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TWO-SCALE CURVED ELEMENT METHOD FOR ELLIPTIC PROBLEMS WITH SMALL PERIODIC COEFFICIENTS*[1)]

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Abstract

This paper is concerned with the second order elliptic problems with small periodic coefficients on a bounded domain with a curved boundary. A two-scale curved element method which couples linear elements and isoparametric elements is proposed. The error estimate is obtained over the given smooth domain. Furthermore an additive Schwarz method is provided for the isoparametric element method.

Key words: Two-scale, Curred element, Small periodic coefficients.

1. Introduction

Multiscale phenomenon is often encountered in science and engineering. Typical examples include composite materials and flows in porous media. They are usually described by partial differential equations with highly oscillatory coefficients. Solving these problems by standard element methods is difficult because achieving an approximate solution needs very fine triangulation in general and hence tremendous amount of computer memory and CPU time. Thus it is desirable to have a numerical method that can capture the effect of small scales on large scales without resolving the small scale details.

Two-scale method is a very promising method for solving the above problems (see [4] [5] [6], and references therein). It couples macroscopic scale and microscopic scale together, and not only reflects the global mechanical and physical properties of structure, but also the effect of micro-configuration of composite materials and flows. Using this method, we can slove elliptic problems with small periodic coefficients by solving a homogenization problem with coarse meshes in whole domain and a periodic problem with fine meshes only in one small periodic subdomain.

The objective of this paper is to study the elliptic problems with small periodic coefficients on a bounded domain with a curved boundary. A dual coupled expression is used to approximate the exact solution. Since the homogenization problem is solved with coarse meshes in whole smooth domain, while the periodic problem is solved with fine meshes only in one small periodic subdomain, in order to match the errors of two problems, it is natural to solve the homogenization problem using high order elements and the periodic problem using low order elements. If we use straight side element method to solve the homogenization problem, the smooth domain is approximated by a ploygonal domain. In this case, the error is not optimal, since the best error in the H^1 norm is $O(h_0^{3/2})$, where parameter h_0 is the mesh size (see [11]).

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To overcome this shortcoming, we use isoparametric element method to solve the homogenization problem on smooth domain. But if one uses isoparametric element method in the usual way as in [2], the approximate solutions and the error estimates can be obtained only over an approximate domain Ω_{h_0}. In general, the approximate domain Ω_{h_0} is different from the given smooth domain Ω. To obtain the approximate solutions and the error estimates over the given smooth domain, we use the method given in [9] to define isoparametric element space. Based on this idea, the error of two-scale element method is derived over the given smooth domain. Finally, an additive Schwarz method is proposed for the isoparametric element method. Note that for isoparametric elements both triangulations and finite element spaces are nonnested. Moreover isoparametric element spaces do not contain usual linear conforming element space which is defined on same mesh or coarse meshes in a natural way as a subspace of H_0^1. So we choose a special linear conforming element space as coarse mesh space (for details see section 4).

The remainder of this paper is outlined as follows. Section 2 presents the continuous problems and some notations. Section 3 gives the two-scale curved element method and estimates the error. Section 4 provides an additive Schwarz method.

In this paper, C (with or without subscripts) denotes a generic positive constant with different values in different contexts. For any domain D, we use Sobolev space $W_p^m(D)$ with Sobolev norm $\|\cdot\|_{W_p^m(D)}$ and seminorm $|\cdot|_{W_p^m(D)}$ (see [1]). If $D=\Omega$, we omit D. Moreover if $D=\Omega$ and $p=2$, we denote the usual L^2 inner product by $(\cdot,\cdot)$, the Sobolev norm by $\|\cdot\|_m$ and seminorm by $|\cdot|_m$. Also we use Einstein summation notation, i.e., repeated index indicates to sum.

2. Preliminaries

Consider the following elliptic boundary value problem on a bounded domain $\Omega \subset \mathcal{R}^2$ with a sufficiently smooth curved boundary $\Gamma = \partial\Omega$:

$$\begin{cases} L^\epsilon u^\epsilon \equiv -\bigtriangledown \cdot (a^\epsilon \bigtriangledown u^\epsilon) = f, & \text{in} \quad \Omega, \\ u^\epsilon = 0, & \text{on} \quad \Gamma, \end{cases} \tag{2.1}$$

where $a^\epsilon = (a_{ij}^\epsilon(x))$ is a bounded symmetric positive definite matrix with small period ϵ, and f is a sufficiently smooth function.

Let $y = \frac{x}{\epsilon}$ and $a = (a_{ij}(y)) = (a_{ij}^\epsilon(x))$, then $a_{ij}(y)$ is a periodic function with period 1. Let $Q = (0,1)\times(0,1)$. Assume $a_{ij}(y) \in W_\infty^1(Q)$. First we introduce a periodic function $N_k(y)$ which is the solution of the following equation

$$\begin{cases} -\frac{\partial}{\partial y_i}(a_{ij}\frac{\partial N_k}{\partial y_j}) = \frac{\partial}{\partial y_i}a_{ik}, & \text{in} \quad Q, \\ N_k = 0, & \text{on} \quad \partial Q. \end{cases} \tag{2.2}$$

From [8] we know that problem (2.2) has H^2 regularity, i.e., problem (2.2) has a solution $N_k \in H^2(Q)$ satisfying

$$\|N_k\|_{H^2(Q)} \le C\|\frac{\partial}{\partial y_i}a_{ik}\|_{L^2(Q)}.$$

Then we define a constant matrix $a^0 = (a_{ij}^0)$ by

$$a_{ij}^0 = \int_Q (a_{ij} + a_{ik}\frac{\partial N_j}{\partial y_k})dy.$$

Also we define a function $u^0(x)$ to be the solution of the following homogenization equation

$$\begin{cases} -\nabla\cdot(a^0\nabla u^0)=f, & \text{in} \quad \Omega, \\ u^0=0, & \text{on} \quad \Gamma. \end{cases} \tag{2.3}$$

Define dual approximate expression of the solution u^ϵ for problem (2.1) by

$$u_1^\epsilon = u^0+\epsilon u^1 \equiv u^0+\epsilon N_k\frac{\partial u^0}{\partial x_k}. \tag{2.4}$$

3. Two-Scale Curved Element Method and Error Estimate

By (1.51) in [13], the following result holds.

Lemma 3.1. $\|u^\epsilon-u_1^\epsilon\|_1\le C\epsilon^{1/2}$.

According to the expression of u_1^ϵ, in order to obtain the numerical solution of equation (2.1), we only need to compute u^0 in smooth domain Ω, and N_k in one periodic subdomain. To match the numerical approximate errors of u^0 and N_k, we use linear conforming element method to compute N_k and isoparametric element method to compute u_0.

First we consider the finite element approximation of periodic solution N_k. The weak form of problem (2.2) is to find $N_k\in H_0^1(Q)$ such that

$$(a_{ij}\frac{\partial N_k}{\partial y_j},\frac{\partial v}{\partial y_i})_Q=-(a_{ik},\frac{\partial v}{\partial y_i})_Q,\quad \forall v\in H_0^1(Q). \tag{3.1}$$

Let $\mathcal{T}_h$ be a quasi-uniform partition of Q with grid size h and $V_h\subset H_0^1(Q)$ be a linear conforming finite element space defined on $\mathcal{T}_h$. The finite element approximation of problem (3.1) is to find $N_k^h\in V_h$ satisfying

$$(a_{ij}\frac{\partial N_k^h}{\partial y_j},\frac{\partial v}{\partial y_i})_Q=-(a_{ik},\frac{\partial v}{\partial y_i})_Q,\quad \forall v\in V_h. \tag{3.2}$$

Using standard argument, we can obtain the following result.

Lemma 3.2. *Assume N_k and N_k^h to be the solutions of problems (3.1) and (3.2) respectively, $N_k\in W_\infty^2(Q)$, then*

$$\begin{aligned}&\|N_k-N_k^h\|_{L^2(Q)}+h|N_k-N_k^h|_{H^1(Q)}\le Ch^2|N_k|_{H^2(Q)},\\ &\|N_k-N_k^h\|_{L^\infty(Q)}+h|N_k-N_k^h|_{W_\infty^1(Q)}\le Ch^2|\ln h||N_k|_{W_\infty^2(Q)}.\end{aligned}$$

Next we consider the finite element approximation of the homogenization solution u^0. The weak form of problem (2.3) is to find $u^0\in H_0^1(\Omega)$ such that

$$(a^0\nabla u^0,\nabla v)=(f,v),\quad \forall v\in H_0^1(\Omega). \tag{3.3}$$

According to the definition of a^0, a^0 can be computed by a and N_j. Unfortunately, N_j is unknown, but we can use N_j^h to substitute N_j. Based on this idea, we define the approximate matrix a_h^0 of a^0 as follows:

$$a_h^0=(a_{ij}^{0,h}),\quad a_{ij}^{0,h}=\int_Q(a_{ij}+a_{ik}\frac{\partial N_j^h}{\partial y_k})dy.$$

Substituting a^0 by a_h^0 in problm (3.3), we obtain the following problem: find $\bar{u}^0\in H_0^1(\Omega)$ such that

$$a(\bar{u}^0,v)=(f,v),\quad \forall v\in H_0^1(\Omega), \tag{3.4}$$

where and hereinafter $a(v,w) = (a_h^0 \bigtriangledown v, \bigtriangledown w)$, for all $v, w \in H^1(\Omega)$.

The following Lemma says that problems (3.3) and (3.4) are uniformly elliptic, which is restatement of Lemma 4.2 in [3].

Lemma 3.3. *If $N_j \in W_\infty^1(Q)$, then the matrix a^0 is bounded symmetric positive definite, moreover if $N_j \in H^2(Q)$, then for sufficiently small h, a_h^0 is also bounded symmetric positive definite.*

According to Lemma 3.3, we know that problem (3.3) or (3.4) has a unique solution.

Now we consider isoparametric element approximation of problem (3.4). Here we only discuss isoparametric 2-simplices of type 2.

As in [9], we use a special triangulation $\mathcal{T}_{h_0}$, such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_{h_0}} \bar{K}$, where the interior finite element (K, P_K, Σ_K) $(K \in \mathcal{T}_{h_0})$ is obtained from a reference finite element $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ through a affine mapping $F_K(\widehat{K})$ which is uniquely determined by the data of the nodes of the finite element K (see [2]), while the boundary finite element (K, P_K, Σ_K) $(K \in \mathcal{T}_{h_0})$ can be obtained from a modified reference finite element $(\widetilde{K}, \widetilde{P}, \widetilde{\Sigma})$ through an isoparametric mapping $F_K \in P_2(\widetilde{K})$ which can be obtained by generalizing original mapping $F_K(\widehat{K})$, where $\widetilde{K}$ is expanded from $\widehat{K}$ and the mapping $F_K(\widetilde{K})$ satisfies $K \subset F_K(\widetilde{K})$ (for details see [9]). Assume the isoparametric family to be regular. For convenience, we still denote modified reference finite element by $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$. Let E_{h_0} denote the set of nodes of the boundary finite elements corresponding to $\mathcal{T}_{h_0}$ which are on Γ. Thus we define

$$\begin{aligned} V_{h_0} = \{ v_h \quad | \quad & v_h \in C^0(\bar{\Omega}), \quad v_h(x) = 0, \quad \forall x \in E_{h_0}, \\ & v_h = \widehat{v}_h \circ F_K^{-1}, \quad \text{on} \quad K, \quad \widehat{v}_h \in P_2(\widehat{K}) \}, \end{aligned}$$

where $\widehat{K}$ corresponding to boundary element K is a modified reference finite element.

The isoparametric element approximation of problem (3.4) is: find $u_{h_0}^0 \in V_{h_0}$ satisfying

$$a(u_{h_0}^0, v_h) = (f, v_h), \quad \forall v_h \in V_{h_0}. \tag{3.5}$$

By Lemma 3.2 in [9], the following H^1 error estimate holds over domain Ω.

Lemma 3.4. *Assume $\bar{u}^0$ and $u_{h_0}^0$ to be the solutions of problems (3.4) and (3.5) respectively, $\bar{u}^0 \in H^3(\Omega)$, then*

$$\|\bar{u}^0 - u_{h_0}^0\|_1 \leq C h_0^2.$$

Now we estimate $\|u^0 - u_{h_0}^0\|_1$.

Lemma 3.5. *Assume u^0 and $u_{h_0}^0$ to be the solutions of problems (3.3) and (3.5) respectively, $\bar{u}^0 \in H^3(\Omega)$ and $N_k \in H^2(Q)$ to be the solutions of problems (3.4) and (3.1) respectively, then we have*

$$\|u^0 - u_{h_0}^0\|_1 \leq C(h_0^2 + h^2).$$

Proof. From (3.3) and (3.4) it follows that

$$(a^0 \bigtriangledown (u^0 - \bar{u}^0), \bigtriangledown v) = ((a_h^0 - a^0) \bigtriangledown \bar{u}^0, \bigtriangledown v), \quad \forall v \in H_0^1(\Omega). \tag{3.6}$$

Taking $v = N_l^h - N_l$ in (3.1) and using the definitions of a^0 and a_0^h yield

$$\begin{aligned} (a_{ij} \frac{\partial N_k}{\partial y_j}, \frac{\partial (N_l^h - N_l)}{\partial y_i})_Q &= -(a_{ik}, \frac{\partial (N_l^h - N_l)}{\partial y_i})_Q \\ &= a_{kl}^0 - a_{kl}^{0,h}. \end{aligned} \tag{3.7}$$

Subtracting (3.1) from (3.2) and taking $v = N_l^h$, we obtain

$$(a_{ij}\frac{\partial(N_k^h - N_k)}{\partial y_j}, \frac{\partial N_l^h}{\partial y_i})_Q = 0. \tag{3.8}$$

From (3.7) and (3.8) it follows that

$$\begin{aligned} |a_{ij}^{0,h} - a_{ij}^0| &= |\int_Q \frac{\partial(N_i^h - N_i)}{\partial y_k} a_{kl} \frac{\partial(N_j^h - N_j)}{\partial y_l} dy| \\ &\leq C|N_i^h - N_i|_{H^1(Q)}|N_j^h - N_j|_{H^1(Q)} \\ &\leq Ch^2|N_i|_{H^2(Q)}|N_j|_{H^2(Q)}. \end{aligned} \tag{3.9}$$

Taking $v = u^0 - \bar{u}^0$ in (3.6), using Lemma 3.3, (3.9), and H^2 regularity of problem (2.2), noting $a_{ij}(y) \in W_\infty^1(Q)$, we obtain

$$\begin{aligned} C_1\|u^0 - \bar{u}^0\|_1^2 &\leq (a^0 \bigtriangledown (u^0 - \bar{u}^0), \bigtriangledown(u^0 - \bar{u}^0)) \\ &= ((a_h^0 - a^0) \bigtriangledown \bar{u}^0, \bigtriangledown(u^0 - \bar{u}^0)) \\ &\leq Ch^2 \max_{1\leq j\leq 2} |N_j|^2_{H^2(Q)}|\bar{u}^0|_1|u^0 - \bar{u}^0|_1 \\ &\leq Ch^2 \max_{1\leq j\leq 2} |\frac{\partial}{\partial y_i} a_{ij}|^2_{L^2(Q)}|\bar{u}^0|_1|u^0 - \bar{u}^0|_1 \\ &\leq Ch^2|\bar{u}^0|_1|u^0 - \bar{u}^0|_1, \end{aligned}$$

which implies

$$\|u^0 - \bar{u}^0\|_1 \leq Ch^2.$$

By Lemma 3.4 we derive

$$\begin{aligned} \|u^0 - u_{h_0}^0\|_1 &\leq \|u^0 - \bar{u}^0\|_1 + \|\bar{u}^0 - u_{h_0}^0\|_1 \\ &\leq C(h_0^2 + h^2). \end{aligned}$$

Finally we give a numerical computational formula of u^ϵ as follows.

$$u_{h_0}^{\epsilon,h} = u_{h_0}^0 + \epsilon N_k^h \frac{\partial u_{h_0}^0}{\partial x_k}.$$

For any $v|_\tau \in H^1(\tau)$, $\tau \in \mathcal{T}_h$, define

$$|v|_{1,h_0} = (\sum_{\tau\in\mathcal{T}_{h_0}} |v|^2_{H^1(\tau)})^{1/2}, \quad |v|_{2,h_0} = (\sum_{\tau\in\mathcal{T}_{h_0}} |v|^2_{H^2(\tau)})^{1/2}.$$

The main result of this section is the following Theorem.

Theorem 3.6. *Assume $u^0 \in H^3(\Omega) \cap W_\infty^2(\Omega)$ to be the solution of problem (3.3), $\bar{u}^0 \in H^3(\Omega)$ to be the solution of problem (3.4), and $N_k(y) \in W_\infty^2(Q)$ to be the solution of problem (3.1). Then*

$$|u^\epsilon - u_{h_0}^{\epsilon,h}|_{1,h_0} \leq C(\epsilon^{1/2} + h + h_0^2).$$

Proof. Let $\Pi_{h_0} : H^2(\Omega) \cap H_0^1(\Omega) \to V_{h_0}$ to be the nodal value interpolation operator. According to the definitions of u_1^ϵ and $u_{h_0}^{\epsilon,h}$, using inverse inequality, Lemma 3.2, Lemma 3.5, and the interpolation error estimate (see (3.10) in [9]), we deduce that

$$\begin{aligned}
|u_1^\epsilon - u_{h_0}^{\epsilon,h}|_{1,h_0} &\le |u^0 - u_{h_0}^0|_1 + \epsilon|(N_k - N_k^h)\frac{\partial u^0}{\partial x_k}|_1 + \epsilon|N_k^h(\frac{\partial u^0}{\partial x_k} - \frac{\partial u_{h_0}^0}{\partial x_k})|_{1,h_0} \\
&\le |u^0 - u_{h_0}^0|_1 + \epsilon \max_{1\le k\le 2}\{|N_k - N_k^h|_1 |u^0|_{W_\infty^1} \\
&+ |N_k - N_k^h|_0 |u^0|_{W_\infty^2}\} + \epsilon \max_{1\le k\le 2} |N_k^h|_{W_\infty^1} |u^0 - u_{h_0}^0|_1 \\
&+ \epsilon |N_k^h|_{L^\infty}\{|u^0 - \Pi_{h_0}u^0|_{2,h_0} + Ch_0^{-1}|\Pi_{h_0}u^0 - u_{h_0}^0|_1\} \\
&\le |u^0 - u_{h_0}^0|_1 + \epsilon \max_{1\le k\le 2}\{\epsilon^{-1}|N_k - N_k^h|_{H^1(Q)}|u^0|_{W_\infty^1} \\
&+ |N_k - N_k^h|_{L^2(Q)}|u^0|_{W_\infty^2}\} + \max_{1\le k\le 2}(|N_k^h - N_k|_{W_\infty^1(Q)} \\
&+ |N_k|_{W_\infty^1(Q)})|u^0 - u_{h_0}^0|_1 + \epsilon(|N_k^h - N_k|_{L^\infty(Q)} + |N_k|_{L^\infty(Q)}) \\
&\cdot\{|u^0 - \Pi_{h_0}u^0|_{2,h_0} + Ch_0^{-1}(|\Pi_{h_0}u^0 - u^0|_1 + |u^0 - u_{h_0}^0|_1)\} \\
&\le C \max_{1\le k\le 2}\{h_0^2 + h^2 + h|N_k|_{H^2(Q)}|u^0|_{W_\infty^1} + \epsilon h^2 |N_k|_{H^2(Q)}|u^0|_{W_\infty^2} \\
&+ (h|\ln h||N_k|_{W_\infty^2(Q)} + |N_k|_{W_\infty^1(Q)}) \cdot (h_0^2 + h^2) \\
&+ \epsilon(h^2|\ln h||N_k|_{W_\infty^2(Q)} + |N_k|_{L^\infty(Q)}) \\
&\cdot\{h_0|u^0|_3 + h_0^{-1}(h_0^2|u^0|_3 + h_0^2 + h^2)\} \\
&\le C(h_0^2 + h + \epsilon h^2 + \epsilon h_0 + \epsilon h_0^{-1}h^2) \\
&\le C(h_0^2 + h + \epsilon^2).
\end{aligned}$$

Combining above estimate with Lemma 3.1 completes the proof.

4. Additive Schwarz Method

In this section, we discuss two-level additive Schwarz method for solving problem (3.5). As usual, we first decompose Ω into N non-overlapping subdomains Ω_i $(i = 1, \cdots, N)$, and obtain coarse partition $\mathcal{T}_H$ with grid size H. Then we refine the partition $\mathcal{T}_H$ as follows: if a triangle has two vertices in Ω, we connect the midpoints of the sides of the triangle and form four new triangles; if a triangle has two vertices on the boundary, then we take a new boundary point to be the midpoint of curved side and connect the midpoint of the curved side and the midpoints of other sides to form four new triangles. In this way, we obtain a fine partition $\mathcal{T}_{h_0}$ with grid size h_0. Let V_{h_0} be isoparametric element space defined on $\mathcal{T}_{h_0}$ as in section 3. To obtain the decomposition of the space V_{h_0}, we extend each subdomain Ω_i to a larger one $\Omega_i^{'} \supset \Omega_i$. Assume that the distance between $\partial\Omega_i \cap \Omega$ and $\partial\Omega_i^{'} \cap \Omega$ is bounded from below by $\delta > 0$ and $\partial\Omega_i^{'}$ does not cut through any element $E \in \mathcal{T}_{h_0}$. Let $N(x)$ be the number of subdomains $\{\Omega_i^{'}\}$ which contain point $x \in \Omega$. We assume that $\max_{x\in\Omega} N(x)$ is less than a constant N_c which is independent of $\mathcal{T}_H$ and $\mathcal{T}_{h_0}$. Let N_i be the set of vertices and midpoints of sides which belong to $\partial\Omega_i^{'}$. Define subspace $V_{h_0}^i$ of V_{h_0} by

$$V_{h_0}^i = \{v \in V_{h_0} \ | \ v = 0 \ \text{ on } \ \Omega\backslash\overline{\Omega_i^{'}} \ \text{ and } \ v(m) = 0, \ \forall m \in N_i\}.$$

Let $\bar{V}_H \subset H_0^1(\Omega)$ be piecewise linear subspace of $H_0^1(\Omega)$ defined on $\mathcal{T}_H$ with zero extension to Ω. Obviously $\bar{V}_H \not\subset V_{h_0}$. In order to obtain a subspace of V_{h_0} as coarse space, let Ω_0 be the interior of the union of the closed triangles of $\mathcal{T}_H$ which do not have vertices on $\partial\Omega$. The space V_H is defined by

$$V_H = \{v \in \bar{V}_H \ | \ supp(v) \subset \bar{\Omega}_0\}.$$

One can see $V_H \subset V_{h_0}$. For convenience, denote V_H by $V_{h_0}^0$. Define operator $P_i : V_{h_0} \to V_{h_0}^i$, $i = 0, 1, \cdots, N$, by

$$a(P_i v, w) = a(v, w), \quad \forall w \in V_{h_0}^i, \ \ i = 0, 1, \cdots, N.$$

Let $P = \sum_{i=0}^{N} P_i$, $b = \sum_{i=0}^{N} P_i u_{h_0}$. Then problem (3.5) is equivalent to the following problem

$$P u_{h_0} = b. \tag{4.1}$$

It is important to understand that $b \in V_{h_0}$ can be computed without a priori knowledge of the solution u_{h_0} of (3.5). In fact, the computation can be completed by first solving the following equations

$$a(P_i u_{h_0}, v) = (f, v), \quad \forall v \in V_{h_0}^i, \ \ i = 0, 1, \cdots, N,$$

and then taking the sum of the above solutions.

Define operators $\bar{Q}_H : L^2(\Omega) \to \bar{V}_H$ and $Q_H : L^2(\Omega) \to V_H$ by

$$(\bar{Q}_H v, w) = (v, w), \quad \forall w \in \bar{V}_H,$$
$$(Q_H v, w) = (v, w), \quad \forall w \in V_H.$$

In order to prove the L^2-approximation of operator Q_H, we need the following two Lemmas.

Lemma 4.1. *For any $v \in V_{h_0}$, we have*

$$\int_\Gamma v^2 ds \le C h_0^5 |v|_1^2.$$

Proof. See Theorem 2.1 in [9].

Lemma 4.2. *Let $\Omega^\eta = \{x \in \Omega \ | \ dist(x, \Gamma) < \eta\}$, then*

$$\|v\|_{L^2(\Omega^\eta)}^2 \le C_1 \eta \int_\Gamma v^2 ds + C_2 \eta^2 |v|_1^2, \quad \forall v \in H^1(\Omega).$$

Proof. Divide Ω^η into some curved quadrilateral subregions Ω_i^η, such that $\overline{\Omega^\eta} = \bigcup \overline{\Omega_i^\eta}$, $\Omega_i^\eta \cap \Omega_j^\eta = \phi$ for $i \ne j$, $diam(\Omega_i^\eta) = O(\eta)$, and $mes(\Gamma_i) = mes(\partial\Omega_i^\eta \cap \partial\Omega) = O(\eta)$. For each Ω_i^η, we have the following Friedichs inequality

$$\begin{aligned} \|v\|_{L^2(\Omega_i^\eta)}^2 &\le C\{(\int_{\Gamma_i} v ds)^2 + \eta^2 |v|_{H^1(\Omega_i^\eta)}^2\} \\ &\le C_1 \eta \int_{\Gamma_i} v^2 ds + C_2 \eta^2 |v|_{H^1(\Omega_i^\eta)}^2. \end{aligned}$$

Summing over all Ω_i^η completes the proof.

Combining Lemma 4.1 with Lemma 4.2 yields the following result.

Corollary 4.3. *Let Ω^η be defined as in Lemma 4.2, $h_0^5 \le C\eta$, then*

$$\|v\|_{L^2(\Omega^\eta)}^2 \le C\eta^2 |v|_1^2, \quad \forall v \in V_{h_0}.$$

By Corollary 4.3, standard interpolation estimate, and interpolation space theory (see [10]), we derive the L^2-approximation of operator $\bar{Q}_H$.

Lemma 4.4. $\|(I - \bar{Q}_H)v\|_0 \le CH\|v\|_1, \quad \forall v \in V_{h_0}$.

Now we prove the L^2-approximation of operator Q_H.

Lemma 4.5. $\|(I-Q_H)v\|_0 \le CH|v|_1$, $\forall v \in V_{h_0}$.

Proof. Let $w \in V_H$ be equal to $\bar{Q}_H v$ at the interior nodes of Ω_0. Note that $w = 0$ in $\Omega\backslash\Omega_0$, by Corollary 4.3 and Lemma 4.4, we have

$$\begin{aligned}
\|(I-Q_H)v\|_0^2 &\le \|v-w\|_0^2 \\
&\le 2\{\|v-w\|^2_{L^2(\Omega_0)} + \|v-w\|^2_{L^2(\Omega\backslash\Omega_0)}\} \\
&\le C\{\|v-\bar{Q}_H v\|^2_{L^2(\Omega_0)} + \|\bar{Q}_H v - w\|^2_{L^2(\Omega_0)} \\
&\quad + \|v\|^2_{L^2(\Omega\backslash\Omega_0)}\} \\
&\le C\{H^2\|v\|_1^2 + \|\bar{Q}_H v - w\|^2_{L^2(\Omega_0)}\}.
\end{aligned} \tag{4.2}$$

Note that $\bar{Q}_H v - w \in \bar{V}_H$ and vanishes at all vertices except those on $\partial\Omega_0$. Denote by N_0 the set of all vertices on $\partial\Omega_0$. Using the discrete norm, Corollary 4.3 and Lemma 4.4 yields

$$\begin{aligned}
\|\bar{Q}_H v - w\|^2_{L^2(\Omega_0)} &\le CH^2 \sum_{x_i \in N_0} (\bar{Q}_H v(x_i))^2 \\
&\le C\|\bar{Q}_H v\|^2_{L^2(\Omega\backslash\Omega_0)} \\
&\le C\{\|\bar{Q}_H v - v\|^2_{L^2(\Omega\backslash\Omega_0)} + \|v\|^2_{L^2(\Omega\backslash\Omega_0)}\} \\
&\le CH^2\|v\|_1^2.
\end{aligned} \tag{4.3}$$

According to (4.2) and (4.3), we obtain

$$\|(I-Q_H)v\|_0 \le CH\|v\|_1. \tag{4.4}$$

By Lemma 4.1 and Friedichs inequality

$$\|v\|_0^2 \le C_1\{|v|_1^2 + (\int_\Gamma v ds)^2\} \le C|v|_1^2, \quad \forall v \in V_{h_0}. \tag{4.5}$$

Combining (4.4) with (4.5) completes the proof.

Next we prove the H^1-stability of operator Q_H.

Lemma 4.6. $|Q_H v|_1 \le C|v|_1$, $\forall v \in V_{h_0}$.

Proof. In fact for each $v \in V_{h_0}$, we can find a piecewise constant w on $\mathcal{T}_H$ (for example local L^2-projection) such that

$$\|v-w\|_0 \le CH|v|_1.$$

By inverse inequality, above inequality, and Lemma 4.5,

$$\begin{aligned}
|Q_H v|_1 &= (\sum_{E\in\mathcal{T}_H} |Q_H v - w|^2_{H^1(E)})^{1/2} \\
&\le CH^{-1}\|Q_H v - w\|_0 \\
&\le CH^{-1}(\|Q_H v - v\|_0 + \|v-w\|_0) \\
&\le C|v|_1.
\end{aligned}$$

Using Lemmas 4.5-4.6, we can obtain following result. Since its proof is standard, we refer readers to Lemma 7.1 in [12].

Lemma 4.7. *For any $v \in V_{h_0}$, there exist $v_i \in V^i_{h_0}$, $i = 0,1,\cdots,N$, such that $v = \sum_{i=0}^N v_i$, and*

$$\sum_{i=0}^N |v_i|_1^2 \le C(1+\frac{H}{\delta})^2|v|_1^2.$$

Now we prove the main result of this section.

Theorem 4.8. *For any* $v \in V_{h_0}$*, we have*

$$C_1(1+\frac{H}{\delta})^{-2}a(v,v) \le a(Pv,v) \le C_2 a(v,v).$$

Proof. By Lemma 2.1 in [7], for any $v_i \in V_{h_0}^i$, $v = \sum_{i=0}^{N} v_i$,

$$a(P^{-1}v,v) = \min \sum_{i=0}^{N} a(v_i,v_i).$$

From Lemma 4.7 it follows

$$a(P^{-1}v,v) \le C(1+\frac{H}{\delta})^2 a(v,v),$$

which means

$$C_1(1+\frac{H}{\delta})^{-2}a(v,v) \le a(Pv,v).$$

To prove the upper bound, using the fact $\max_{x\in\Omega} N(x) \le N_c$ yields

$$\begin{aligned} a^2(Pv,v) &\le a(Pv,Pv)a(v,v) \\ &\le C\sum_{i=0}^{N} a(P_iv,P_iv)a(v,v) \\ &= Ca(Pv,v)a(v,v), \end{aligned}$$

which implies

$$a(Pv,v) \le Ca(v,v).$$

Therefore the proof is completed.

From Theorem 4.8 we know $cond(P) \le C(1+\frac{H}{\delta})^2$. If we choose $\delta = O(H)$, the condition number of operator P is optimal.

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