(h^2)$, then for scheme 3.1 with $\delta = dist(\gamma_1,\gamma_2) > 0$ we have error estimate*

$$\|V^n - U^n\|_{L^\infty(\Omega)} \le c_7(\rho(\tau,h))^{2k}(\tau+h)$$

where constant c_7 indepedent of τ and h; convergence factor $0<\rho(\tau,h)<1$ depends on δ,h,τ exponentially and

$$\lim_{\tau,h\to 0}\frac{\rho(\tau,h)}{\tau^\alpha h^\beta} = 0, \quad \forall\ \alpha,\beta>0.$$

Proof. With (4.18), (4.20)

$$\begin{aligned}\max_{x\in\overline{\Omega}_1'}|e^{n+1}(x)| &= \max_{x\in\overline{\Omega}_1'}|e^{n+1}_{(2k+1)}(x)|\\ &\le \|e^n\|_{L^\infty(\Omega)} + \rho(\tau,h)\max_{x\in\gamma_1}|e^{n+1}_{(2k)}(x)|\end{aligned} \tag{4.22}$$

$$\begin{aligned}\max_{x\in\overline{\Omega}_2'}|e^{n+1}(x)| &= \max_{x\in\overline{\Omega}_2'}|e^{n+1}_{(2k+2)}(x)|\\ &\le \|e^n\|_{L^\infty(\Omega)} + \rho(\tau,h)\max_{x\in\gamma_2}|e^{n+1}_{(2k+1)}(x)|\end{aligned} \tag{4.23}$$

By (4.19), (4.21) and lemma 4.1

$$\begin{aligned}\max_{x\in\gamma_2}\ & |e^{n+1}_{(2k+1)}(x)| \le \rho(\tau,h)\max_{x\in\gamma_1}|e^{n+1}_{(2k)}(x)| + \|e^n\|_{L^\infty(\Omega)}\\ &\le (\rho(\tau,h))^2\max_{x\in\gamma_2}|e^{n+1}_{(2k-1)}| + (1+\rho(\tau,h))\|e^n\|_{L^\infty(\Omega)}\\ &\le \cdots\\ &\le (\rho(\tau,h))^{2k+1}\max_{x\in\gamma_1}|e^{n+1}_0(x)| + (1+\rho(\tau,h)+\cdots+(\rho(\tau,h))^{2k})\|e^n\|_{L^\infty(\Omega)}\\ &\le c_6(\tau+h)(\rho(\tau,h))^{2k+1} + \frac{1}{1-\rho(\tau,h)}\|e^n\|_{L^\infty(\Omega)}\end{aligned} \tag{4.24}$$

Similarly for $e^{n+1}_{(2k)}$:

$$\max_{x\in\gamma_1}|e^{n+1}_{(2k)}| \le c_6(\tau+h)(\rho(\tau,h))^{2k} + \frac{1}{1-\rho(\tau,h)}\|e^n\|_{L^\infty(\Omega)} \tag{4.25}$$

Now summarizing (4.23) and (4.24) ,(4.22) and (4.25) we see that

$$\max_{x\in\overline{\Omega}_1'}|e^{n+1}(x)| \le c_6(\tau+h)(\rho(\tau,h))^{2k+1} + \frac{1}{1-\rho(\tau,h)}\|e^n\|_{L\infty(\Omega)} \tag{4.26}$$

$$\max_{x\in\overline{\Omega}_2'}|e^{n+1}(x)| \le c_6(\tau+h)(\rho(\tau,h))^{2k+2} + \frac{1}{1-\rho(\tau,h)}\|e^n\|_{L^\infty(\Omega)} \tag{4.27}$$

By (4.26), (4.27) we obtain

$$\|e^{n+1}\|_{L^\infty(\Omega)} \le \frac{1}{1-\rho(\tau,h)}\|e^n\|_{L^\infty(\Omega)} + c_6(\tau+h)(\rho(\tau,h))^{2k+1} \tag{4.28}$$

Using successive recurrence we have

$$\|e^{n+1}\|_{L^\infty(\Omega)} \le (\frac{1}{1-\rho(\tau,h)})^n\|e^0\|_{L^\infty(\Omega)} + \sum_{i=0}^{n-1}(\frac{1}{1-\rho(\tau,h)})^i(\rho(\tau,h))^{2k+1}c_6(\tau+h), \tag{4.29}$$

where geometric series is bounded by $M(\rho(\tau,h))^{-1}$ and $\|e^0\|_{L^\infty(\Omega)} = 0$, hence

$$\|e^{n+1}\|_{L^\infty(\Omega)} \le c_6 M(\rho(\tau,h))^{2k}(\tau+h)$$

Let $c_7 = c_6 M$ and notice that

$$\rho(\tau, h) = (\frac{1}{1 + c_3 h^2/\tau})^{\delta/2h-1},$$

and $\tau = O(h^2)$, theorem follows.

With $u^n - V^n = u^n - U^n + U^n - V^n$, theorem 2.2 and theorem 4.1, we have error estimate for solution of scheme 3.1 and weak solution of problem (P):

Theorem 4.2. *Assume that the condition of theorem 4.1 holds, then*

$$\|V^n - u^n\|_{L^\infty(\Omega)} \le C(1 + \rho(\tau, h)^{2k})(h + \tau).$$

Where constant C *independent of* τ *and* h*, convergence factor* $0 < \rho(\tau, h) < 1$ *depends on* δ, h, τ *exponentially and*

$$\lim_{\tau, h \to 0} \frac{\rho(\tau, h)}{\tau^\alpha h^\beta} = 0, \quad \forall\ \alpha, \beta > 0.$$

It is similar to get the error estimate for scheme 3.2. In that case the estimates (4.18), (4.19) still hold but (4.20), (4.21) will be replaced by

$$\max_{x \in \overline{\Omega}_2'} |e^{n+1}_{(2m+2)}(x)| \le \rho(\tau, h) \max_{x \in \gamma_2} |e^{n+1}_{(2m-1)}(x)| + \|e^n\|_{L^\infty(\Omega)} \tag{4.20$'$}$$

$$\max_{x \in \gamma_1} |e^{n+1}_{(2m+2)}(x)| \le \rho(\tau, h) \max_{x \in \gamma_2} |e^{n+1}_{(2m-1)}(x)| + \|e^n\|_{L^\infty(\Omega)} \tag{4.21$'$}$$

Therefore we have following error estimate for scheme 3.2:

Theorem 4.3. *Assume that the condition of theorem 4.1 holds, then*

$$\|V^n - u^n\|_{L^\infty(\Omega)} \le C(1 + \rho(\tau, h)^{k})(h + \tau).$$

Where constant C *independent of* τ *and* h*; convergence factor* $0 < \rho(\tau, h) < 1$ *depends on* δ, h, τ *exponentially and*

$$\lim_{\tau, h \to 0} \frac{\rho(\tau, h)}{\tau^\alpha h^\beta} = 0, \quad \forall\ \alpha, \beta > 0.$$

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