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# EXPANSION OF STEP-TRANSITION OPERATOR OF MULTI-STEP METHOD AND ITS APPLICATIONS (II)[*1)]

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**Abstract**

We give some formulae for calculation of the expansions for (1) composition of step-transition operators (STO) of any two difference schemes (DS) for ODE's, (2) inverse operator of STO of any DS, and (3) conjugate operator of STO of any DS.

*Key words*: Step-transition operator, Expansion, Composition, Inverse operator, Conjugate operator.

## 1. Introduction

For an ordinarily differential equation (ODE)

$$\frac{d}{dt}Z = f(Z), \quad Z \in R^p, \tag{1}$$

any compatible linear $m$-step difference scheme (DS)

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left( \sum_{k=0}^{m} \beta_k \neq 0 \right), \tag{2}$$

can be characterized by a step-transition operator (STO) $G$ (also denoted by $G^\tau$): $R^p \to R^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where $G^k$ stands for $k$-time composition of $G$: $G \circ G \cdots \circ G$ (refer to [2,5,6,10,11]). This operator $G^\tau$ can be represented as a power series in $\tau$ with first term equal to *identity* $I$. More precisely, one can expand[13] the STO $G^\tau(Z)$ of any linear multi-step method (LMSM)[2)] of form (2) with order $s \geq 2$ up to $O(\tau^{s+5})$:

$$G^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1}A(Z) + \tau^{s+2}B(Z) + \tau^{s+3}C(Z) + \tau^{s+4}D(Z) + O(\tau^{s+5}) \tag{4}$$

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2)More generally, one can use an STO to characterize any DS compatible with (1), and obviously the STO can be written in form (4).

(where $Z^{[0]} = Z$, $Z^{[1]} = f(Z)$, $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]}$ for $k = 1, 2, \cdots$) with complete formulae for calculation of $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$.

Thus, the STO $G^\tau$ satisfying equation (3) completely characterizes the LMSM (2) as: $Z_1 = G^\tau(Z_0), \cdots, Z_m = G^\tau(Z_{m-1}) = [G^\tau]^m(Z_0), \cdots$.

In the present paper, we study the composition of any two STO's, the inverse operator and the conjugate operator of STO for any DS. In §2, for any two DS's of order $w-1 (w \geq 2)$, we expand the composition of their STO's up to $O(\tau^{w+5})$ (Theorem 1). In §3 and §4, we do the same things for the inverse operator and the conjugate operator of STO for any DS, respectively (Theorems 2-3). And examples for calculation for these three cases (composition, inverse, conjugation) are given in §3 ( Examples 1-2) and §4 (Remark 1) respectively.

## 2. COMPOSITION OF TWO STEP-TRANSITION OPERATORS

**Theorem 1.** *The composition of two STO's ($w \geq 2$, $\lambda$ and $\mu$ are real numbers)*

$$E^{\mu\tau}(Z) = \sum_{i=0}^{+\infty} \frac{(\mu\tau)^i}{i!} Z^{[i]} + \tau^w B + \tau^{w+1} B_1 + \tau^{w+2} B_2 + \tau^{w+3} B_3 + \tau^{w+4} B_4 + O(\tau^{w+5}) \tag{5}$$

*and*

$$F^{\lambda\tau}(Z) = \sum_{j=0}^{+\infty} \frac{(\lambda\tau)^j}{j!} Z^{[j]} + \tau^w A + \tau^{w+1} A_1 + \tau^{w+2} A_2 + \tau^{w+3} A_3 + \tau^{w+4} A_4 + O(\tau^{w+5}) \tag{6}$$

*can be expressed as follows:*

$$\begin{aligned}
& E^{\mu\tau} \circ F^{\lambda\tau}(Z) \\
&= \sum_{l=0}^{+\infty} \frac{(\lambda+\mu)^l \tau^l}{l!} Z^{[l]} + \tau^w \{A + B\} + \tau^{w+1} \left\{ A_1 + \mu Z_z^{[1]} A + \lambda B_z Z^{[1]} + B_1 \right\} \\
&\quad + \tau^{w+2} \left\{ A_2 + \mu Z_z^{[1]} A_1 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A + \frac{\mu^2}{2} Z_z^{[2]} A \right. \\
&\qquad \left. + \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2 \right\} \\
&\quad + \tau^{2w} \{B_z A\} \\
&\quad + \tau^{w+3} \left\{ A_3 + \mu Z_z^{[1]} A_2 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \right. \\
&\qquad + \frac{\mu^2}{2} Z_z^{[2]} A_1 + \frac{\lambda\mu^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\mu^3}{6} Z_z^{[3]} A + \frac{\lambda^3}{6} B_z Z^{[3]} + \frac{\lambda^3}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&\qquad \left. + \frac{\lambda^3}{6} B_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_1)_z Z^{[2]} + \frac{\lambda^2}{2} (B_1)_{z^2} (Z^{[1]})^2 + \lambda (B_2)_z Z^{[1]} + B_3 \right\} \\
&\quad + \tau^{2w+1} \left\{ \frac{\mu}{2} Z_{z^2}^{[1]} A^2 + B_z A_1 + \lambda B_{z^2} Z^{[1]} A + (B_1)_z A \right\} \qquad\qquad (7) \\
&\quad + \tau^{w+4} \left\{ A_4 + \mu Z_z^{[1]} A_3 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda^3\mu}{6} Z_{z^2}^{[1]} Z^{[3]} A \right. \\
&\qquad + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda^3\mu}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda^3\mu}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\mu^2}{2} Z_z^{[2]} A_2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2A+\frac{\mu^3}{6}Z^{[3]}_{z}A_1\\
&+\frac{\lambda\mu^3}{6}Z^{[3]}_{z^2}Z^{[1]}A+\frac{\mu^4}{24}Z^{[4]}_{z}A+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4\Big\}\\
&+\tau^{2w+2}\Big\{\mu Z^{[1]}_{z^2}AA_1+\frac{\lambda\mu}{2}Z^{[1]}_{z^3}Z^{[1]}A^2+\frac{\mu^2}{4}Z^{[2]}_{z^2}A^2+B_zA_2+\lambda B_{z^2}Z^{[1]}A_1\\
&+\frac{\lambda^2}{2}B_{z^2}Z^{[2]}A+\frac{\lambda^2}{2}B_{z^3}(Z^{[1]})^2A+(B_1)_zA_1+\lambda(B_1)_{z^2}Z^{[1]}A+(B_2)_zA\Big\}\\
&+\tau^{3w}\left\{\frac{1}{2}B_{z^2}A^2\right\}+O(\tau^{w+5}).
\end{aligned}
$$

*Concretely, when $w=2$:*

$$
\begin{aligned}
&E^{\mu\tau}\circ F^{\lambda\tau}(Z)\\
&=\sum_{l=0}^{+\infty}\frac{(\lambda+\mu)^l\tau^l}{l!}Z^{[l]}+\tau^w\{A+B\}+\tau^{w+1}\left\{A_1+\mu Z^{[1]}_zA+\lambda B_zZ^{[1]}+B_1\right\}\\
&+\tau^{w+2}\Big\{A_2+\mu Z^{[1]}_zA_1+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A+\frac{\mu^2}{2}Z^{[2]}_zA\\
&+\frac{\lambda^2}{2}B_zZ^{[2]}+\frac{\lambda^2}{2}B_{z^2}(Z^{[1]})^2+\lambda(B_1)_zZ^{[1]}+B_2+B_zA\Big\}\\
&+\tau^{w+3}\Big\{A_3+\mu Z^{[1]}_zA_2+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A\\
&+\frac{\mu^2}{2}Z^{[2]}_zA_1+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A+\frac{\mu^3}{6}Z^{[3]}_zA+\frac{\lambda^3}{6}B_zZ^{[3]}+\frac{\lambda^3}{2}B_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}B_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_1)_zZ^{[2]}+\frac{\lambda^2}{2}(B_1)_{z^2}(Z^{[1]})^2+\lambda(B_2)_zZ^{[1]}+B_3\\
&+\frac{\mu}{2}Z^{[1]}_{z^2}A^2+B_zA_1+\lambda B_{z^2}Z^{[1]}A+(B_1)_zA\Big\} \qquad (7.2)\\
&+\tau^{w+4}\Big\{A_4+\mu Z^{[1]}_zA_3+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A_2+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^2}Z^{[2]}A_1+\frac{\lambda^3\mu}{6}Z^{[1]}_{z^2}Z^{[3]}A\\
&+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z^{[1]}_{z^3}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z^{[1]}_{z^4}(Z^{[1]})^3A+\frac{\mu^2}{2}Z^{[2]}_zA_2\\
&+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2A+\frac{\mu^3}{6}Z^{[3]}_zA_1\\
&+\frac{\lambda\mu^3}{6}Z^{[3]}_{z^2}Z^{[1]}A+\frac{\mu^4}{24}Z^{[4]}_zA+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4\\
&+\mu Z^{[1]}_{z^2}AA_1+\frac{\lambda\mu}{2}Z^{[1]}_{z^3}Z^{[1]}A^2+\frac{\mu^2}{4}Z^{[2]}_{z^2}A^2+B_zA_2+\lambda B_{z^2}Z^{[1]}A_1
\end{aligned}
$$

$$+ \frac{\lambda^2}{2} B_{z^2} Z^{[2]} A + \frac{\lambda^2}{2} B_{z^3} (Z^{[1]})^2 A + (B_1)_z A_1 + \lambda (B_1)_{z^2} Z^{[1]} A + (B_2)_z A + \frac{1}{2} B_{z^2} A^2 \Big\} + O(\tau^{w+5});$$

*when* $w = 3$*:*

$$\begin{aligned}
&E^{\mu\tau} \circ F^{\lambda\tau}(Z) \\
&= \sum_{l=0}^{+\infty} \frac{(\lambda+\mu)^l \tau^l}{l!} Z^{[l]} + \tau^w \{A + B\} + \tau^{w+1} \left\{ A_1 + \mu Z_z^{[1]} A + \lambda B_z Z^{[1]} + B_1 \right\} \\
&+ \tau^{w+2} \Big\{ A_2 + \mu Z_z^{[1]} A_1 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A + \frac{\mu^2}{2} Z_z^{[2]} A \\
&\quad + \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2 \Big\} \\
&+ \tau^{w+3} \Big\{ B_z A + A_3 + \mu Z_z^{[1]} A_2 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\
&\quad + \frac{\mu^2}{2} Z_z^{[2]} A_1 + \frac{\lambda\mu^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\mu^3}{6} Z_z^{[3]} A + \frac{\lambda^3}{6} B_z Z^{[3]} + \frac{\lambda^3}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&\quad + \frac{\lambda^3}{6} B_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_1)_z Z^{[2]} + \frac{\lambda^2}{2} (B_1)_{z^2} (Z^{[1]})^2 + \lambda (B_2)_z Z^{[1]} + B_3 \Big\} \\
&+ \tau^{w+4} \Big\{ \frac{\mu}{2} Z_{z^2}^{[1]} A^2 + B_z A_1 + \lambda B_{z^2} Z^{[1]} A + (B_1)_z A \\
&\quad + A_4 + \mu Z_z^{[1]} A_3 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda^3\mu}{6} Z_{z^2}^{[1]} Z^{[3]} A \\
&\quad + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda^3\mu}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda^3\mu}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\mu^2}{2} Z_z^{[2]} A_2 \\
&\quad + \frac{\lambda\mu^2}{2} Z_{z^2}^{[2]} Z^{[1]} A_1 + \frac{\lambda^2\mu^2}{4} Z_{z^2}^{[2]} Z^{[2]} A + \frac{\lambda^2\mu^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 A + \frac{\mu^3}{6} Z_z^{[3]} A_1 \\
&\quad + \frac{\lambda\mu^3}{6} Z_{z^2}^{[3]} Z^{[1]} A + \frac{\mu^4}{24} Z_z^{[4]} A + \frac{\lambda^4}{24} B_z Z^{[4]} + \frac{\lambda^4}{8} B_{z^2} (Z^{[2]})^2 + \frac{\lambda^4}{6} B_{z^2} Z^{[1]} Z^{[3]} \\
&\quad + \frac{\lambda^4}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{\lambda^4}{24} B_{z^4} (Z^{[1]})^4 + \frac{\lambda^3}{6} (B_1)_z Z^{[3]} + \frac{\lambda^3}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]} \\
&\quad + \frac{\lambda^3}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_2)_z Z^{[2]} + \frac{\lambda^2}{2} (B_2)_{z^2} (Z^{[1]})^2 + \lambda (B_3)_z Z^{[1]} + B_4 \Big\} \\
&+ O(\tau^{w+5});
\end{aligned} \tag{7.3}$$

*when* $w = 4$*:*

$$\begin{aligned}
&E^{\mu\tau} \circ F^{\lambda\tau}(Z) \\
&= \sum_{l=0}^{+\infty} \frac{(\lambda+\mu)^l \tau^l}{l!} Z^{[l]} + \tau^w \{A + B\} + \tau^{w+1} \left\{ A_1 + \mu Z_z^{[1]} A + \lambda B_z Z^{[1]} + B_1 \right\} \\
&+ \tau^{w+2} \Big\{ A_2 + \mu Z_z^{[1]} A_1 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A + \frac{\mu^2}{2} Z_z^{[2]} A \\
&\quad + \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2 \Big\} \\
&+ \tau^{w+3} \Big\{ A_3 + \mu Z_z^{[1]} A_2 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A
\end{aligned} \tag{7.4}$$

$$
\begin{aligned}
&+\frac{\mu^2}{2}Z_z^{[2]}A_1+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A+\frac{\mu^3}{6}Z_z^{[3]}A+\frac{\lambda^3}{6}B_zZ^{[3]}+\frac{\lambda^3}{2}B_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}B_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_1)_zZ^{[2]}+\frac{\lambda^2}{2}(B_1)_{z^2}(Z^{[1]})^2+\lambda(B_2)_zZ^{[1]}+B_3\Big\}\\
&+\tau^{w+4}\Big\{B_zA+A_4+\mu Z_z^{[1]}A_3+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A_2+\frac{\lambda^2\mu}{2}Z_{z^2}^{[1]}Z^{[2]}A_1+\frac{\lambda^3\mu}{6}Z_{z^2}^{[1]}Z^{[3]}A\\
&+\frac{\lambda^2\mu}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z_{z^3}^{[1]}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z_{z^4}^{[1]}(Z^{[1]})^3A+\frac{\mu^2}{2}Z_z^{[2]}A_2\\
&+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z_{z^2}^{[2]}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z_{z^3}^{[2]}(Z^{[1]})^2A+\frac{\mu^3}{6}Z_z^{[3]}A_1\\
&+\frac{\lambda\mu^3}{6}Z_{z^2}^{[3]}Z^{[1]}A+\frac{\mu^4}{24}Z_z^{[4]}A+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4\Big\}\\
&+O(\tau^{w+5});
\end{aligned}
$$

*when $w>4$:*

$$
\begin{aligned}
&E^{\mu\tau}\circ F^{\lambda\tau}(Z)\\
&=\sum_{l=0}^{+\infty}\frac{(\lambda+\mu)^l\tau^l}{l!}Z^{[l]}+\tau^w\{A+B\}+\tau^{w+1}\Big\{A_1+\mu Z_z^{[1]}A+\lambda B_zZ^{[1]}+B_1\Big\}\\
&+\tau^{w+2}\Big\{A_2+\mu Z_z^{[1]}A_1+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A+\frac{\mu^2}{2}Z_z^{[2]}A \qquad (7.5)\\
&+\frac{\lambda^2}{2}B_zZ^{[2]}+\frac{\lambda^2}{2}B_{z^2}(Z^{[1]})^2+\lambda(B_1)_zZ^{[1]}+B_2\Big\}\\
&+\tau^{w+3}\Big\{A_3+\mu Z_z^{[1]}A_2+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A_1+\frac{\lambda^2\mu}{2}Z_{z^2}^{[1]}Z^{[2]}A+\frac{\lambda^2\mu}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A\\
&+\frac{\mu^2}{2}Z_z^{[2]}A_1+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A+\frac{\mu^3}{6}Z_z^{[3]}A+\frac{\lambda^3}{6}B_zZ^{[3]}+\frac{\lambda^3}{2}B_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}B_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_1)_zZ^{[2]}+\frac{\lambda^2}{2}(B_1)_{z^2}(Z^{[1]})^2+\lambda(B_2)_zZ^{[1]}+B_3\Big\}\\
&+\tau^{w+4}\Big\{A_4+\mu Z_z^{[1]}A_3+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A_2+\frac{\lambda^2\mu}{2}Z_{z^2}^{[1]}Z^{[2]}A_1+\frac{\lambda^3\mu}{6}Z_{z^2}^{[1]}Z^{[3]}A\\
&+\frac{\lambda^2\mu}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z_{z^3}^{[1]}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z_{z^4}^{[1]}(Z^{[1]})^3A+\frac{\mu^2}{2}Z_z^{[2]}A_2\\
&+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z_{z^2}^{[2]}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z_{z^3}^{[2]}(Z^{[1]})^2A+\frac{\mu^3}{6}Z_z^{[3]}A_1\\
&+\frac{\lambda\mu^3}{6}Z_{z^2}^{[3]}Z^{[1]}A+\frac{\mu^4}{24}Z_z^{[4]}A+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4\Big\}
\end{aligned}
$$

$+ O(\tau^{w+5})$.

*We use the notation*

$$B_{z^3}\left(Z^{[1]}\right)^2 Z^{[2]} = \sum_{i,j,k=1}^{p} \frac{\partial^3 B}{\partial z_i \partial z_j \partial z_k}\left[Z^{[1]}\right]_{(i)}\left[Z^{[1]}\right]_{(j)}\left[Z^{[2]}\right]_{(k)}$$

*where* $z_i$ *is the* $i$*-th component of* $p$*-dim vector* $Z$*, and* $\left[Z^{[1]}\right]_{(j)}$ *stands for the* $j$*-th component of* $p$*-dim vector* $Z^{[1]}$ *(refer to [11,13]).* □

*Proof.* The proof of Theorem 1 is merely tedious but straightforward calculation, so we omit it here.

## 3. Inverse Operator of Step-Transition Operator

According to Theorem 1, we have immediately

**Theorem 2.** *If two STOs (*$w \geq 2$*,* $\lambda$ *and* $\mu$ *are real numbers)*

$$E^{\mu\tau}(Z) = \sum_{i=0}^{+\infty} \frac{(\mu\tau)^i}{i!} Z^{[i]} + \tau^w B + \tau^{w+1} B_1 + \tau^{w+2} B_2 + \tau^{w+3} B_3 + \tau^{w+4} B_4 + O(\tau^{w+5})$$

*and*

$$F^{\lambda\tau}(Z) = \sum_{j=0}^{+\infty} \frac{(\lambda\tau)^j}{j!} Z^{[j]} + \tau^w A + \tau^{w+1} A_1 + \tau^{w+2} A_2 + \tau^{w+3} A_3 + \tau^{w+4} A_4 + O(\tau^{w+5})$$

*are inverse operators reciprocally (i.e.,* $E^{\mu\tau} \circ F^{\lambda\tau} = identity$*), then*

*(i) when* $w = 2$*:*

$$\lambda + \mu = 0, \tag{8.21}$$

$$A + B = 0, \tag{8.22}$$

$$A_1 + \mu Z_z^{[1]} A + \lambda B_z Z^{[1]} + B_1 = 0, \tag{8.23}$$

$$\begin{aligned} & A_2 + \mu Z_z^{[1]} A_1 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A + \frac{\mu^2}{2} Z_z^{[2]} A \\ & + \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2 + B_z A = 0, \end{aligned} \tag{8.24}$$

$$\begin{aligned} & A_3 + \mu Z_z^{[1]} A_2 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\ & + \frac{\mu^2}{2} Z_z^{[2]} A_1 + \frac{\lambda\mu^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\mu^3}{6} Z_z^{[3]} A + \frac{\lambda^3}{6} B_z Z^{[3]} + \frac{\lambda^3}{2} B_{z^2} Z^{[1]} Z^{[2]} \\ & + \frac{\lambda^3}{6} B_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_1)_z Z^{[2]} + \frac{\lambda^2}{2} (B_1)_{z^2} (Z^{[1]})^2 + \lambda (B_2)_z Z^{[1]} + B_3 \\ & + \frac{\mu}{2} Z_{z^2}^{[1]} A^2 + B_z A_1 + \lambda B_{z^2} Z^{[1]} A + (B_1)_z A = 0, \end{aligned} \tag{8.25}$$

$$A_4 + \mu Z_z^{[1]} A_3 + \lambda\mu Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda^2\mu}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda^3\mu}{6} Z_{z^2}^{[1]} Z^{[3]} A$$

$$\begin{aligned}
&+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z^{[1]}_{z^3}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z^{[1]}_{z^4}(Z^{[1]})^3A+\frac{\mu^2}{2}Z^{[2]}_{z}A_2\\
&+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2A+\frac{\mu^3}{6}Z^{[3]}_{z}A_1\\
&+\frac{\lambda\mu^3}{6}Z^{[3]}_{z^2}Z^{[1]}A+\frac{\mu^4}{24}Z^{[4]}_{z}A+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4\\
&+\mu Z^{[1]}_{z^2}AA_1+\frac{\lambda\mu}{2}Z^{[1]}_{z^3}Z^{[1]}A^2+\frac{\mu^2}{4}Z^{[2]}_{z^2}A^2+B_zA_2+\lambda B_{z^2}Z^{[1]}A_1\\
&+\frac{\lambda^2}{2}B_{z^2}Z^{[2]}A+\frac{\lambda^2}{2}B_{z^3}(Z^{[1]})^2A+(B_1)_zA_1+\lambda(B_1)_{z^2}Z^{[1]}A+(B_2)_zA\\
&+\frac{1}{2}B_{z^2}A^2=0;
\end{aligned}\tag{8.26}$$

*(ii) when $w=3$:*

$$\lambda+\mu=0,\tag{8.31}$$

$$A+B=0,\tag{8.32}$$

$$A_1+\mu Z^{[1]}_{z}A+\lambda B_zZ^{[1]}+B_1=0,\tag{8.33}$$

$$\begin{aligned}
&A_2+\mu Z^{[1]}_{z}A_1+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A+\frac{\mu^2}{2}Z^{[2]}_{z}A\\
&+\frac{\lambda^2}{2}B_zZ^{[2]}+\frac{\lambda^2}{2}B_{z^2}(Z^{[1]})^2+\lambda(B_1)_zZ^{[1]}+B_2=0,
\end{aligned}\tag{8.34}$$

$$\begin{aligned}
&B_zA+A_3+\mu Z^{[1]}_{z}A_2+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A\\
&+\frac{\mu^2}{2}Z^{[2]}_{z}A_1+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A+\frac{\mu^3}{6}Z^{[3]}_{z}A+\frac{\lambda^3}{6}B_zZ^{[3]}+\frac{\lambda^3}{2}B_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}B_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_1)_zZ^{[2]}+\frac{\lambda^2}{2}(B_1)_{z^2}(Z^{[1]})^2+\lambda(B_2)_zZ^{[1]}+B_3=0,
\end{aligned}\tag{8.35}$$

$$\begin{aligned}
&\frac{\mu}{2}Z^{[1]}_{z^2}A^2+B_zA_1+\lambda B_{z^2}Z^{[1]}A+(B_1)_zA\\
&+A_4+\mu Z^{[1]}_{z}A_3+\lambda\mu Z^{[1]}_{z^2}Z^{[1]}A_2+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^2}Z^{[2]}A_1+\frac{\lambda^3\mu}{6}Z^{[1]}_{z^2}Z^{[3]}A\\
&+\frac{\lambda^2\mu}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z^{[1]}_{z^3}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z^{[1]}_{z^4}(Z^{[1]})^3A+\frac{\mu^2}{2}Z^{[2]}_{z}A_2\\
&+\frac{\lambda\mu^2}{2}Z^{[2]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^2}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2A+\frac{\mu^3}{6}Z^{[3]}_{z}A_1
\end{aligned}\tag{8.36}$$

$$+ \frac{\lambda\mu^3}{6} Z^{[3]}_{z^2} Z^{[1]} A + \frac{\mu^4}{24} Z^{[4]}_z A + \frac{\lambda^4}{24} B_z Z^{[4]} + \frac{\lambda^4}{8} B_{z^2} (Z^{[2]})^2 + \frac{\lambda^4}{6} B_{z^2} Z^{[1]} Z^{[3]}$$
$$+ \frac{\lambda^4}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{\lambda^4}{24} B_{z^4} (Z^{[1]})^4 + \frac{\lambda^3}{6} (B_1)_z Z^{[3]} + \frac{\lambda^3}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]}$$
$$+ \frac{\lambda^3}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_2)_z Z^{[2]} + \frac{\lambda^2}{2} (B_2)_{z^2} (Z^{[1]})^2 + \lambda (B_3)_z Z^{[1]} + B_4 = 0;$$

*(iii) when $w = 4$:*

$$\lambda + \mu = 0, \quad (8.41)$$

$$A + B = 0, \quad (8.42)$$

$$A_1 + \mu Z^{[1]}_z A + \lambda B_z Z^{[1]} + B_1 = 0, \quad (8.43)$$

$$A_2 + \mu Z^{[1]}_z A_1 + \lambda\mu Z^{[1]}_{z^2} Z^{[1]} A + \frac{\mu^2}{2} Z^{[2]}_z A \quad (8.44)$$
$$+ \frac{\lambda^2}{2} B_z Z^{[2]} + \frac{\lambda^2}{2} B_{z^2} (Z^{[1]})^2 + \lambda (B_1)_z Z^{[1]} + B_2 = 0,$$

$$A_3 + \mu Z^{[1]}_z A_2 + \lambda\mu Z^{[1]}_{z^2} Z^{[1]} A_1 + \frac{\lambda^2\mu}{2} Z^{[1]}_{z^2} Z^{[2]} A + \frac{\lambda^2\mu}{2} Z^{[1]}_{z^3} (Z^{[1]})^2 A$$
$$+ \frac{\mu^2}{2} Z^{[2]}_z A_1 + \frac{\lambda\mu^2}{2} Z^{[2]}_{z^2} Z^{[1]} A + \frac{\mu^3}{6} Z^{[3]}_z A + \frac{\lambda^3}{6} B_z Z^{[3]} + \frac{\lambda^3}{2} B_{z^2} Z^{[1]} Z^{[2]} \quad (8.45)$$
$$+ \frac{\lambda^3}{6} B_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_1)_z Z^{[2]} + \frac{\lambda^2}{2} (B_1)_{z^2} (Z^{[1]})^2 + \lambda (B_2)_z Z^{[1]} + B_3 = 0,$$

$$B_z A + A_4 + \mu Z^{[1]}_z A_3 + \lambda\mu Z^{[1]}_{z^2} Z^{[1]} A_2 + \frac{\lambda^2\mu}{2} Z^{[1]}_{z^2} Z^{[2]} A_1 + \frac{\lambda^3\mu}{6} Z^{[1]}_{z^2} Z^{[3]} A$$
$$+ \frac{\lambda^2\mu}{2} Z^{[1]}_{z^3} (Z^{[1]})^2 A_1 + \frac{\lambda^3\mu}{2} Z^{[1]}_{z^3} Z^{[1]} Z^{[2]} A + \frac{\lambda^3\mu}{6} Z^{[1]}_{z^4} (Z^{[1]})^3 A + \frac{\mu^2}{2} Z^{[2]}_z A_2$$
$$+ \frac{\lambda\mu^2}{2} Z^{[2]}_{z^2} Z^{[1]} A_1 + \frac{\lambda^2\mu^2}{4} Z^{[2]}_{z^2} Z^{[2]} A + \frac{\lambda^2\mu^2}{4} Z^{[2]}_{z^3} (Z^{[1]})^2 A + \frac{\mu^3}{6} Z^{[3]}_z A_1 \quad (8.46)$$
$$+ \frac{\lambda\mu^3}{6} Z^{[3]}_{z^2} Z^{[1]} A + \frac{\mu^4}{24} Z^{[4]}_z A + \frac{\lambda^4}{24} B_z Z^{[4]} + \frac{\lambda^4}{8} B_{z^2} (Z^{[2]})^2 + \frac{\lambda^4}{6} B_{z^2} Z^{[1]} Z^{[3]}$$
$$+ \frac{\lambda^4}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{\lambda^4}{24} B_{z^4} (Z^{[1]})^4 + \frac{\lambda^3}{6} (B_1)_z Z^{[3]} + \frac{\lambda^3}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]}$$
$$+ \frac{\lambda^3}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (B_2)_z Z^{[2]} + \frac{\lambda^2}{2} (B_2)_{z^2} (Z^{[1]})^2 + \lambda (B_3)_z Z^{[1]} + B_4 = 0;$$

*(iv) when $w > 4$:*

$$\lambda + \mu = 0, \quad (8.51)$$

$$A + B = 0, \quad (8.52)$$

$$A_1 + \mu Z^{[1]}_z A + \lambda B_z Z^{[1]} + B_1 = 0, \quad (8.53)$$

$$A_2 + \mu Z^{[1]}_z A_1 + \lambda\mu Z^{[1]}_{z^2} Z^{[1]} A + \frac{\mu^2}{2} Z^{[2]}_z A \quad (8.54)$$

$$+\frac{\lambda^2}{2}B_z Z^{[2]}+\frac{\lambda^2}{2}B_{z^2}(Z^{[1]})^2+\lambda(B_1)_z Z^{[1]}+B_2=0,$$

$$\begin{aligned}&A_3+\mu Z_z^{[1]}A_2+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A_1+\frac{\lambda^2\mu}{2}Z_{z^2}^{[1]}Z^{[2]}A+\frac{\lambda^2\mu}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A\\&+\frac{\mu^2}{2}Z_z^{[2]}A_1+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A+\frac{\mu^3}{6}Z_z^{[3]}A+\frac{\lambda^3}{6}B_zZ^{[3]}+\frac{\lambda^3}{2}B_{z^2}Z^{[1]}Z^{[2]}\\&+\frac{\lambda^3}{6}B_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_1)_zZ^{[2]}+\frac{\lambda^2}{2}(B_1)_{z^2}(Z^{[1]})^2+\lambda(B_2)_zZ^{[1]}+B_3=0,\end{aligned}\tag{8.55}$$

$$\begin{aligned}&A_4+\mu Z_z^{[1]}A_3+\lambda\mu Z_{z^2}^{[1]}Z^{[1]}A_2+\frac{\lambda^2\mu}{2}Z_{z^2}^{[1]}Z^{[2]}A_1+\frac{\lambda^3\mu}{6}Z_{z^2}^{[1]}Z^{[3]}A\\&+\frac{\lambda^2\mu}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A_1+\frac{\lambda^3\mu}{2}Z_{z^3}^{[1]}Z^{[1]}Z^{[2]}A+\frac{\lambda^3\mu}{6}Z_{z^4}^{[1]}(Z^{[1]})^3A+\frac{\mu^2}{2}Z_z^{[2]}A_2\\&+\frac{\lambda\mu^2}{2}Z_{z^2}^{[2]}Z^{[1]}A_1+\frac{\lambda^2\mu^2}{4}Z_{z^2}^{[2]}Z^{[2]}A+\frac{\lambda^2\mu^2}{4}Z_{z^3}^{[2]}(Z^{[1]})^2A+\frac{\mu^3}{6}Z_z^{[3]}A_1\\&+\frac{\lambda\mu^3}{6}Z_{z^2}^{[3]}Z^{[1]}A+\frac{\mu^4}{24}Z_z^{[4]}A+\frac{\lambda^4}{24}B_zZ^{[4]}+\frac{\lambda^4}{8}B_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}B_{z^2}Z^{[1]}Z^{[3]}\\&+\frac{\lambda^4}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}B_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(B_1)_zZ^{[3]}+\frac{\lambda^3}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\&+\frac{\lambda^3}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(B_2)_zZ^{[2]}+\frac{\lambda^2}{2}(B_2)_{z^2}(Z^{[1]})^2+\lambda(B_3)_zZ^{[1]}+B_4=0.\end{aligned}\tag{8.56}$$

□

**Example 1.** We know that the Euler-forward scheme (denoted by $G_{ef}^{\tau}$)

$$\widetilde{Z}=Z+\tau f(Z)\tag{9}$$

and the Euler-backward scheme (denoted by $G_{eb}^{\tau}$)

$$\widetilde{Z}=Z+\tau f(\widetilde{Z})\tag{10}$$

are both of order 1. It's easy to see (refer to [4])

$$G_{eb}^{-\tau}\circ G_{ef}^{\tau}=identity.\tag{11}$$

If we write their STO's ($w=2$) as

$$\begin{aligned}G_{eb}^{-\tau}(Z)=&\sum_{i=0}^{+\infty}\frac{(-\tau)^i}{i!}Z^{[i]}+\tau^wB+\tau^{w+1}B_1\\&+\tau^{w+2}B_2+\tau^{w+3}B_3+\tau^{w+4}B_4+O(\tau^{w+5})\end{aligned}\tag{12}$$

and

$$\begin{aligned}G_{ef}^{\tau}(Z)=&\sum_{j=0}^{+\infty}\frac{\tau^j}{j!}Z^{[j]}+\tau^wA+\tau^{w+1}A_1\\&+\tau^{w+2}A_2+\tau^{w+3}A_3+\tau^{w+4}A_4+O(\tau^{w+5})\end{aligned}\tag{13}$$

respectively, then obviously ($A_0=A$)

$$A_k=-\frac{Z^{[k+2]}}{(k+2)!},\quad k=0,1,\cdots,4,\tag{14}$$

i.e.,

$$A = -\frac{1}{2}Z^{[2]}; \tag{14.0}$$

$$A_1 = -\frac{1}{6}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{6}Z^{[1]}_{z}Z^{[2]}; \tag{14.1}$$

$$\begin{aligned} A_2 = &- \frac{1}{24}Z^{[1]}_{z^3}(Z^{[1]})^3 - \frac{1}{8}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} - \frac{1}{24}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 \\ &- \frac{1}{24}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]}; \end{aligned} \tag{14.2}$$

$$\begin{aligned} A_3 = &- \frac{1}{120}Z^{[1]}_{z^4}(Z^{[1]})^4 - \frac{1}{20}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[2]} - \frac{1}{40}Z^{[1]}_{z^2}(Z^{[2]})^2 \\ &- \frac{1}{30}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{30}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[2]} - \frac{1}{120}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^3 \\ &- \frac{1}{40}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} - \frac{1}{120}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{120}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]}; \end{aligned} \tag{14.3}$$

$$\begin{aligned} A_4 = &- \frac{1}{720}Z^{[1]}_{z^5}(Z^{[1]})^5 - \frac{1}{72}Z^{[1]}_{z^4}(Z^{[1]})^3Z^{[2]} - \frac{1}{48}Z^{[1]}_{z^3}Z^{[1]}(Z^{[2]})^2 \\ &- \frac{1}{72}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{72}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[1]}_{z}Z^{[2]} - \frac{1}{72}Z^{[1]}_{z^2}Z^{[2]}Z^{[1]}_{z^2}(Z^{[1]})^2 \\ &- \frac{1}{72}Z^{[1]}_{z^2}Z^{[2]}Z^{[1]}_{z}Z^{[2]} - \frac{1}{144}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^3}(Z^{[1]})^3 - \frac{1}{48}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} \\ &- \frac{1}{144}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{144}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]} - \frac{1}{720}Z^{[1]}_{z}Z^{[1]}_{z^4}(Z^{[1]})^4 \\ &- \frac{1}{120}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[2]} - \frac{1}{240}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[2]})^2 - \frac{1}{180}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}(Z^{[1]})^2 \\ &- \frac{1}{180}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[2]} - \frac{1}{720}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^3 - \frac{1}{240}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} \\ &- \frac{1}{720}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{1}{720}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]}. \end{aligned} \tag{14.4}$$

And then from (8.22)–(8.26) we obtain

$$B = \frac{1}{2}Z^{[2]}; \tag{15.0}$$

$$B_1 = -\frac{1}{3}Z^{[1]}_{z^2}(Z^{[1]})^2 - \frac{5}{6}Z^{[1]}_{z}Z^{[2]}; \tag{15.1}$$

$$\begin{aligned} B_2 = &\frac{1}{8}Z^{[1]}_{z^3}(Z^{[1]})^3 + \frac{7}{8}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} + \frac{11}{24}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 \\ &+ \frac{23}{24}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]}; \end{aligned} \tag{15.2}$$

$$B_3 = - \frac{1}{30}Z^{[1]}_{z^4}(Z^{[1]})^4 - \frac{9}{20}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[2]} - \frac{19}{40}Z^{[1]}_{z^2}(Z^{[2]})^2$$

$$- \frac{14}{30} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z^2} (Z^{[1]})^2 - \frac{29}{30} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z} Z^{[2]} - \frac{19}{120} Z^{[1]}_{z} Z^{[1]}_{z^3} (Z^{[1]})^3 \qquad (15.3)$$
$$- \frac{39}{40} Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[2]} - \frac{59}{120} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z^2} (Z^{[1]})^2 - \frac{119}{120} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[2]};$$

$$\begin{aligned}
B_4 =& \frac{1}{144} Z^{[1]}_{z^5} (Z^{[1]})^5 + \frac{11}{72} Z^{[1]}_{z^4} (Z^{[1]})^3 Z^{[2]} + \frac{23}{48} Z^{[1]}_{z^3} Z^{[1]} (Z^{[2]})^2 \\
&+ \frac{17}{72} Z^{[1]}_{z^3} (Z^{[1]})^2 Z^{[1]}_{z^2} (Z^{[1]})^2 + \frac{35}{72} Z^{[1]}_{z^3} (Z^{[1]})^2 Z^{[1]}_{z} Z^{[2]} + \frac{35}{72} Z^{[1]}_{z^2} Z^{[2]} Z^{[1]}_{z^2} (Z^{[1]})^2 \\
&+ \frac{71}{72} Z^{[1]}_{z^2} Z^{[2]} Z^{[1]}_{z} Z^{[2]} + \frac{23}{144} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z^3} (Z^{[1]})^3 + \frac{47}{48} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{71}{144} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z} Z^{[1]}_{z^2} (Z^{[1]})^2 + \frac{143}{144} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[2]} + \frac{29}{720} Z^{[1]}_{z} Z^{[1]}_{z^4} (Z^{[1]})^4 \qquad (15.4) \\
&+ \frac{59}{120} Z^{[1]}_{z} Z^{[1]}_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{119}{240} Z^{[1]}_{z} Z^{[1]}_{z^2} (Z^{[2]})^2 + \frac{89}{180} Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z^2} (Z^{[1]})^2 \\
&+ \frac{179}{180} Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[1]}_{z} Z^{[2]} + \frac{119}{720} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z^3} (Z^{[1]})^3 + \frac{239}{240} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{359}{720} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z^2} (Z^{[1]})^2 + \frac{719}{720} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[2]}.
\end{aligned}$$

**Example 2.** We know that the 2nd-order mid-point rule (denoted by $G^{\tau}_{mp}$)

$$\widetilde{Z} = Z + \tau f\left(\frac{\widetilde{Z} + Z}{2}\right) \qquad (16)$$

is exactly the composition of the Euler-forward scheme $G^{\frac{\tau}{2}}_{ef}$ and the Euler-backward scheme $G^{\frac{\tau}{2}}_{eb}$ (refer to [4]):

$$G^{\tau}_{mp} = G^{\frac{\tau}{2}}_{ef} \circ G^{\frac{\tau}{2}}_{eb}. \qquad (17)$$

If we write the expansion ($w = 2$) of $G^{\tau}_{mp}$ as

$$\begin{aligned}
G^{\tau}_{mp}(Z) = & \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^w M + \tau^{w+1} M_1 \qquad (18) \\
& + \tau^{w+2} M_2 + \tau^{w+3} M_3 + \tau^{w+4} M_4 + O(\tau^{w+5}),
\end{aligned}$$

then from (12-14), (15.0-15.4) and (7.2) we have

$$M = 0; \qquad (19.0)$$

$$M_1 = -\frac{1}{24} Z^{[1]}_{z^2} (Z^{[1]})^2 + \frac{1}{12} Z^{[1]}_{z} Z^{[2]}; \qquad (19.1)$$

$$\begin{aligned}
M_2 = & -\frac{1}{48} Z^{[1]}_{z^3} (Z^{[1]})^3 + \frac{1}{48} Z^{[1]}_{z} Z^{[1]}_{z^2} (Z^{[1]})^2 \\
& + \frac{1}{12} Z^{[1]}_{z} Z^{[1]}_{z} Z^{[2]}; \qquad (19.2)
\end{aligned}$$

$$\begin{aligned}M_3 = &-\frac{11}{1920}Z^{[1]}_{z^4}(Z^{[1]})^4 - \frac{3}{160}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[2]} + \frac{1}{160}Z^{[1]}_{z^2}(Z^{[2]})^2\\ &-\frac{1}{480}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}(Z^{[1]})^2 + \frac{7}{240}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[2]} + \frac{1}{480}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^3\\ &+\frac{3}{80}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]} + \frac{11}{480}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 + \frac{13}{240}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]};\end{aligned} \tag{19.3}$$

$$\begin{aligned}M_4 = &-\frac{13}{11520}Z^{[1]}_{z^5}(Z^{[1]})^5 - \frac{5}{576}Z^{[1]}_{z^4}(Z^{[1]})^3Z^{[2]} - \frac{1}{192}Z^{[1]}_{z^3}Z^{[1]}(Z^{[2]})^2\\ &-\frac{7}{1152}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[1]}_{z^2}(Z^{[1]})^2 + \frac{1}{576}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[1]}_{z}Z^{[2]} + \frac{1}{576}Z^{[1]}_{z^2}Z^{[2]}Z^{[1]}_{z^2}(Z^{[1]})^2\\ &+\frac{5}{288}Z^{[1]}_{z^2}Z^{[2]}Z^{[1]}_{z}Z^{[2]} - \frac{1}{576}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^3}(Z^{[1]})^3 + \frac{1}{96}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]}\\ &+\frac{5}{576}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 + \frac{7}{288}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]} - \frac{1}{11520}Z^{[1]}_{z}Z^{[1]}_{z^4}(Z^{[1]})^4\\ &+\frac{7}{960}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^2Z^{[2]} + \frac{11}{960}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[2]})^2 + \frac{29}{2880}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z^2}(Z^{[1]})^2\\ &+\frac{37}{1440}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[1]}_{z}Z^{[2]} + \frac{11}{2880}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^3}(Z^{[1]})^3 + \frac{13}{480}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}Z^{[1]}Z^{[2]}\\ &+\frac{41}{2880}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z^2}(Z^{[1]})^2 + \frac{43}{1440}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[1]}_{z}Z^{[2]}.\end{aligned} \tag{19.4}$$

## 4. Conjugate Operator of Step-Transition Operator

In the beginning of this section, let's introduce the definition of *conjugate operator*:

**Definition 1** (see [12]). *Providing $E^\tau$, $F^\tau$ and $G^\tau$ are three operators of form (5), $E^\tau$ is said to be conjugate to $F^\tau$ through $G^\tau$ iff*

$$G^{\lambda\tau} \circ E^\tau = F^\tau \circ G^{\lambda\tau} \tag{20}$$

*for some $\lambda \neq 0$ and for any function $f$ and any sufficiently small step-size $\tau$.*

From Theorem 1, we obtain straightforwardly

**Theorem 3.** *Given ($w \geq 2$)*

$$E^\tau(Z) = \sum_{j=0}^{+\infty}\frac{\tau^j}{j!}Z^{[j]} + \tau^w A + \tau^{w+1}A_1 + \tau^{w+2}A_2 + \tau^{w+3}A_3 + \tau^{w+4}A_4 + O(\tau^{w+5}), \tag{21}$$

$$F^\tau(Z) = \sum_{j=0}^{+\infty}\frac{\tau^j}{j!}Z^{[j]} + \tau^w M + \tau^{w+1}M_1 + \tau^{w+2}M_2 + \tau^{w+3}M_3 + \tau^{w+4}M_4 + O(\tau^{w+5}) \tag{22}$$

*and*

$$G^{\lambda\tau}(Z) = \sum_{i=0}^{+\infty}\frac{(\lambda\tau)^i}{i!}Z^{[i]} + \tau^w B + \tau^{w+1}B_1 + \tau^{w+2}B_2 + \tau^{w+3}B_3 + \tau^{w+4}B_4 + O(\tau^{w+5}), \tag{23}$$

*if $E^\tau$ is conjugate to $F^\tau$ through $G^\tau$ with conjugate coefficient $\lambda$: $G^{\lambda\tau} \circ E^\tau = F^\tau \circ G^{\lambda\tau}$, then* *(i) when $w = 2$:*

$$A = M, \tag{24.22}$$

$$A_1 + \lambda Z_z^{[1]} A + B_z Z^{[1]} = Z_z^{[1]} B + \lambda M_z Z^{[1]} + M_1, \tag{24.23}$$

$$\begin{aligned}
&A_2 + \lambda Z_z^{[1]} A_1 + \lambda Z_{z^2}^{[1]} Z^{[1]} A + \frac{\lambda^2}{2} Z_z^{[2]} A \\
&+ \frac{1}{2} B_z Z^{[2]} + \frac{1}{2} B_{z^2} (Z^{[1]})^2 + (B_1)_z Z^{[1]} + B_z A = \\
&Z_z^{[1]} B_1 + \lambda Z_{z^2}^{[1]} Z^{[1]} B + \frac{1}{2} Z_z^{[2]} B \\
&+ \frac{\lambda^2}{2} M_z Z^{[2]} + \frac{\lambda^2}{2} M_{z^2} (Z^{[1]})^2 + \lambda (M_1)_z Z^{[1]} + M_2 + M_z B,
\end{aligned} \tag{24.24}$$

$$\begin{aligned}
&A_3 + \lambda Z_z^{[1]} A_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\
&+ \frac{\lambda^2}{2} Z_z^{[2]} A_1 + \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\lambda^3}{6} Z_z^{[3]} A + \frac{1}{6} B_z Z^{[3]} + \frac{1}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} B_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_1)_z Z^{[2]} + \frac{1}{2} (B_1)_{z^2} (Z^{[1]})^2 + (B_2)_z Z^{[1]} \\
&+ \frac{\lambda}{2} Z_{z^2}^{[1]} A^2 + B_z A_1 + B_{z^2} Z^{[1]} A + (B_1)_z A = \\
&Z_z^{[1]} B_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} B_1 + \frac{\lambda^2}{2} Z_{z^2}^{[1]} Z^{[2]} B + \frac{\lambda^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 B \\
&+ \frac{1}{2} Z_z^{[2]} B_1 + \frac{\lambda}{2} Z_{z^2}^{[2]} Z^{[1]} B + \frac{1}{6} Z_z^{[3]} B + \frac{\lambda^3}{6} M_z Z^{[3]} + \frac{\lambda^3}{2} M_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{\lambda^3}{6} M_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (M_1)_z Z^{[2]} + \frac{\lambda^2}{2} (M_1)_{z^2} (Z^{[1]})^2 + \lambda (M_2)_z Z^{[1]} + M_3 \\
&+ \frac{1}{2} Z_{z^2}^{[1]} B^2 + M_z B_1 + \lambda M_{z^2} Z^{[1]} B + (M_1)_z B,
\end{aligned} \tag{24.25}$$

$$\begin{aligned}
&A_4 + \lambda Z_z^{[1]} A_3 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda}{6} Z_{z^2}^{[1]} Z^{[3]} A \\
&+ \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\lambda^2}{2} Z_z^{[2]} A_2 \\
&+ \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A_1 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} Z^{[2]} A + \frac{\lambda^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 A + \frac{\lambda^3}{6} Z_z^{[3]} A_1 \\
&+ \frac{\lambda^3}{6} Z_{z^2}^{[3]} Z^{[1]} A + \frac{\lambda^4}{24} Z_z^{[4]} A + \frac{1}{24} B_z Z^{[4]} + \frac{1}{8} B_{z^2} (Z^{[2]})^2 + \frac{1}{6} B_{z^2} Z^{[1]} Z^{[3]} \\
&+ \frac{1}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{1}{24} B_{z^4} (Z^{[1]})^4 + \frac{1}{6} (B_1)_z Z^{[3]} + \frac{1}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_2)_z Z^{[2]} + \frac{1}{2} (B_2)_{z^2} (Z^{[1]})^2 + (B_3)_z Z^{[1]} \\
&+ \lambda Z_{z^2}^{[1]} A A_1 + \frac{\lambda}{2} Z_{z^3}^{[1]} Z^{[1]} A^2 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} A^2 + B_z A_2 + B_{z^2} Z^{[1]} A_1
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}B_{z^2}Z^{[2]}A+\frac{1}{2}B_{z^3}(Z^{[1]})^2A+(B_1)_zA_1+(B_1)_{z^2}Z^{[1]}A+(B_2)_zA\\
&+\frac{1}{2}B_{z^2}A^2=\\
&Z_z^{[1]}B_3+\lambda Z_{z^2}^{[1]}Z^{[1]}B_2+\frac{\lambda^2}{2}Z_{z^2}^{[1]}Z^{[2]}B_1+\frac{\lambda^3}{6}Z_{z^2}^{[1]}Z^{[3]}B\\
&+\frac{\lambda^2}{2}Z_{z^3}^{[1]}(Z^{[1]})^2B_1+\frac{\lambda^3}{2}Z_{z^3}^{[1]}Z^{[1]}Z^{[2]}B+\frac{\lambda^3}{6}Z_{z^4}^{[1]}(Z^{[1]})^3B+\frac{1}{2}Z_z^{[2]}B_2\\
&+\frac{\lambda}{2}Z_{z^2}^{[2]}Z^{[1]}B_1+\frac{\lambda^2}{4}Z_{z^2}^{[2]}Z^{[2]}B+\frac{\lambda^2}{4}Z_{z^3}^{[2]}(Z^{[1]})^2B+\frac{1}{6}Z_z^{[3]}B_1\\
&+\frac{\lambda}{6}Z_{z^2}^{[3]}Z^{[1]}B+\frac{1}{24}Z_z^{[4]}B+\frac{\lambda^4}{24}M_zZ^{[4]}+\frac{\lambda^4}{8}M_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}M_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}M_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}M_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(M_1)_zZ^{[3]}+\frac{\lambda^3}{2}(M_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(M_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(M_2)_zZ^{[2]}+\frac{\lambda^2}{2}(M_2)_{z^2}(Z^{[1]})^2+\lambda(M_3)_zZ^{[1]}+M_4\\
&+Z_{z^2}^{[1]}BB_1+\frac{\lambda}{2}Z_{z^3}^{[1]}Z^{[1]}B^2+\frac{1}{4}Z_{z^2}^{[2]}B^2+M_zB_2+\lambda M_{z^2}Z^{[1]}B_1\\
&+\frac{\lambda^2}{2}M_{z^2}Z^{[2]}B+\frac{\lambda^2}{2}M_{z^3}(Z^{[1]})^2B+(M_1)_zB_1+\lambda(M_1)_{z^2}Z^{[1]}B+(M_2)_zB\\
&+\frac{1}{2}M_{z^2}B^2;
\end{aligned}
\tag{24.26}
$$

*(ii) when $w=3$:*

$$A=M, \tag{24.32}$$

$$A_1+\lambda Z_z^{[1]}A+B_zZ^{[1]}=Z_z^{[1]}B+\lambda M_zZ^{[1]}+M_1, \tag{24.33}$$

$$
\begin{aligned}
&A_2+\lambda Z_z^{[1]}A_1+\lambda Z_{z^2}^{[1]}Z^{[1]}A+\frac{\lambda^2}{2}Z_z^{[2]}A\\
&+\frac{1}{2}B_zZ^{[2]}+\frac{1}{2}B_{z^2}(Z^{[1]})^2+(B_1)_zZ^{[1]}=\\
&Z_z^{[1]}B_1+\lambda Z_{z^2}^{[1]}Z^{[1]}B+\frac{1}{2}Z_z^{[2]}B\\
&+\frac{\lambda^2}{2}M_zZ^{[2]}+\frac{\lambda^2}{2}M_{z^2}(Z^{[1]})^2+\lambda(M_1)_zZ^{[1]}+M_2,
\end{aligned}
\tag{24.34}
$$

$$
\begin{aligned}
&B_zA+A_3+\lambda Z_z^{[1]}A_2+\lambda Z_{z^2}^{[1]}Z^{[1]}A_1+\frac{\lambda}{2}Z_{z^2}^{[1]}Z^{[2]}A+\frac{\lambda}{2}Z_{z^3}^{[1]}(Z^{[1]})^2A\\
&+\frac{\lambda^2}{2}Z_z^{[2]}A_1+\frac{\lambda^2}{2}Z_{z^2}^{[2]}Z^{[1]}A+\frac{\lambda^3}{6}Z_z^{[3]}A+\frac{1}{6}B_zZ^{[3]}+\frac{1}{2}B_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{1}{6}B_{z^3}(Z^{[1]})^3+\frac{1}{2}(B_1)_zZ^{[2]}+\frac{1}{2}(B_1)_{z^2}(Z^{[1]})^2+(B_2)_zZ^{[1]}=\\
&M_zB+Z_z^{[1]}B_2+\lambda Z_{z^2}^{[1]}Z^{[1]}B_1+\frac{\lambda^2}{2}Z_{z^2}^{[1]}Z^{[2]}B+\frac{\lambda^2}{2}Z_{z^3}^{[1]}(Z^{[1]})^2B\\
&+\frac{1}{2}Z_z^{[2]}B_1+\frac{\lambda}{2}Z_{z^2}^{[2]}Z^{[1]}B+\frac{1}{6}Z_z^{[3]}B+\frac{\lambda^3}{6}M_zZ^{[3]}+\frac{\lambda^3}{2}M_{z^2}Z^{[1]}Z^{[2]}
\end{aligned}
\tag{24.35}
$$

$$+\frac{\lambda^3}{6}M_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(M_1)_zZ^{[2]}+\frac{\lambda^2}{2}(M_1)_{z^2}(Z^{[1]})^2+\lambda(M_2)_zZ^{[1]}+M_3,$$

$$\begin{aligned}
&\frac{\lambda}{2}Z^{[1]}_{z^2}A^2+B_zA_1+B_{z^2}Z^{[1]}A+(B_1)_zA\\
&+A_4+\lambda Z^{[1]}_zA_3+\lambda Z^{[1]}_{z^2}Z^{[1]}A_2+\frac{\lambda}{2}Z^{[1]}_{z^2}Z^{[2]}A_1+\frac{\lambda}{6}Z^{[1]}_{z^2}Z^{[3]}A\\
&+\frac{\lambda}{2}Z^{[1]}_{z^3}(Z^{[1]})^2A_1+\frac{\lambda}{2}Z^{[1]}_{z^3}Z^{[1]}Z^{[2]}A+\frac{\lambda}{6}Z^{[1]}_{z^4}(Z^{[1]})^3A+\frac{\lambda^2}{2}Z^{[2]}_zA_2\\
&+\frac{\lambda^2}{2}Z^{[2]}_{z^2}Z^{[1]}A_1+\frac{\lambda^2}{4}Z^{[2]}_{z^2}Z^{[2]}A+\frac{\lambda^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2A+\frac{\lambda^3}{6}Z^{[3]}_zA_1\\
&+\frac{\lambda^3}{6}Z^{[3]}_{z^2}Z^{[1]}A+\frac{\lambda^4}{24}Z^{[4]}_zA+\frac{1}{24}B_zZ^{[4]}+\frac{1}{8}B_{z^2}(Z^{[2]})^2+\frac{1}{6}B_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{1}{4}B_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{1}{24}B_{z^4}(Z^{[1]})^4+\frac{1}{6}(B_1)_zZ^{[3]}+\frac{1}{2}(B_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{1}{6}(B_1)_{z^3}(Z^{[1]})^3+\frac{1}{2}(B_2)_zZ^{[2]}+\frac{1}{2}(B_2)_{z^2}(Z^{[1]})^2+(B_3)_zZ^{[1]}=\\
&\frac{1}{2}Z^{[1]}_{z^2}B^2+M_zB_1+\lambda M_{z^2}Z^{[1]}B+(M_1)_zB\\
&Z^{[1]}_zB_3+\lambda Z^{[1]}_{z^2}Z^{[1]}B_2+\frac{\lambda^2}{2}Z^{[1]}_{z^2}Z^{[2]}B_1+\frac{\lambda^3}{6}Z^{[1]}_{z^2}Z^{[3]}B\\
&+\frac{\lambda^2}{2}Z^{[1]}_{z^3}(Z^{[1]})^2B_1+\frac{\lambda^3}{2}Z^{[1]}_{z^3}Z^{[1]}Z^{[2]}B+\frac{\lambda^3}{6}Z^{[1]}_{z^4}(Z^{[1]})^3B+\frac{1}{2}Z^{[2]}_zB_2\\
&+\frac{\lambda}{2}Z^{[2]}_{z^2}Z^{[1]}B_1+\frac{\lambda^2}{4}Z^{[2]}_{z^2}Z^{[2]}B+\frac{\lambda^2}{4}Z^{[2]}_{z^3}(Z^{[1]})^2B+\frac{1}{6}Z^{[3]}_zB_1\\
&+\frac{\lambda}{6}Z^{[3]}_{z^2}Z^{[1]}B+\frac{1}{24}Z^{[4]}_zB+\frac{\lambda^4}{24}M_zZ^{[4]}+\frac{\lambda^4}{8}M_{z^2}(Z^{[2]})^2+\frac{\lambda^4}{6}M_{z^2}Z^{[1]}Z^{[3]}\\
&+\frac{\lambda^4}{4}M_{z^3}(Z^{[1]})^2Z^{[2]}+\frac{\lambda^4}{24}M_{z^4}(Z^{[1]})^4+\frac{\lambda^3}{6}(M_1)_zZ^{[3]}+\frac{\lambda^3}{2}(M_1)_{z^2}Z^{[1]}Z^{[2]}\\
&+\frac{\lambda^3}{6}(M_1)_{z^3}(Z^{[1]})^3+\frac{\lambda^2}{2}(M_2)_zZ^{[2]}+\frac{\lambda^2}{2}(M_2)_{z^2}(Z^{[1]})^2+\lambda(M_3)_zZ^{[1]}+M_4;
\end{aligned}\tag{24.36}$$

*(iii) when $w=4$:*

$$A=M,\tag{24.42}$$

$$A_1+\lambda Z^{[1]}_zA+B_zZ^{[1]}=Z^{[1]}_zB+\lambda M_zZ^{[1]}+M_1,\tag{24.43}$$

$$\begin{aligned}
&A_2+\lambda Z^{[1]}_zA_1+\lambda Z^{[1]}_{z^2}Z^{[1]}A+\frac{\lambda^2}{2}Z^{[2]}_zA\\
&+\frac{1}{2}B_zZ^{[2]}+\frac{1}{2}B_{z^2}(Z^{[1]})^2+(B_1)_zZ^{[1]}=\\
&Z^{[1]}_zB_1+\lambda Z^{[1]}_{z^2}Z^{[1]}B+\frac{1}{2}Z^{[2]}_zB\\
&+\frac{\lambda^2}{2}M_zZ^{[2]}+\frac{\lambda^2}{2}M_{z^2}(Z^{[1]})^2+\lambda(M_1)_zZ^{[1]}+M_2,
\end{aligned}\tag{24.44}$$

$$\begin{aligned}
&A_3 + \lambda Z_z^{[1]} A_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\
&+ \frac{\lambda^2}{2} Z_z^{[2]} A_1 + \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\lambda^3}{6} Z_z^{[3]} A + \frac{1}{6} B_z Z^{[3]} + \frac{1}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} B_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_1)_z Z^{[2]} + \frac{1}{2} (B_1)_{z^2} (Z^{[1]})^2 + (B_2)_z Z^{[1]} = \\
&Z_z^{[1]} B_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} B_1 + \frac{\lambda^2}{2} Z_{z^2}^{[1]} Z^{[2]} B + \frac{\lambda^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 B \\
&+ \frac{1}{2} Z_z^{[2]} B_1 + \frac{\lambda}{2} Z_{z^2}^{[2]} Z^{[1]} B + \frac{1}{6} Z_z^{[3]} B + \frac{\lambda^3}{6} M_z Z^{[3]} + \frac{\lambda^3}{2} M_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{\lambda^3}{6} M_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (M_1)_z Z^{[2]} + \frac{\lambda^2}{2} (M_1)_{z^2} (Z^{[1]})^2 + \lambda (M_2)_z Z^{[1]} + M_3,
\end{aligned} \tag{24.45}$$

$$\begin{aligned}
&B_z A + A_4 + \lambda Z_z^{[1]} A_3 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda}{6} Z_{z^2}^{[1]} Z^{[3]} A \\
&+ \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\lambda^2}{2} Z_z^{[2]} A_2 \\
&+ \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A_1 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} Z^{[2]} A + \frac{\lambda^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 A + \frac{\lambda^3}{6} Z_z^{[3]} A_1 \\
&+ \frac{\lambda^3}{6} Z_{z^2}^{[3]} Z^{[1]} A + \frac{\lambda^4}{24} Z_z^{[4]} A + \frac{1}{24} B_z Z^{[4]} + \frac{1}{8} B_{z^2} (Z^{[2]})^2 + \frac{1}{6} B_{z^2} Z^{[1]} Z^{[3]} \\
&+ \frac{1}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{1}{24} B_{z^4} (Z^{[1]})^4 + \frac{1}{6} (B_1)_z Z^{[3]} + \frac{1}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_2)_z Z^{[2]} + \frac{1}{2} (B_2)_{z^2} (Z^{[1]})^2 + (B_3)_z Z^{[1]} = \\
&M_z B + Z_z^{[1]} B_3 + \lambda Z_{z^2}^{[1]} Z^{[1]} B_2 + \frac{\lambda^2}{2} Z_{z^2}^{[1]} Z^{[2]} B_1 + \frac{\lambda^3}{6} Z_{z^2}^{[1]} Z^{[3]} B \\
&+ \frac{\lambda^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 B_1 + \frac{\lambda^3}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} B + \frac{\lambda^3}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 B + \frac{1}{2} Z_z^{[2]} B_2 \\
&+ \frac{\lambda}{2} Z_{z^2}^{[2]} Z^{[1]} B_1 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} Z^{[2]} B + \frac{\lambda^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 B + \frac{1}{6} Z_z^{[3]} B_1 \\
&+ \frac{\lambda}{6} Z_{z^2}^{[3]} Z^{[1]} B + \frac{1}{24} Z_z^{[4]} B + \frac{\lambda^4}{24} M_z Z^{[4]} + \frac{\lambda^4}{8} M_{z^2} (Z^{[2]})^2 + \frac{\lambda^4}{6} M_{z^2} Z^{[1]} Z^{[3]} \\
&+ \frac{\lambda^4}{4} M_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{\lambda^4}{24} M_{z^4} (Z^{[1]})^4 + \frac{\lambda^3}{6} (M_1)_z Z^{[3]} + \frac{\lambda^3}{2} (M_1)_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{\lambda^3}{6} (M_1)_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (M_2)_z Z^{[2]} + \frac{\lambda^2}{2} (M_2)_{z^2} (Z^{[1]})^2 + \lambda (M_3)_z Z^{[1]} + M_4;
\end{aligned} \tag{24.46}$$

*(iv) when $w > 4$:*

$$A = M, \tag{24.52}$$

$$A_1 + \lambda Z_z^{[1]} A + B_z Z^{[1]} = Z_z^{[1]} B + \lambda M_z Z^{[1]} + M_1, \tag{24.53}$$

$$\begin{aligned}
&A_2 + \lambda Z_z^{[1]} A_1 + \lambda Z_{z^2}^{[1]} Z^{[1]} A + \frac{\lambda^2}{2} Z_z^{[2]} A \\
&+ \frac{1}{2} B_z Z^{[2]} + \frac{1}{2} B_{z^2} (Z^{[1]})^2 + (B_1)_z Z^{[1]} =
\end{aligned} \tag{24.54}$$

$$
\begin{aligned}
&Z_z^{[1]}B_1 + \lambda Z_{z^2}^{[1]} Z^{[1]} B + \frac{1}{2} Z_z^{[2]} B \\
&+ \frac{\lambda^2}{2} M_z Z^{[2]} + \frac{\lambda^2}{2} M_{z^2} (Z^{[1]})^2 + \lambda (M_1)_z Z^{[1]} + M_2,
\end{aligned}
$$

$$
\begin{aligned}
&A_3 + \lambda Z_z^{[1]} A_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_1 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A + \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A \\
&+ \frac{\lambda^2}{2} Z_z^{[2]} A_1 + \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A + \frac{\lambda^3}{6} Z_z^{[3]} A + \frac{1}{6} B_z Z^{[3]} + \frac{1}{2} B_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} B_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_1)_z Z^{[2]} + \frac{1}{2} (B_1)_{z^2} (Z^{[1]})^2 + (B_2)_z Z^{[1]} = \\
&Z_z^{[1]} B_2 + \lambda Z_{z^2}^{[1]} Z^{[1]} B_1 + \frac{\lambda^2}{2} Z_{z^2}^{[1]} Z^{[2]} B + \frac{\lambda^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 B \\
&+ \frac{1}{2} Z_z^{[2]} B_1 + \frac{\lambda}{2} Z_{z^2}^{[2]} Z^{[1]} B + \frac{1}{6} Z_z^{[3]} B + \frac{\lambda^3}{6} M_z Z^{[3]} + \frac{\lambda^3}{2} M_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{\lambda^3}{6} M_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (M_1)_z Z^{[2]} + \frac{\lambda^2}{2} (M_1)_{z^2} (Z^{[1]})^2 + \lambda (M_2)_z Z^{[1]} + M_3,
\end{aligned}
\tag{24.55}
$$

$$
\begin{aligned}
&A_4 + \lambda Z_z^{[1]} A_3 + \lambda Z_{z^2}^{[1]} Z^{[1]} A_2 + \frac{\lambda}{2} Z_{z^2}^{[1]} Z^{[2]} A_1 + \frac{\lambda}{6} Z_{z^2}^{[1]} Z^{[3]} A \\
&+ \frac{\lambda}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 A_1 + \frac{\lambda}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} A + \frac{\lambda}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 A + \frac{\lambda^2}{2} Z_z^{[2]} A_2 \\
&+ \frac{\lambda^2}{2} Z_{z^2}^{[2]} Z^{[1]} A_1 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} Z^{[2]} A + \frac{\lambda^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 A + \frac{\lambda^3}{6} Z_z^{[3]} A_1 \\
&+ \frac{\lambda^3}{6} Z_{z^2}^{[3]} Z^{[1]} A + \frac{\lambda^4}{24} Z_z^{[4]} A + \frac{1}{24} B_z Z^{[4]} + \frac{1}{8} B_{z^2} (Z^{[2]})^2 + \frac{1}{6} B_{z^2} Z^{[1]} Z^{[3]} \\
&+ \frac{1}{4} B_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{1}{24} B_{z^4} (Z^{[1]})^4 + \frac{1}{6} (B_1)_z Z^{[3]} + \frac{1}{2} (B_1)_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{1}{6} (B_1)_{z^3} (Z^{[1]})^3 + \frac{1}{2} (B_2)_z Z^{[2]} + \frac{1}{2} (B_2)_{z^2} (Z^{[1]})^2 + (B_3)_z Z^{[1]} = \\
&Z_z^{[1]} B_3 + \lambda Z_{z^2}^{[1]} Z^{[1]} B_2 + \frac{\lambda^2}{2} Z_{z^2}^{[1]} Z^{[2]} B_1 + \frac{\lambda^3}{6} Z_{z^2}^{[1]} Z^{[3]} B \\
&+ \frac{\lambda^2}{2} Z_{z^3}^{[1]} (Z^{[1]})^2 B_1 + \frac{\lambda^3}{2} Z_{z^3}^{[1]} Z^{[1]} Z^{[2]} B + \frac{\lambda^3}{6} Z_{z^4}^{[1]} (Z^{[1]})^3 B + \frac{1}{2} Z_z^{[2]} B_2 \\
&+ \frac{\lambda}{2} Z_{z^2}^{[2]} Z^{[1]} B_1 + \frac{\lambda^2}{4} Z_{z^2}^{[2]} Z^{[2]} B + \frac{\lambda^2}{4} Z_{z^3}^{[2]} (Z^{[1]})^2 B + \frac{1}{6} Z_z^{[3]} B_1 \\
&+ \frac{\lambda}{6} Z_{z^2}^{[3]} Z^{[1]} B + \frac{1}{24} Z_z^{[4]} B + \frac{\lambda^4}{24} M_z Z^{[4]} + \frac{\lambda^4}{8} M_{z^2} (Z^{[2]})^2 + \frac{\lambda^4}{6} M_{z^2} Z^{[1]} Z^{[3]} \\
&+ \frac{\lambda^4}{4} M_{z^3} (Z^{[1]})^2 Z^{[2]} + \frac{\lambda^4}{24} M_{z^4} (Z^{[1]})^4 + \frac{\lambda^3}{6} (M_1)_z Z^{[3]} + \frac{\lambda^3}{2} (M_1)_{z^2} Z^{[1]} Z^{[2]} \\
&+ \frac{\lambda^3}{6} (M_1)_{z^3} (Z^{[1]})^3 + \frac{\lambda^2}{2} (M_2)_z Z^{[2]} + \frac{\lambda^2}{2} (M_2)_{z^2} (Z^{[1]})^2 + \lambda (M_3)_z Z^{[1]} + M_4.
\end{aligned}
\tag{24.56}
$$

□

A very interesting example[1,3,4,7,8,9,14] is that the trapezoid rule (denoted by $G_{tz}^{\tau}$)

$$\widetilde{Z} = Z + \frac{\tau}{2}[f(\widetilde{Z}) + f(Z)] \tag{25}$$

is conjugate to the mid-point rule (16) through the Euler-forward scheme $G_{ef}^{\tau}$ (see equation

(9)) with conjugate coefficient $\lambda = \frac{1}{2}$:

$$G_{ef}^{\frac{\tau}{2}} \circ G_{tz}^{\tau} = G_{mp}^{\tau} \circ G_{ef}^{\frac{\tau}{2}}. \tag{26}$$

**Remark 1.** From the expansions of $G_{ef}^{\frac{\tau}{2}}$ and $G_{tz}^{\tau}$ (refer to (14) and the expansion for the trapezoid rule in [13]), one can also obtain that of $G_{mp}^{\tau}$. The result is exactly the same as (19.0-19.4).

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