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THE SOLVABILITY CONDITIONS FOR INVERSE EIGENVALUE PROBLEM OF ANTI-BISYMMETRIC MATRICES[*1)]

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Abstract

This paper is mainly concerned with solving the following two problems:

Problem I. Given $X \in C^{n\times m}$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m) \in C^{m\times m}$. Find $A \in ABSR^{n\times n}$ such that

$$AX = X\Lambda$$

where $ABSR^{n\times n}$ is the set of all real $n \times n$ anti-bisymmetric matrices.

Problem II. Given $A^* \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where $\|\cdot\|_F$ is Frobenius norm, and S_E denotes the solution set of Problem I.

The necessary and sufficient conditions for the solvability of Problem I have been studied. The general form of S_E has been given. For Problem II the expression of the solution has been provided.

Key words: Eigenvalue problem, Norm, Approximate solution.

1. Introduction

Inverse eigenvalue problem has widely been used in engineering. For example inverse eigenvalue method is a useful means in vibration design and vibration control of flyer. In recent years a serial of good conclussions have been made for inverse eigenvalue problem. However, inverse problems of anti-bisymmetric matrices have not be concerned yet. In this paper we will discuss this problem.

We denote the complex $n \times m$ matrix space by $C^{n\times m}$, the real $n \times m$ matrix space by $R^{n\times m}$, and $R^n = R^{n\times 1}$, the set of all matrices in $R^{n\times m}$ with rank r by $R_r^{n\times m}$, the set of all $n\times n$ orthogonal matrices by $OR^{n\times n}$, the set of all $n\times n$ anti-symmetric matrices by $ASR^{n\times n}$, the column space, the null space and the Moore–Penrose generalized inverse of a matrix A by $R(A), N(A), A^+$ respectively, the identity matrix of order n by I_n, the Frobenius norm of A by $\|A\|_F$. We define inner product in space $R^{n\times m}$, $(A, B) = \text{tr}(B^T A) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}b_{ij}, \quad \forall A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix produced by the

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inner product is Frobenius norm. Let $S_k = (e_k, e_{k-1}, \cdots, e_1) \in R^{k\times k}$ in which e_i is the i-th Cloumn of the identity matrix I_k.

Definition 1. *$A = (a_{ij}) \in R^{n\times n}$, if*

$$a_{ij} = -a_{ji}, \qquad a_{ij} = -a_{n-j+1,n-i+1}, \qquad i,j = 1,2,\ldots,n$$

then A is called a anti-bisymmetric matrix. The set of all anti-bisymmetric matrices is denoted by $ABSR^{n\times n}$.

Now we consider the following problems:

Problem I. Given $X \in C^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Find $A \in ABSR^{n\times n}$ such that

$$AX = X\Lambda.$$

Problem II. Given $A^* \in R^{n\times n}$. Find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\|_F = \min_{A\in S_E} \|A^* - A\|_F,$$

where S_E is the solution set of problem I.

At first, in this paper, we will discuss the character of eigenvector for anti-bisymmetric matrices. Then we will give the necessary and sufficient conditions for the solvability of Problem I and the expression of the general solution of Problem I in real number field, and point out S_E is a closed convex set. At last, we will prove that there exists a unique solution of Problem II and give an expression of the solution for Problem II.

2. The Solvability Conditions and General Form of the Solutions for Problem I in Real Number Field

At first we discuss the construction of $ABSR^{n\times n}$ and the character of eigenvector for matrices in $ABSR^{n\times n}$.

Let

$$k = [\frac{n}{2}], \qquad \text{[x] is integer number that is not greater than x.} \tag{2.1}$$

When $n = 2k$,
$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}, \quad D^TD = I_n; \tag{2.2}$$

and when $n = 2k+1$,
$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}, \qquad D^TD = I_n. \tag{2.3}$$

Lemma 1. *$A \in ABSR^{n\times n}$ if and only if*

$$A = S_nAS_n, \qquad A = -A^T$$

Theorem 1.

$$ABSR^{2k\times 2k} = \left\{ \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} \middle| \; M, H \in ASR^{k\times k} \right\}. \tag{2.4}$$

$$ABSR^{(2k+1)\times(2k+1)} = \left\{ \begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} \middle| \; N, H \in ASR^{k\times k}, C \in R^k \right\}. \tag{2.5}$$

whether n is odd or even number, the general form of elements in $ABSR^{n\times n}$ is

$$D\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D^T \tag{2.6}$$

Where A_{11}, A_{22} are anti-symmetric matrices, D is the same as (2.2) or (2.3).

Proof. We only prove(2.4). If

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in ABSR^{2k\times 2k}.$$

Then

$$A = -A^T, \quad \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{2.7}$$

(2.7) is equivalent to

$$\begin{array}{cc} A_{11} = -A_{11}^T, \quad A_{12} = -A_{21}^T, & A_{22} = -A_{22}^T, \\ A_{22} = S_k A_{11} S_k & (A_{12}S_k)^T = -A_{12}S_k \end{array}. \tag{2.8}$$

Let $\qquad A_{12}S_k = H, \qquad A_{11} = M.$
Then $\quad H = -H^T, \quad M = -M^T, \quad A_{12} = HS_k, \quad A_{21} = S_kH, \quad A_{22} = S_kMS_k.$
It implies that

$$A = \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix}.$$

Then $ABSR^{2k\times 2k} \subseteq \left\{ \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} \middle| M, H \in ASR^{k\times k} \right\}.$

Conversely, for every $M, H \in ASR^{k\times k}$ it is easy to see

$$\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} = -\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix}^T$$

and

$$\begin{aligned} &\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix}\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix}\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} \\ &= \begin{pmatrix} H & MS_k \\ S_kM & S_kHS_k \end{pmatrix}\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix}. \end{aligned}$$

From Lemma 1, it follows that $\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} \subseteq ABSR^{2k\times 2k}$. Thus (2.4) holds. The form (2.5) can be obtained by the similar method.

Futhermore, when $n = 2k$

$$D^T\begin{pmatrix} M & HS_k \\ S_kH & S_kMS_k \end{pmatrix} D = \begin{pmatrix} M+H & 0 \\ 0 & M-H \end{pmatrix} \tag{2.9}$$

we have the form (2.6). When $n = 2k+1$, D is the same as (2.3)

$$D^T \begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} D$$
$$= \frac{1}{2}\begin{pmatrix} I_k & 0 & S_k \\ 0 & \sqrt{2} & 0 \\ I_k & 0 & -S_k \end{pmatrix}\begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix}\begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}$$
$$= \begin{pmatrix} N+H & \sqrt{2}C & 0 \\ -\sqrt{2}C^T & 0 & 0 \\ 0 & 0 & N-H \end{pmatrix}.$$

Hence
$$\begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} = D\begin{pmatrix} N+H & \sqrt{2}C & 0 \\ -\sqrt{2}C^T & 0 & 0 \\ 0 & 0 & N-H \end{pmatrix}D^T \tag{2.10}$$

It implies that the elements in $ABSR^{n\times n}$ have the form (2.6) when $n = 2k+1$.

On the other hand, it can be directly verified that matrices in form (2.6) belong to $ABSR^{n\times n}$ from Lemma 1

Next we consider the problem I in real number field.

When $n = 2k$ in (2.9) suppose eigenvalues of $M+H$ are $\pm\lambda_1 i$, $\pm\lambda_2 i$, $\cdots$, $\pm\lambda_{t_1} i$, $\underbrace{0\cdots,0}_{k-2t_1}$, where $\lambda_j, j = 1,2\cdots,t_1$ are real numbers, the corresponding eigenvectors are $\alpha_1\pm\beta_1 i$, $\alpha_2\pm\beta_2 i$, $\cdots$,$\alpha_{t_1}\pm\beta_{t_1} i$, x_{2t_1+1}, $\cdots$, x_k, where $i^2 = -1$ and $\alpha_j,\beta_j \in R^k, j = 1,\cdots t_1$, $x_s \in R^k$, s$=2t_1+1,\ldots,k$ and eigenvalues of $M-H$ are $\pm\mu_1 i$, $\pm\mu_2 i$, $\cdots$, $\pm\mu_{t_2} i$, $\underbrace{0,\cdots,0}_{k-2t_2}$,$\mu_j \in R, j = 1,\ldots,t_2$, the corresponding eigenvectors are $\theta_1\pm\gamma_1 i$, $\theta_2\pm\gamma_2 i,\cdots$, $\theta_{t_1}\pm\gamma_{t_2} i$, $y_{2t_2+1},\cdots,y_k$, $\theta_j,\gamma_j \in R^k$, $j = 1,\ldots,t_2$, $y_s \in R^k$, $s = 2t_2+1,\ldots,k$. Then eigenvalues of A are $\pm\lambda_1 i$, $\pm\lambda_2 i$, $\cdots$, $\pm\lambda_{t_1} i$, $\underbrace{0,\cdots,0}_{k-2t_1}$, $\pm\mu_1 i$, $\pm\mu_2 i$, $\cdots$, $\pm\mu_{t_2} i$, $\underbrace{0,\cdots,0}_{k-2t_2}$.

Let

$z_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1+\beta_1 i \\ S_k(\alpha_1+\beta_1 i) \end{pmatrix}, \bar{z}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_1-\beta_1 i \\ S_k(\alpha_1-\beta_1 i) \end{pmatrix}, \cdots, z_{t_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_{t_1}+\beta_{t_1} i \\ S_k(\alpha_{t_1}+\beta_{t_1} i) \end{pmatrix},$
$\bar{z}_{t_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_{t_1}-\beta_{t_1} i \\ S_k(\alpha_{t_1}-\beta_{t_1} i) \end{pmatrix}, z_{2t_1+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} x_{t_1+1} \\ S_kx_{t_1+1} \end{pmatrix}, \cdots, z_k = \frac{1}{\sqrt{2}}\begin{pmatrix} x_k \\ S_kx_k \end{pmatrix},$
$z_{k+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \theta_1+\gamma_1 i \\ -S_k(\theta_1+\gamma_1 i) \end{pmatrix}, \bar{z}_{k+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \theta_1-\gamma_1 i \\ -S_k(\theta_1-\gamma_1 i) \end{pmatrix}, \cdots, z_{k+2t_2+1} = \frac{1}{\sqrt{2}}$
$\begin{pmatrix} y_{t_2+1} \\ -S_ky_{t_2+1} \end{pmatrix}, \cdots, z_n = \frac{1}{\sqrt{2}}\begin{pmatrix} y_k \\ -S_ky_k \end{pmatrix}.$

Then $z_1, z_2, \ldots, z_n$ are the corresponding eigenvectors of A.

Definition 2. *If $x \in C^n$ satisfies $S_nx = x$ then x is called a symmetric vector, if $x \in C^n$ satisfies $S_nx = -x$ then x is called an anti-symmetric vector.*

It is easy to verify that $S_nz_1 = z_1$, $S_n\bar{z}_1 = \bar{z}_1$, $\cdots$, $S_nz_k = z_k$, $S_nz_{k+1} = -z_{k+1}$, $S_n\bar{z}_{k+1} = -\bar{z}_{k+1}$, $\ldots$, $S_nz_n = -z_n$. Therefore it is that there exist k symmetric vectors and k anti-symmetric vectors for every $A \in ABSR^{2k\times 2k}$.

On the orther hand, if λ is an eigenvalue of A and u is a corresponding eigenvector, i.e $Au=\lambda u$. By Lemma 1 we have

$$AS_{2k}u=S_{2k}Au=\lambda S_{2k}u$$

it implies that $S_{2k}u$ is an eigenvector of A corresponding to λ. Then $u\pm S_{2k}u$ is also an eigenvector corresponding to λ. Where $u+S_{2k}u$ is a symmetric vector and $u-S_{2k}u$ is an anti-symmetric vector. Hence we can obtain a group of symmetric eigenvectors and anti-symmetric eigenvectors from the given eigenvectors.

Because

$$A\begin{pmatrix}\alpha_j+\beta_j i\\ S_k(\alpha_j+\beta_j i)\end{pmatrix}=\lambda_j i\begin{pmatrix}\alpha_j+\beta_j i\\ S_k(\alpha_j+\beta_j i)\end{pmatrix}$$

$$A\begin{pmatrix}\alpha_j-\beta_j i\\ S_k(\alpha_j-\beta_j i)\end{pmatrix}=-\lambda_j i\begin{pmatrix}\alpha_j-\beta_j i\\ S_k(\alpha_j-\beta_j i)\end{pmatrix}$$

Hence

$$A\begin{pmatrix}\alpha_j\\ S_k\alpha_j\end{pmatrix}=-\lambda_j\begin{pmatrix}\beta_j\\ S_k\beta_j\end{pmatrix}$$

$$A\begin{pmatrix}\beta_j\\ S_k\beta_j\end{pmatrix}=\lambda_j\begin{pmatrix}\alpha_j\\ S_k\alpha_j\end{pmatrix}$$

$$A\begin{pmatrix}\alpha_j & \beta_j\\ S_k\alpha_j & S_k\beta_j\end{pmatrix}=\begin{pmatrix}\alpha_j & \beta_j\\ S_k\alpha_j & S_k\beta_j\end{pmatrix}\begin{pmatrix}0 & \lambda_j\\ -\lambda_j & 0\end{pmatrix}.$$

Suppose $\pm\lambda_j i$, $j=1,\ldots,t_1$, $\pm\mu_j i$, $j=1,\ldots,t_2$, $\underbrace{0,\cdots,0}_{m-2(t_1+t_2)}$ are eigenvalues of A .

Let

$$\Lambda=\mathrm{diag}(B_1,\cdots,B_{t_1},0\cdot I_{l-2t_1},C_1,\cdots,C_{t_2},0\cdot I_{m-l-2t_2}),\tag{2.11}$$

where $B_j=\begin{pmatrix}0 & \lambda_j\\ -\lambda_j & 0\end{pmatrix}$, $C_j=\begin{pmatrix}0 & \mu_j\\ -\mu_j & 0\end{pmatrix}\in R^{2\times 2}$

According to above analysis we obtain corresponding problem in real numbers field as follows:

Problem I_0. Given $X\in R^{n\times m}$, Λ is the same as (2.11). Find $A\in ABSR^{n\times n}$ such that $AX=X\Lambda$,

where $ABSR^{n\times n}$ is the set of all real $n\times n$ anti-bisymmetric matrices.

when $n=2k$, X can be supposed the following form

$$X=\begin{pmatrix}X_1 & Y_1\\ S_kX_1 & -S_kY_1\end{pmatrix}.\tag{2.12}$$

Lemma 2. [3] *Suppose $X=(X_1,X_2,\cdots,X_t,X_{t+1})\in R^{n\times m}$, every column of X is nonzero vector, $X_j\in R^{n\times 2}$, $j=1,2,\cdots,t$, $X_{t+1}\in R^{n\times(m-2t)}$, $rank(X)=r$, $\Lambda=diag(B_1,\cdots,B_t,0\cdot I_{m-2t})$. Then there is $A\in ASR^{n\times n}$ such that $AX=X\Lambda$ if and only if*

$$X_j^TX_l=0,\quad B_j\neq B_l, j,l=1,\cdots,t+1.\tag{2.13}$$

and the solutions of $AX=X\Lambda$ can be respresented as

$$A=X\Lambda X^{+}+(X^{+})^T\Lambda X^T-(X^{+})^T\Lambda X^TXX^{+}+U_2ZU_2^T,\quad Z\in ASR^{n\times n},\tag{2.14}$$

where $U_2 \in R^{n\times(n-r)}, U_2^TU_2 = I_{n-r}, N(X^T) = R(U_2)$.

Theorem 2. *Given* $X \in R^{n\times m}$, *and* X *is the same as (2.12). Suppose* $X_1 = (X_1' \vdots X_2' \vdots \cdots \vdots X_{t_1+1}') \in R^{k\times l}$, *every column of* X_1 *is a non-vanishing vector,* $X_j' \in R^{k\times 2}$, $(j = 1, 2, \ldots, t_1)$, *rank* $(X_1) = r_1$, $X_{t_1+1}' \in R^{k\times(l-2t_1)}$, $\Lambda_1 = diag(B_1, B_2, \ldots, B_{t_1}, 0\cdot I_{l-2t_1})$, $B_j = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}$, $j = 1, 2, \ldots, t_1$; $Y_1 = (Y_1' \vdots Y_2' \vdots \cdots \vdots Y_{t_2+1}') \in R^{k\times(m-l)}$, *every column of* Y_1 *is a non-vanishing vector,* $Y_j' \in R^{k\times 2}$, $(j = 1, 2, \ldots, t_2)$, $Y_{t_2+1} \in R^{k\times(m-l-2t_1)}$, $rank(Y_1) = r_2$, $\Lambda_2 = diag(C_1, C_2, \ldots, C_{t_2}, 0\cdot I_{m-l-2t_2})$, $C_j = \begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}$, $j = 1, 2, \ldots, t_2$. *Let* $\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$. *Then there exists* $A \in ABSR^{n\times n}(n = 2k)$ *such that* $AX = X\Lambda$ *if and only if*

$$X_j'^TX_l' = 0, \quad \lambda_l \neq \lambda_j, l, j = 1, 2 \ldots, t_1 + 1 \tag{2.15}$$

$$Y_j'^TY_l' = 0, \quad \mu_l \neq \mu_j, l, j = 1, 2 \ldots, t_2 + 1 \tag{2.16}$$

and the general solution can be respresented as

$$A = A_0 + D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix}D^T \tag{2.17}$$

where

$$A_0 = D\begin{pmatrix} X_1\Lambda_1X_1^+ & 0 \\ 0 & Y_1\Lambda_2Y_1^+ \end{pmatrix}D^T \tag{2.18}$$

$$\forall G_1 \in ASR^{(k-r_1)\times(k-r_1)}, \quad \forall G_2 \in ASR^{(k-r_2)\times(k-r_2)}$$
$$U_2 \in R^{k\times(k-r_1)}, \quad U_2^TU_2 = I_{k-r_1}, \quad N(X_1^T) = R(U_2),$$
$$P_2 \in R^{k\times(k-r_2)}, \quad P_2^TP_2 = I_{k-r_2}, \quad N(Y_1^T) = R(P_2),$$

D *is the same as (2.2).*

Proof. **Necessity:** Suppose A is a solution of Problem I_0. From Theorem 1 it is verified that there exist A_{11} , $A_{22} \in ASR^{k\times k}$ such that

$$A = D\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}D^T, \qquad AX = X\Lambda$$

i. e

$$D\begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}D^T\begin{pmatrix} X_1 & Y_1 \\ S_kX_1 & -S_kY_1 \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 \\ S_kX_1 & -S_kY_1 \end{pmatrix}\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \tag{2.19}$$

(2.19) is equivalent to

$$A_{11}X_1 = X_1\Lambda_1, \quad A_{11} \in ASR^{k\times k} \tag{2.20}$$

$$A_{22}Y_1 = Y_1\Lambda_2, \quad A_{22} \in ASR^{k\times k} \tag{2.21}$$

From Lemma 2 we know (2.20) , (2.21) have a solution if and only if

$$X_j'^TX_l' = 0, \quad \lambda_j \neq \lambda_l, j, l = 1, 2 \ldots, t_1 + 1 \tag{2.22}$$

$$Y_j'^T Y_l' = 0, \quad \mu_j \neq \mu_l, j, l = 1, 2 \ldots, t_2 + 1 \tag{2.23}$$

and there are respectively $G_1 \in ASR^{(k-r_1)\times(k-r_1)}$, $G_2 \in ASR^{(k-r_2)\times(k-r_2)}$ such that

$$A_{11} = X_1 \Lambda_1 X_1^+ + U_2 G_1 U_2^T \tag{2.24}$$

$$A_{22} = Y_1 \Lambda_2 Y_1^+ + P_2 G_2 P_2^T \tag{2.25}$$

where $U_2 \in R^{k\times(k-r_1)}, U_2^T U_2 = I_{k-r_1}$, $N(X_1^T) = R(U_2)$, $P_2 \in R^{k\times(k-r_2)}$, $P_2^T P_2 = I_{k-r_2}$, $N(Y_1^T) = R(P_2)$.

From (2.2), (2.6), (2.24) and (2.25) we have

$$A = D \begin{pmatrix} X_1\Lambda_1 X_1^+ & 0 \\ 0 & Y_1 \Lambda_2 Y_1^+ \end{pmatrix} D^T + D \begin{pmatrix} U_2 G_1 U_2^T & 0 \\ 0 & P_2 G_2 P_2^T \end{pmatrix} D^T, \tag{2.26}$$

for some $G_1 \in ASR^{(k-r_1)\times(k-r_1)}$, $\quad G_2 \in ASR^{(k-r_2)\times(k-r_2)}$.

Sufficiency: In (2.26) taking $G_1 = 0 \in ASR^{(k-r_1)\times(k-r_1)}$, $G_2 = 0 \in ASR^{(k-r_2)\times(k-r_2)}$. We have $A \in ABSR^{2k\times 2k}$

$$\begin{aligned} AX &= D \begin{pmatrix} X_1\Lambda_1 X_1^+ & 0 \\ 0 & Y_1\Lambda_2 Y_1^+ \end{pmatrix} D^T \begin{pmatrix} X_1 & Y_1 \\ S_k X_1 & -S_1 Y_1 \end{pmatrix} \\ &= D \begin{pmatrix} X_1\Lambda_1 X_1^+ & 0 \\ 0 & Y_1\Lambda_2 Y_1^+ \end{pmatrix} \begin{pmatrix} \sqrt{2} X_1 & 0 \\ 0 & \sqrt{2} Y_1 \end{pmatrix} \end{aligned}$$

From $U_2^T X_1 = 0$, $P_2^T Y_1 = 0$ and (2.15), (2.16) we obtain [4]

$$X_1 \Lambda_1 X_1^+ X_1 = X_1 \Lambda_1, \quad Y_1 \Lambda_2 Y_1^+ Y_1 = Y_1 \Lambda_2.$$

Hence

$$AX = \sqrt{2} D \begin{pmatrix} X_1 \Lambda_1 & 0 \\ 0 & Y_1 \Lambda_2 \end{pmatrix} = X\Lambda.$$

Since $(X_1\Lambda_1 X_1^+)^T = -X_1\Lambda_1 X_1^+$, $(Y_1\Lambda_2 Y_1^+)^T = -Y_1\Lambda_2 Y_1^+$[3], then $A \in ABSR^{2k\times 2k}$. Therefore A is a solution of Problem I_0.

When $n = 2k+1$, D is the same as (2.3). Then

$$D^T A D = D^T \begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^T S_k \\ S_k H & S_k C & S_k N S_k \end{pmatrix} D = \begin{pmatrix} N+H & \sqrt{2}C & 0 \\ \sqrt{2}C^T & 0 & 0 \\ 0 & 0 & N-H \end{pmatrix} \tag{2.27}$$

Suppose eigenvalues of main submatrix $\begin{pmatrix} N+H & \sqrt{2}C \\ -\sqrt{2}C^T & 0 \end{pmatrix}$ are $\pm\lambda_1 i$, $\pm\lambda_2 i$, …, $\pm\lambda_{t_1} i$, $\underbrace{0,\ldots,0}_{k+1-2t_1}$, $\lambda_j, j = 1, 2, \ldots, t_1$ are real numbers and the corresponding eigenvectors are $\begin{pmatrix} \alpha_1 \pm \beta_1 i \\ e_1 \pm f_1 i \end{pmatrix}$, …, $\begin{pmatrix} \alpha_{t_1} \pm \beta_{t_1} i \\ e_{t_1} \pm f_{t_1} i \end{pmatrix}$, $\begin{pmatrix} x_{2t_1+1} \\ e_{2t_1+1} \end{pmatrix}$, …, $\begin{pmatrix} x_{k+1} \\ e_{k+1} \end{pmatrix}$, $\alpha_j, \beta_j \in R^k$, $j = 1, 2 \ldots, t_1$, $x_l \in R^k$, $l = 2t_1+1, \cdots, k+1$, $e_j, j = 1, \cdots_{k+1}$ and $f_j, j = 1, \ldots t_1$ are real numbers; eigenvalues of $N-H$ are $\pm\mu_1 i$, $\pm\mu_2 i$, … , $\pm\mu_{t_2} i$, $\underbrace{0,\ldots,0}_{k-2t_2}$, $\mu_j, j = 1, \cdots t_2$ are real numbers and the corresponding eigenvectors are $\theta_1 \pm \gamma_1 i$, $\theta_2 \pm \gamma_2 i$, …, $\theta_{t_2} \pm \gamma_{t_2} i$, y_{2t_2+1}, …, y_k, $\theta_j, \gamma_j \in R^k$, $j = 1, 2, \cdots, t_2$, $y_l \in R^k$,

$l = 2t_2+1,\cdots,k$. Then eigenvalues of A are $\pm\lambda_1 i$, $\pm\lambda_2 i$, ..., $\pm\lambda_{t_1} i$, $\underbrace{0,\ldots,0}_{k+1-2t_1}$, $\pm\mu_1 i$, $\pm\mu_2 i$, ..., $\pm\mu_{t_2} i$, $\underbrace{0,\ldots,0}_{k-2t_2}$, whereas symmetric vectors $\begin{pmatrix} \alpha_1 \pm \beta i \\ e_1 \pm \sqrt{2} f_1 i \\ S_k(\alpha_1 \pm \beta i) \end{pmatrix}$, ..., $\begin{pmatrix} \alpha_{t_1} \pm \beta_{t_1} i \\ e_{t_1} \pm f_{t_1} i \\ S_k(\alpha_{t_1} \pm \beta_{t_1} i) \end{pmatrix}$, $\begin{pmatrix} x_{2t_1+1} \\ e_{2t_1+1} \\ S_k x_{2t_1+1} \end{pmatrix}$, ..., $\begin{pmatrix} x_{k+1} \\ e_{k+1} \\ S_k x_{k+1} \end{pmatrix}$ and anti-symmetric vectors $\begin{pmatrix} \theta_1 \pm \gamma_1 i \\ w_1 \pm q_1 i \\ -S_k(\theta_1 \pm \gamma_1 i) \end{pmatrix}$, ..., $\begin{pmatrix} \theta_{t_2} \pm \gamma_{t_2} i \\ w_{t_2} \pm q_{t_2} i \\ -S_k(\theta_{t_2} \pm \gamma_{t_2} i) \end{pmatrix}$, $\begin{pmatrix} y_{2t_2+1} \\ 0 \\ -S_k y_{2t_2+1} \end{pmatrix}$, ..., $\begin{pmatrix} y_k \\ 0 \\ -S_k y_k \end{pmatrix}$ are eignvectors corresponding to $\pm\lambda_1 i$, $\pm\lambda_2 i$, ..., $\pm\lambda_{t_1} i$, $\underbrace{0,\ldots,0}_{k+1-2t_1}$, $\pm\mu_1 i$, ..., $\pm\mu_2 i$, ..., $\pm\mu_{t_2} i$, $\underbrace{0,\ldots,0}_{k-2t_2}$. Hence it is easy to see that exist $k+1$ symmetric eigenvectors and k anti-symmetric eigenvectors for every $A \in ABSR^{(2k+1)\times(2k+1)}$.

Similary to case $n = 2k$ in Problem I_0 we can suppose

$$X = \begin{pmatrix} X_1 & Y_1 \\ e^T & 0^T \\ S_k X_1 & -S_k Y_1 \end{pmatrix} \tag{2.28}$$

where $X_1 \in R^{k\times l}$, $Y_1 \in R^{k\times(m-l)}$, $e^T = (\sqrt{2}e_1, \sqrt{2}e_2 \ldots, \sqrt{2}e_l) \in R^{1\times l}$, $0^T = (0,0,\ldots,0) \in R^{1\times(m-l)}$.

Theorem 3. *Given $X \in R^{(2k+1)\times m}$, and X is the same as (2.28). Suppose $X' = \begin{pmatrix} \sqrt{2}X_1 \\ e^T \end{pmatrix} = (Z_1' \vdots Z_2' \vdots \cdots \vdots Z_{t_1+1}') \in R^{(k+1)\times l}$, every column of X' is a non-vanishing vector, $Z_j' \in R^{(k+1)\times 2}$, $j = 1,2,\ldots,t_1$, $Z_{t_1+1}' \in R^{(k+1)\times(l-2t_1)}$, $rank(X') = r_1$, $\Lambda_1 = diag(B_1, B_2, \ldots, B_{t_1}, 0\cdot I_{l-2t_1})$, $B_j = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}$, $j = 1,2,\ldots,t_1$; $Y_1 = (Y_1' \vdots Y_2' \vdots \cdots \vdots Y_{t_2+1}') \in R^{k\times(m-l)}$, every column of Y_1 is a non-vanishing vector, $Y_j' \in R^{k\times 2}$, $(j = 1,2,\ldots,t_2)$, $Y_{t_2+1} \in R^{k\times(m-l-2t_2)}$, $\sum_{j=1}^{t_2+1} m_j = m - l$, $rank(Y_1) = r_2$, $\Lambda_2 = diag(C_1, C_2, \ldots, C_{t_2}, 0\cdot I_{k-2t_2})$, $C_j = \begin{pmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{pmatrix}$, $j = 1,2,\ldots,t_2$. Let $\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$. Then there exists $A \in ABSR^{n\times n}(n = 2k+1)$ such that $AX = X\Lambda$ if and only if*

$${Z_j'}^T Z_l' = 0, \quad \lambda_l \neq \lambda_j, l, j = 1, 2 \ldots, t_1 + 1 \tag{2.29}$$

$${Y_j'}^T Y_l' = 0, \quad \mu_l \neq \mu_j, l, j = 1, 2 \ldots, t_2 + 1 \tag{2.30}$$

and the general solution can be respresented as

$$A = A_0' + D \begin{pmatrix} {U'}_2 E_1 {U'}_2^T & 0 \\ 0 & {P'}_2 E_2 {P'}_2^T \end{pmatrix} D^T \tag{2.31}$$

where

$$A_0' = D \begin{pmatrix} X'\Lambda_1 {X'}^+ & 0 \\ 0 & Y_1 \Lambda_2 Y_1^+ \end{pmatrix} D^T \tag{2.32}$$

$$\forall E_1 \in ASR^{(k+1-r_1)\times(k+1-r_1)}, \quad \forall E_2 \in ASR^{(k-r_2)\times(k-r_2)}$$

$$U'_2 \in R^{(k+1)\times(k+1-r_1)}, \quad {U'}_2^T U'_2 = I_{k+1-r_1}, \quad N({X'}^T) = R(U'_2),$$

$$P'_2 \in R^{k\times(k-r_2)}, \quad {P'_2}^T P'_2 = I_{k-r_1}, \quad N(Y_1^T) = R(P'_2),$$

D is the same as (2.3).

Proof. **Necessity:** Suppose A is a solution of problem I_0. From Theorem 1 we know that exist N , $H \in ASR^{k\times k}$, $C \in R^n$, such that

$$\begin{pmatrix} N & C & HS_k \\ -C^T & 0 & -C^TS_k \\ S_kH & S_kC & S_kNS_k \end{pmatrix} \begin{pmatrix} X_1 & Y_1 \\ e^T & 0 \\ S_kX_1 & -S_kY_1 \end{pmatrix} = \begin{pmatrix} X_1 & Y_1 \\ e^T & 0 \\ S_kX_1 & -S_kY_1 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \tag{2.33}$$

i. e

$$\begin{cases} NX_1 + Ce^T + HX_1 & = X_1\Lambda_1 \\ -2C^TX_1 & = e^T\Lambda_1 \\ (N-H)Y_1 & = Y_1\Lambda_2 \end{cases} \tag{2.34}$$

(2.34) is equivalent to

$$\begin{cases} \begin{pmatrix} N+H & \sqrt{2}C \\ -\sqrt{2}C^T & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}X_1 \\ e^T \end{pmatrix} = \begin{pmatrix} \sqrt{2}X_1 \\ e^T \end{pmatrix} \Lambda_1 \\ (N-H)Y_1 = Y_1\Lambda_2 \end{cases} \tag{2.35}$$

Let

$$X' = \begin{pmatrix} \sqrt{2}X_1 \\ e^T \end{pmatrix} = (Z'_1 \vdots Z'_2 \vdots \cdots \vdots Z'_{t_1+1}) \in R^{(k+1)\times l}$$
$$Y_1 = (Y'_1 \vdots Y'_2 \vdots \cdots \vdots Y'_{t_2+1}) \in R^{k\times(m-l)} \tag{2.36}$$

(2.29) and (2.30) hold from Lemma 2. And there is $E_1 \in ASR^{(k+1-r_1)\times(k+1-r_1)}$, $E_2 \in ASR^{(k-r_2)\times(k-r_2)}$ such that

$$\begin{pmatrix} N+H & \sqrt{2}C \\ -\sqrt{2}C^T & 0 \end{pmatrix} = X'\Lambda_1{X'}^+ + U'_2E_1{U'_2}^T \tag{2.37}$$

$$N - H = Y_1\Lambda_2Y_1^+ + P'_2E_2{P'_2}^T \tag{2.38}$$

where $U'_2 \in R^{(k+1)\times(k+1-r_1)}, {U'_2}^TU'_2 = I_{k+1-r_1}$, $N({X'}^T) = R(U'_2)$; $P'_2 \in R^{k\times(k-r_2)}$, ${P'_2}^TP'_2 = I_{k-r_2}$, $N(Y_1^T) = R(P'_2)$

By (2.27), (2.37) and (2.38) we know

$$A = D\begin{pmatrix} X'\Lambda_1{X'}^+ & 0 \\ 0 & Y_1\Lambda_2Y_1^+ \end{pmatrix} D^T + D\begin{pmatrix} U'_2E_1{U'_2}^T & 0 \\ 0 & P'_2E_2{P'_2}^T \end{pmatrix} D^T, \tag{2.39}$$

$$\forall E_1 \in ASR^{(k+1-r_1)\times(k+1-r_1)}, \quad \forall E_2 \in ASR^{(k-r_2)\times(k-r_2)}.$$

Sufficiency: In (2.39) taking $E_1 = 0 \in ASR^{(k+1-r_1)\times(k+1-r_1)}$, $E_2 = 0 \in ASR^{(k-r_2)\times(k-r_2)}$. Because ${U'_2}^TX' = 0$, ${P'_2}^TY_1 = 0$ and from (2.29), (2.30) we obtain [4]

$$X'\Lambda_1{X'}^+X' = X'\Lambda_1, \quad Y_1\Lambda_2Y_1^+Y_1 = Y_1\Lambda_2.$$

Similary to the demonstration of Theorem 2 it is directly verified that $AX = X\Lambda$. Because $(X'\Lambda_1 X'^{+})^T = -X'\Lambda_1 X'^{+}$, $(Y_1\Lambda_2 Y_1^{+})^T = -Y_1`\Lambda_2 Y_1^{+}$[3]. We have $A \in ABSR^{(2k+1)\times(2k+1)}$. Therefore A is a solution of Problem I_0.

3. The Expression of Solution for Problem II

When solution set of Problem I_0 is nonempty it is easily verified that S_E is a closed convex set. Therefore when n is even number we have

Theorem 4. *Given $A^* \in R^{n\times n}$, $X \in R^{n\times m}$ $(n = 2k)$ and the notation of X, Λ and conditions are the same as Theorem 2. Then there is a unique soluiton $\hat{A} \in S_E$ for Problem II and $\hat{A}$ can be respresented as*

$$\hat{A} = A_0 + D \begin{pmatrix} \frac{(I-X_1X_1^{+})(\tilde{A}_{11}-\tilde{A}_{11}^T)(I-X_1X_1^{+})}{2} & 0 \\ 0 & \frac{(I-Y_1Y_1^{+})(\tilde{A}_{22}-\tilde{A}_{22}^T)(I-Y_1Y_1^{+})}{2} \end{pmatrix} D^T \tag{3.1}$$

where A_0 is the same as (2.18) and D is the same as (2.2)

$$\tilde{A}_{11} = \frac{1}{2}\begin{pmatrix} I_k & S_k \end{pmatrix}(A^* - A_0)\begin{pmatrix} I_k \\ S_k \end{pmatrix}, \quad \tilde{A}_{22} = \frac{1}{2}\begin{pmatrix} I_k & -S_k \end{pmatrix}(A^* - A_0)\begin{pmatrix} I_k \\ -S_k \end{pmatrix} \tag{3.2}$$

Proof. Because S_E is a closed convex set there is a unique solution $\hat{A}$ for problem II. According to (2.17) every element A in S_E can be respresented as

$$A = A_0 + D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} D^T$$

Taking $U_1 \in R^{k\times(k-r_1)}$, $P_1 \in R^{k\times(k-r_1)}$ such that $U \triangleq (U_1 \vdots U_2) \in R^{k\times r_1}$, $P \triangleq (P_1 \vdots P_2) \in R^{k\times r_2}$

Let

$$\tilde{A} = D^T(A^* - A_0)D = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \tag{3.3}$$

where

$$\tilde{A}_{11} = \frac{1}{2}(I_k \quad S_k)(A^* - A_0)\begin{pmatrix} I_k \\ S_k \end{pmatrix} \tag{3.4}$$

$$\tilde{A}_{12} = \frac{1}{2}(I_k \quad S_k)(A^* - A_0)\begin{pmatrix} I_k \\ -S_k \end{pmatrix} \tag{3.5}$$

$$\tilde{A}_{21} = \frac{1}{2}\begin{pmatrix} I_k & -S_k \end{pmatrix}(A^* - A_0)\begin{pmatrix} I_k \\ S_k \end{pmatrix} \tag{3.6}$$

$$\tilde{A}_{22} = \frac{1}{2}\begin{pmatrix} I_k & -S_k \end{pmatrix}(A^* - A_0)\begin{pmatrix} I_k \\ -S_k \end{pmatrix} \tag{3.7}$$

Because

$$\begin{aligned} \|A^* - A\|^2 &= \left\| A^* - A_0 - D\begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} D^T \right\|^2 \\ &= \left\| D^T(A^* - A_0)D - \begin{pmatrix} U_2G_1U_2^T & 0 \\ 0 & P_2G_2P_2^T \end{pmatrix} \right\|^2 \\ &= \|\tilde{A}_{11} - U_2G_1U_2^T\|^2 + \|\tilde{A}_{12}\|^2 + \|\tilde{A}_{21}\|^2 + \|\tilde{A}_{22} - P_2G_2P_2^T\|^2. \end{aligned}$$

Hence $\|A^* - A\| = \inf_{A\in ABSR^{n\times n}}$ is equivalent to

$$\|\tilde{A}_{11} - U_2 G_1 U_2^T\| = \min_{G_1\in ASR^{k\times k}} \tag{3.8}$$

$$\|\tilde{A}_{22} - P_2 G_2 P_2^T\| = \min_{G_2\in ASR^{k\times k}} \tag{3.9}$$

But

$$\begin{aligned}&\|\tilde{A}_{11} - U_2G_1U_2^T\|^2 = \|U^T\tilde{A}_{11}U - U^TU_2G_1U_2^TU\|^2\\ &= \|U_1^T\tilde{A}_{11}U_1\|^2 + \|U_1^T\tilde{A}_{11}U_2\|^2 + \|U_2^T\tilde{A}_{11}U_1\|^2 + \|U_2^T\tilde{A}_{11}U_2 - G_1\|^2\\ &= \|U_1^T\tilde{A}_{11}U_1\|^2 + \|U_1^T\tilde{A}_{11}U_2\|^2 + \|U_2^T\tilde{A}_{11}U_1\|^2\\ &+\|U_2^T\frac{\tilde{A}_{11} - \tilde{A}_{11}^T}{2}U_2 - G_1\|^2 + \|U_2^T\frac{\tilde{A}_{11} + \tilde{A}_{11}^T}{2}U_2\|^2\end{aligned}$$

Therefore (3.8) holds if and only if

$$G_1 = U_2^T\frac{\tilde{A}_{11} - \tilde{A}_{11}^T}{2}U_2 \tag{3.10}$$

By the similar method (3.9) holds if and only if

$$G_2 = P_2^T\frac{\tilde{A}_{22} - \tilde{A}_{22}^T}{2}P_2. \tag{3.11}$$

Substituting (3.10), (3.11) to (2.17) we obtain the solution of Problem II is

$$\begin{aligned}\hat{A} &= A_0 + D\begin{pmatrix} U_2U_2^T\frac{\tilde{A}_{11} - \tilde{A}_{11}^T}{2}U_2U_2^T & 0\\ 0 & P_2P_2^T\frac{\tilde{A}_{22} - \tilde{A}_{22}^T}{2}P_2P_2^T\end{pmatrix}D^T\\ &= A_0 + D\begin{pmatrix}(I - X_1X_1^+)\frac{\tilde{A}_{11} - \tilde{A}_{11}^T}{2}(I - X_1X_1^+) & 0\\ 0 & (I - Y_1Y_1^+)\frac{\tilde{A}_{22} - \tilde{A}_{22}^T}{2}(I - Y_1Y_1^+)\end{pmatrix}D^T.\end{aligned}$$

According to similar discussion in Theorem 4 when n is odd number we have

Theorem 5. *Given $A^* \in R^{n\times n}$, $X \in R^{n\times m}(n = 2k + 1)$ and the notation of X, Λ and conditions are the same as Theorem 3. Then there is a unique soluiton $\hat{A} \in S_E$ and $\hat{A}$ can be respresented as*

$$\hat{A} = A_0' + D\begin{pmatrix}\frac{(I-X'X'^+)(\bar{A}_{11}-\bar{A}_{11}^T)(I-X'X'^+)}{2} & 0\\ 0 & \frac{(I-Y_1Y_1^+)(\bar{A}_{22}-\bar{A}_{22}^T)(I-Y_1Y_1^+)}{2}\end{pmatrix}D^T$$

where A_0' is the same as (2.32) and D is the same as (2.3)

$$\bar{A}_{11} = \frac{1}{2}\begin{pmatrix} I_k & 0 & S_k\\ 0 & \sqrt{2} & 0\end{pmatrix}(A^* - A_0')\begin{pmatrix} I_k & 0\\ 0 & \sqrt{2}\\ S_k & 0\end{pmatrix},$$

$$\bar{A}_{22} = \frac{1}{2}\begin{pmatrix} I_k & 0 & -S_k\end{pmatrix}(A^* - A_0')\begin{pmatrix} I_k\\ 0\\ -S_k\end{pmatrix}.$$

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