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ASYMPTOTIC STABILITY FOR GAUSS METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS*[1)]

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Abstract

In [4] we proved that all Gauss methods are$N\tau(0)$-compatible for neutral delay differential equations (NDDEs) of the form :

$$\begin{aligned} &y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau), \quad t > 0, \\ &y(t) = g(t), \qquad -\tau \le t \le 0, \end{aligned} \tag{0.1}$$

where a, b, c are real, $\tau > 0$, $g(t)$ is a continuous real valued function. In this paper we are going to use the theory of order stars to characterize the asymptotic stability properties of Gauss methods for NDDEs. And then proved that all Gauss methods are $N\tau(0)$-stable.

Key words: Delay differential equations, Stability, Runge-Kutta methods.

1. Introduction

In the past, most of the work on the asymptotic stability for delay and neutral delay differential equations dealt with finding the stability region independently of the delay term. Al Mutib[1] and recently N. Guglielmi [8, 9, 10] revisited the investigation of stability region for a fixed but arbitrary delay term for so called $\tau(0)$-stability. Some results have been pointed out for DDEs, which have been reeaximined for NDDEs [4]. It has already been shown [7] all Gauss methods are $\tau(0)$-stability for DDEs. In this paper we pursue our investigation of Gauss methods in a NDDEs case. In order t simplify the notation, without loosing the generality of the problem we can fix the delay equal to 1. For the sake of the simplicity, in the sequel we deal with the following test equation

$$\begin{aligned} &y'(t) = ay(t) + by(t-1) + cy'(t-1), \quad t > 0, \\ &y(t) = g(t), \qquad -1 \le t \le 0, \end{aligned} \tag{1.1}$$

where a, b, c are real, $\tau > 0$, $g(t)$ is a continuous real valued function. Its characteristic equation is given by:

$$\lambda - a - b\exp(-\lambda) - c\lambda\exp(-\lambda) = 0. \tag{1.2}$$

It is known that the set of triplet $(a,\ b,\ c)$ for which the solution $y(t)$ of (1.1) tend to zero when $t \to \quad \infty$ is given by:

$$\Sigma_* = \{(a,b,c) \in I\!R^3 \text{ all root } \ \lambda \ \text{ of (1.2) satisfying Re } [\lambda] < 0, |c| < 1\}.$$

It can be rewriten as $\Sigma_* = \Sigma \cup \mathrm{E}$ where

$$\mathrm{E} = \{(a,b,c) \in I\!R^3, \quad a + |b| < 0 \quad \text{and} \quad |c| < 1\},$$

$$\Sigma = \left\{(a,\ b,c) \in I\!R^3,\ |a| < -b, \text{ and } \sqrt{b^2 - a^2} < \sqrt{1-c^2}\arccos\left(\frac{c+\rho}{1+c\rho}\right) \text{ with } |c| < 1\right\}.$$

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with $\rho = -\frac{a}{b}$. This set is bounded in the right by the plane
$P = \{(a, b, c) \in \mathbb{R}^3,\ a = -b \text{ with } a < 1 - c, |c| < 1\}$ and the transcendental surface

$$\Gamma_* = \{(a_*(\theta, c), b_*(\theta, c), c) \in \mathbb{R}^3 \mid \theta \in (0, \pi) \text{ and } a < 1 - c, |c| < 1\},$$

with

$$a_*(\theta, c) = \frac{\theta \cos\theta - c\theta}{\sin\theta}, \qquad b_*(\theta, c) = \frac{c\theta \cos\theta - \theta}{\sin\theta}.$$

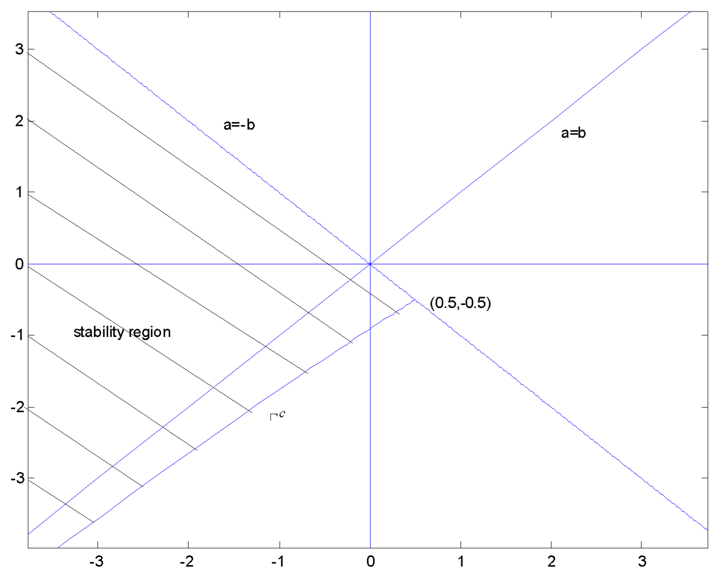

Figure 1. Stability region of analytical solution of equation (1.1) for $c = 0.5$

1.1. Runge -Kutta methods for NDDEs

Let us consider the following s-stage RK method

$$y_{n+1} = y_n + h \sum_{i=1}^{v} w_i K_i^{n+1} \tag{1.3}$$

$$K_i^{n+1} = f\left(t_n + c_i h,\ y_n + h \sum_{j=1}^{v} a_{ij} K_j^{n+1},\ y_{n-m} + h \sum_{j=1}^{v} b_{ij} K_j^{n-m+1},\ \sum_{j=1}^{v} c_{ij} K_j^{n-m+1}\right),$$

$i = 1, 2, \ldots, s$, where $h = \tau/m$, $c_i = \sum_{j=1}^{s} a_{ij}$. Here $W = [w_1, \ldots, w_s]^T$ and the matrix $A = [a_{ij}]_{i,j=1}^{s}$ define a RK method for ODEs. [3, 6, 13]. The second argument in f can be interpreted as an approximation to $y(t)$ at the intermediate point $t_n + c_j h$. Similarly the third argument in f can be interpreted as an approximation to $y(t_{n-m} + c_j h)$ and the fourth to $y'(t_{n-m} + c_j h)$ usually $b_{ij} = w_j(c_i)$ and $c_{ij} = w_j'(c_i)$ where $w_i(\theta)$, $i = 1, \ldots s$ are polynomials which define the natural continuous extension of RK method, i.e. polynomials such that the approximate solution y_h define the whole interval of integration is given by

$$y_h(t_n+\theta h)=y_h(t_n)+h\sum_{i=1}^{s}w_i(\theta)K_i^{n+1},\quad n=0,1,\cdots,\quad \theta\in(0,1].$$

Take $B=[b_{ij}]_{i,j=1}^{s}$, $C=[c_{ij}]_{i,j=1}^{s}$ and $e=[1,\cdots,1]^T$, I is the identity matrix.

In our present work we consider constant step-size and equal to an integer sub-multiple of the delay i.e. $h=1/m$, m is positive number. When apply (1.3) in the case A=B, C=I to the test equation (1.1), the following difference equation equationin some case is also useful for examining the stability c.f. Bellen *et* all [3]

$$P(\zeta)=1+(\frac{a}{m}+\frac{b}{m\zeta^m})W^T((1-\frac{c}{\zeta^m})I-(\frac{a}{m}+\frac{b}{m\zeta^m})A)^{-1}e-\zeta=0, \tag{1.4}$$

$$\zeta=R(z),\quad z=\frac{a+b\zeta^{-m}}{m(1-c\zeta^{-m})}, \tag{1.5}$$

with

$$R(z)=1+zW^T(I-zA)^{-1}e,$$

where $R(z)$ is the stability function of the method [6, 11] the numerical solution of (1.1) is asymptotically stable if and only if $|\zeta|<1$ whenever ζ satisfies (1.5). Or we can present it as

$$\Sigma_m=\left\{(a,b,c)\in\mathbb{R}^3,\ \text{all root }\zeta\text{ of (1.5) satisfying }\ |\zeta|<1\right\}. \tag{1.6}$$

Now we come across with a simple question whether $\Sigma_*\subset\Sigma_m$for all $m\geq 1$ a property, which is called $N\tau(0)-$ stability. The set

$$A=\{z\in C,\ |R(z)|>|e^z|\ \},$$

is called order star of R. The order star does not compare $R(z)$ to 1, as does the stability domain, but to the exact solution $|e^z|=e^x$ and consequently it is hoped that it would give more information. As we always assume that the coefficients of $R(z)$ are real, the order star is symmetric with respect to the real axis. Furthermore since $\left|e^{iy}\right|=1$, A is the complementary set of the stability domain S on the imaginary axis.

In this part we are concerned about Pade'-approximation to the exponential [5, 6, 11, 12]. They are defined by $R_{k,l}(z)$ such that

$$R_{kl}(z)=P_{kl}(z)/Q_{kl}(z),$$

with

$$P_{kl}(z)=1+\frac{k}{k+l}z+\cdots+\frac{k(k-1)\cdots 1}{(k+l)(k+l-1)\cdots(l+1)}\frac{z^k}{k!}$$
$$Q_{kl}(z)=P_{lk}(-z),$$

and the constant error

$$C=(-1)^s\frac{s!(s-j)!}{(2s-j)!(2s-j+1)!}.$$

Lemma 1.1. *$AR_{kl}(z)$ Pade'approximation to the exponential is A-stable if and only if $k\leq l\leq k+2$.*

For its proof consult [12].

2. Stability Analysis with Respect to c

In order to show $\Sigma_*\subseteq\Sigma_m$ we apply the root locus technique [2, 5]. It is hard to represent geometrically the set of all $(a,\,b,\,c)\in\mathbb{R}^3$ satisfying (1.5), so it is more convenient to reduce the dimension and get a 2-dimensional parametrization. We choose c as a parameter and develop the stability analysis in the (a, b)-plane. We define by:

$$\Sigma_*^c = \left\{(a,b) \in I\!R^2,\ \text{all root } \zeta \text{ of (1.2) satisfying } |\zeta| < 1\right\}.$$

$$\Sigma_m^c = \left\{(a,b) \in I\!R^2,\ \text{all root } \zeta \text{ of (1.5) satisfying } |\zeta| < 1\right\}.$$

So we need only to show $\Sigma_*^c \subseteq \Sigma_m^c$ for any fixed $c \in (-1, 1)$. Since z of (1.5) depends continuously on a, b, ($R(z) = P(z)/Q(z)$) it's root of $(mz - a)P(z)^m - (mcz + b)Q(z)^m = 0$, also $\zeta = R(z)$ depends continuously on a, b Therefore it is sufficient to prove for the values of (a, b) for fixed c satisfying (1.5) with $|\zeta| = 1$ all lie outside in the analytical stability region Σ_*^c

Lemma 2.1. *Suppose that the stability domain $S = \{z \in C,\ |R(z)| \leq 1\}$ is connected, let $z(t) = x(t) + iy(t)$for $t \in (-\varepsilon, \varepsilon)$ be a smooth parameterization of ∂S such that $z(-t) = \bar{z}(t), z(0) = 0$and let $z(t)$ be oriented such that S lies to the left. Further more, let $\phi(t)$ be the argument of*

$$R(z(t)) = e^{i\phi(t)}, \tag{2.1}$$

in such way that $\phi(0) = 0$ and $\phi(t)$is continuous. Then the function $\phi(t)$ is strictly monotonically increasing and it satisfies $\phi(-t) = -\phi(t)$ and $\lim\limits_{t \to \varepsilon} \phi(t) = s\pi$, where s is the number of poles $R(z)$.

For the prove see [7, 11].

To investigate the values of (a, b) such that $|\zeta| = 1$ we insert $z = x + iy$ and $\zeta = e^{i\phi}$in (1.5),and comparing real and imaginary part we come out with the following two equations

$$\begin{aligned} a + b\cos m\phi &= (m - cm\cos m\phi)x - cmy\sin m\phi \\ -b\sin m\phi &= cmx\sin m\phi + (m - cm\cos m\phi)y. \end{aligned}$$

For $x = y = \phi = 0$ we get the line$a + b = 0$ (as the analytic stability region), if $\sin m\phi = 0$ and $y = 0$, it happens for $\phi = s\pi$, we get another line $a \pm b = mx(1 \pm c)$. For $0 < \phi < s\pi$ we get

$$a_m(\phi) = my\frac{\cos m\phi - c}{\sin m\phi} + mx, \tag{2.2}$$

$$b_m(\phi) = -my\frac{1 - c\cos m\phi}{\sin m\phi} - cmx.$$

It can be seen from (2.2) that is helpful to compare the values (a_m, b_m) to the values (a_*, b_*) corresponding to $\zeta = im\phi$, which define the root locus for the analytical solution

$$a_*(m\phi) = \frac{m\phi\cos m\phi - cm\phi}{\sin m\phi}, \quad b_*(m\phi) = \frac{cm\phi\cos m\phi - m\phi}{\sin m\phi}.$$

For $0 < m\phi < \pi$ the point (a_*, b_*) is on the boundary of Σ_*. For $m\phi > \pi$ it lies on the right side of the stability region. So we get

$$a_m(\phi) = \frac{y}{\phi}a_*(m\phi) + mx, \tag{2.3}$$

$$b_m(\phi) = \frac{y}{\phi}b_*(m\phi) - cmx.$$

3. Condition for $N\tau(0)-$Stability

It is known from [4] that the necessary condition of $N\tau(0)-$stability is to have $\Sigma_*^c \subset \Sigma_m^c$ in the neighborhood of the double point bifurcation $(a, b) = (1 - c, c - 1)$ which corresponds to $z = 0$ and $\zeta = 1$. Let us assume that the rational function satisfies

$$R(z) = e^z(1 - Cz^{p+1} - Dz^{p+2} - \cdots), \tag{3.1}$$

where $C \neq 0$ and D are the error constants of the approximation then the necessary condition follows.

Theorem 3.1. *We assume that* $p > 1$ *if close to* $(a, b) = (1-c, c-1)$ *we have* $\Sigma_*^c \subset \Sigma_m^c$ *for all* $m \geq 1$ *then:*

$$(-1)^{\frac{p}{2}} C > 0, \quad if\ p\ is\ even,$$

$$(-1)^{\frac{p}{2}} C > 0\ and\ (2+c)(-1)^{\frac{p}{2}} C/6 > (1+c)(-1)^{\frac{p}{2}+1} D,\ if\ p\ is\ odd,$$

holds.

Proof. Close to$z = 0$, we obtain the following parameterize of ∂S

$$x(y) = \begin{cases} (-1)^{k+1} D y^{2k+2} & if\ p = 2k \\ (-1)^k C y^{2k} & if\ p = 2k-1. \end{cases}$$

We obtain $\phi(y)$ from (3.1) by considering the logarithm of (2.1)

$$\phi(y) - y = \begin{cases} (-1)^{k+1} C y^{2k+1} & if\ p = 2k \\ (-1)^{k+1} D y^{2k+1} & if\ p = 2k-1. \end{cases} \tag{3.2}$$

If $(a_m(\phi, c), b_m(\phi, c))$ does not belong to Σ_*^c then

$$(1 - c - cmx - a_m) x(y)/y > (1+c) m(\phi(y) - y), \text{ when } y \to 0.$$

By considering the Taylor expression of a_m we obtain

$$a_m(y) = (1-c) - (2+c) m^2 y^2/6 \cdot,$$

hence we get that

$$(-cm + (2+c) m^2 y^2/6) x(y)/y > (1+c) m(\phi(y) - y). \tag{3.3}$$

The result comes out by using (3.3) and (3.2).

4. Stability Analysis for Gauss-Methods

The stability function of Gauss methods is a symmetric function i.e. $R(z)R(-z) = 1$ and it is given by diagonal pade'-approximation see [9]. Since $x = 0$ for the values on ∂S the relation (2.3) shows the condition $\phi(y) < y$ for $y > 0$ is sufficient to get $\Sigma_*^c \subset \Sigma_m^c$for all$m \geq 1$. For $0 < \phi < \pi$ it's also necessary to have $\phi(y) < y$.

Lemma 4.1. *Let* $R(z)$ *be symmetric and assume that the order star A has the imaginary axis as boundary with A lying to the left. Then the function* $\phi(y)$ *defined by* $R(iy) = e^{i\phi(y)}$ *and* $\phi(0) = 0$ *satisfies*

$$\phi(y) < y \text{ for } y > 0, \tag{4.1}$$

For the proof see [7].

4.1. Description of the Curves Γ_k^c

We consider the curves of the locus roots (2.2) with $x = 0$ (Gauss methods) and denote by

$$\Gamma_k^c = \left\{ (a_m(\phi, c), b_m(\phi, c),) \in R^2,\ m\phi \in ((k-1)\pi\,, k\pi) \right\},$$

with k a positive integer.

Proposition 4.1. *The curves* Γ_k^c *are all separated.*

If k *is even the curve* Γ_k^c *lies in the sector* $|a| < b$, *if* k *is odd the curve* Γ_k^c *lies in the sector*$|a| < -b$. *And the curve* Γ_1^c *starts at the point* $\delta_c = (1-c, c-1)$, *which is the double bifurcation point.*

Proof. We first show that the curves Γ_k^c are all separated, for that sake we consider the parameterization of the form

$$a = \frac{\mu\cos\alpha - c\mu}{\sin\alpha}, \quad b = \frac{c\mu\cos\alpha - \mu}{\sin\alpha},$$

where $\mu \geq 0$ andα fixed For $\alpha \in (0, \pi)$, the curve lies in the sector $|a| < -b$, and it intersects Γ_k^c(k odd) at$\mu = my(\phi)$ and $m\phi = \alpha + (k-1)\pi$. Since $my(\phi)$ is monotonically increasing, different values of k cannot give the same μ. This proves the curves are separated.

For $m\phi \in ((k-1)\pi,\, k\pi)$ we have

$$b_{.m}(\phi) - a_m = -my(1-c)\frac{1+\cos m\phi}{\sin m\phi},$$

$$b_{.m}(\phi) + a_m = -my(1+c)\frac{1-\cos m\phi}{\sin m\phi}.$$

Since the $-my(1-c)(1+\cos m\phi), -my(1+c)(1-\cos m\phi)$, are negative. The sign of $b_m(\phi) - a_m(\phi), b_m(\phi) + a_m(\phi)$, will depend on the sign of $\sin m\phi$. For k even the curves lie on the sector $|a| < b$. Similarly for k odd the curves lie on the sector $|a| < -b$. Therefore we can see that the Γ_2^c, Γ_4^c, $\Gamma_6^c \ldots$ and Γ_1^c, Γ_3^c, $\Gamma_5^c \ldots$ are ordered in the natural way in the sector $|a| < b$ and $|a| < -b$ respectively (see Fig2).

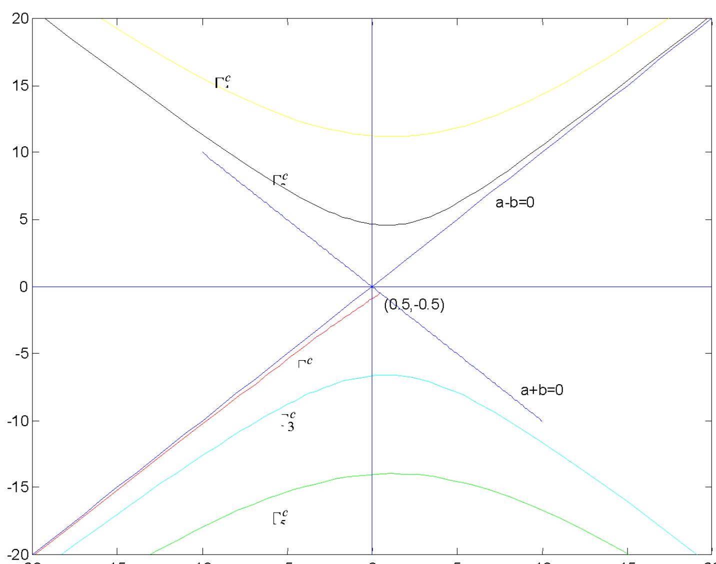

Figure 2. Plot of Γ_k^c for $c = 0.5$, $m = 10$ for implicit midpoint

Theorem 4.1. *All Gauss-methods are $N\tau(0)-$ stable.*

Proof. The stability function of an s-stage Gauss-method is the $R_{ss}(z)$ Pade'-approximation to the exponential which is symmetric and A-stable by Lemma 2.4 with $p = 2s$. It satisfies the necessary condition $(-1)^{p/2}C > 0$. The whole imaginary axis lies on the boundary of the order star because of the symmetry of $R(z)$. Since

$$R(z) = e^z(1 - Cz^{p+1} - O(z^{p+2}))$$

with $z = re^{it}$, hence $1 < |R(z)e^{-z}| \Leftrightarrow \left|1 - Cz^{p+1}\right| > 1$, we have $C\cos(p+1)t < 0$. The fact $(-1)^{p/2}C > 0$, it implies $(-1)^{p/2}C(-1)^{p/2}\cos(p+1)t < 0$, and we obtain $(-1)^{p/2}\cos(p+1)t < 0$. It means that close to $z = 0$ the order star touches the imaginary axis from the left side. There are exactly $p/2+1$ sector of A starting in $\mathbb{C}^-$ and they go to infinity due to A-stability. The surrounded sectors of $\mathbb{C}\backslash A$ leads to exactly $p/2 = s$ zeros of $R(z)$. We suppose that, the surrounded sector touches the imaginary axis from the left side, it proves that, there are two additional zeros which contradicts the fact that $P(z)$is polynomial of order$s$. As $x(y) = 0$ due to the symmetry of $R(z)$ from Lemma 2.1 we obtain $\phi(y) < y$ for all $y > 0$ and the relation (2.3) implies that $(a_m(\phi), b_m(\phi))$ lies outside Σ_*^c for all ϕ

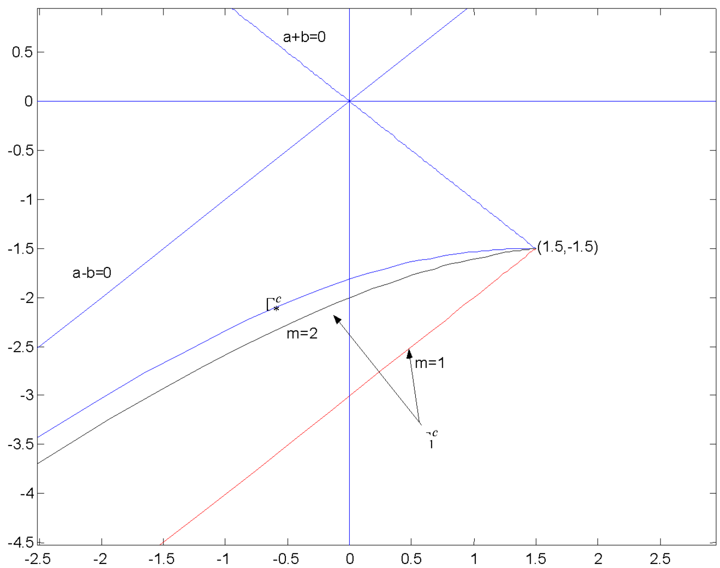

Figure 3. Stability region for implicit midpoint for $c = -0.5$.

Theorem 4.2. *For Gauss-methods the stability region* Σ_m^c *is the set which is bounded to the right by the half line*$a + b = 0$ *with* $a < 1 - c$ *and by the curve* Γ_1^c

Proof. We shall show that if we cross a curve Γ_k^c outward, the roots ζ of (1.5) satisfying $|\zeta| > 1$ increase by 2. By continuity of argument, it is sufficient to prove this $|\zeta| > 1$ on the $b-$axis, so let us take $a \equiv 0$ in the equation (1.5) and by differentiating it we obtain

$$\Delta\zeta = R'(z)\Delta z, \quad m\Delta z = \frac{\zeta^{-m}}{(1 + c\zeta^{-m})}\Delta b - \frac{m\zeta^{-m-1}b\Delta\zeta}{(1 + c\zeta^{-m})^2},$$

hence we get

$$\frac{\Delta b}{b} = \left[\frac{1}{z} + \left(\frac{m^2 cz}{b} + m\right)\frac{R'(z)}{R(z)}\right]\Delta z$$

If b crosses the curve Γ_k^cwe have to consider the point $z = iy$ on the imaginary axis for which $|\zeta| = |R(iy)| = 1$. From $|R(iy)| = e^{i\phi(y)}$ by differentiation we get

$R'(iy)/R(iy) = \phi'(y)$which is positive by Lemma 4.1. Hence $Re\Delta z > 0$ whenever $\frac{\Delta b}{b} > 0$, which means that the root z (and its conjugate) crosses the imaginary axis from the left to the right. Since the method is A-stable, that means ζ leaves the unit circle.

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