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DOMAIN DECOMPOSITION METHODS WITH NONMATCHING GRIDS FOR THE UNILATERAL PROBLEM[*1)]

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Abstract

This paper is devoted to the construction of domain decomposition methods with nonmatching grids based on mixed finite element methods for the unilateral problem. The existence and uniqueness of solution are discussed and optimal error bounds are obtained. Furthermore, global superconvergence estimates are given.

Key words: Domain decomposition, Variational inequality, Global superconvergence, Nonmatching grids.

1. Introduction

Domain decomposition methods (DDMs) with nonmatching grids, which have been developed in recent years, are a quite new class of nonconforming DDMs. As this kind of DDMs can be applied to solving many practical problems which can't be handled by using the usual DDMs, they are been earning particular attention of computational experts and engineers. The advantege of these methods is that they allow non-coincidence of the nodal points at common edges or common faces. Thus they can deal with the problems of moving grids and can design the optimal meshes, namely, one can choose different mesh-sizes and different orders of approximate polynomials in different subdomains according to the different properties of solutions and different requirements of practical problems.

The superconvergence estimates for the finite element methods have been developed in the last ten years. Its mathematical framework is being perfected. We refer to Křižek and Neittanmäki [14], Lin and Xu [16], Lin and Zhu [17,24], Křižek [15] and Wahlbin [23] for details.

The finite element approximations and error analysis for the unilateral problem were studied in many papers. We refer to Brezzi, Hager and Raviart [4,5], Glowinski, Lions and Trémoliérés [9], Haslinger [10], Haslinger and Hlavácek [11,12], Kikuchi and Oden [13] for more details.

We will discuss in this paper the domain decomposition methods with nonmatching grids for the unilateral problem. The nonconforming on the interface of subdomains is handled by introducing Lagrange multipliers. The finite element analysis of the mixed formulation for the unilateral problem is presented and error estimates are derived. Furthermore, we give the global superconvergences if the partition of domain Ω is almost a uniform piecewise strongle regular mesh and the solution is smooth enough.

Let Ω be a bounded domain in R^n with Lipschitz boundary. We will use the usual Sobolev space $W^{m,p}(\Omega)$ consisting of real valued functions defined on Ω with derivatives through order m in $L^p(\Omega)$ and the norm on $W^{m,p}(\Omega)$ is denoted by $||\cdot||_{m,p,\Omega}$. In particular, we define

$$H^m(\Omega) = W^{m,2}(\Omega) \quad \text{and} \quad ||\cdot||_{m,\Omega} = ||\cdot||_{m,2,\Omega}.$$

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Let us consider the following unilateral problem:

$$\begin{cases} -\triangle u + u = f, & \text{in} \quad \Omega, \\ u - g \geq 0, \ \frac{\partial u}{\partial n} \geq 0, & (u-g)\frac{\partial u}{\partial n} = 0, \quad \text{on} \quad \partial\Omega \equiv \Gamma, \end{cases} \tag{1}$$

where n and $\frac{\partial u}{\partial n}$ denote the unit outer normal and the derivative with respect to the outward normal n on Γ, respectively, f and g are given function. Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ inner product. We introduce the convex set

$$K = \{v | \ v \in H^1(\Omega), \nu v - g \geq 0 \ \text{ a.e. on } \Gamma\},$$

where νv denotes the trace of v on the boundary Γ and in the following, we will omit the notation ν without confusion. Then corresponding a variational formulation of the problem (1) can be defined as follows:

$$\begin{cases} \text{find}\, u \in K & \text{such that} \\ a(u, v-u) & \geq (f, v-u), \forall v \in K, \end{cases} \tag{2}$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv),$$

$f \in L^2(\Omega)$ and $g \in H^1(\Omega)$.

We find that the continuous bilinear form $a(\cdot,\cdot)$ on the Hilbert space $H^1(\Omega) \times H^1(\Omega)$ satisfy

$$a(u,v) \leq \delta ||u||_{1,\Omega} ||v||_{1,\Omega}, \quad a(v,v) \geq \kappa ||v||^2_{1,\Omega}, \tag{3}$$

where δ and κ are positive constants. Then we know that the problem (2) exists a unique solution $u \in K$. Moreover, for g and f sufficiently smooth, $u \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ and the pointwise relations (1) hold (cf., [4,pp.440], [5,pp.12]).

2. Domain Decomposition

Now we shall consider domain decomposition methods with nonmatching grids. For simplicity, we assume that Ω is a bounded and convex polygomal domain in R^n. We first will divide the domain Ω into some subdomains Ω_i $(i = 1, \ldots, n_d)$ with size d_i (for simplicity $d_i = d$) and then subdivide these subdomains Ω_i and its boundary $\partial\Omega_i$ into quasi-uniform finite element meshes $T_{h_i} = \{e\}$ with size h_i and $T_{H_i} = \{\tau\}$ with size H_i, respectively. Let $h = \max\{h_i\}$ and $H = \max\{H_i\}$. We will use the following notations:

$$T^h = \cup_{i=1}^{n_d} T_{h_i}, \quad T^H = \cup_{i=1}^{n_d} T_{H_i}, \quad \Gamma_j = \Gamma \cap \partial\Omega_j \neq \phi, \ \Gamma = \cup_{j=1}^{m_d} \Gamma_j,$$

$$\Sigma = \cup_{i=1}^{n_d} \partial\Omega_i, \ \Sigma^i_{int} = \partial\Omega_i \backslash \Gamma, \qquad \Sigma_{int} = \cup_{i=1}^{n_d} \Sigma^i_{int},$$

and define the functional spaces

$$H(\Omega) = \prod_{i=1}^{n_d} H^1(\Omega_i) \ \text{ and } \ H(\Sigma) = \prod_{i=1}^{n_d} H^{-\frac{1}{2}}(\partial\Omega_i)$$

with the norm

$$|||v|||^2_{1,\Omega} = \sum_{i=1}^{n_d} ||v||^2_{1,\Omega_i} \quad \text{and} \quad |||\mu|||^2_{-\frac{1}{2},\Sigma} = \sum_{i=1}^{n_d} ||\mu||^2_{-\frac{1}{2},\partial\Omega_i}$$

respectively, where

$$||u||^2_{\frac{1}{2},\partial\Omega_i} = d^{-1}||u||^2_{0,\partial\Omega_i} + |u|^2_{\frac{1}{2},\partial\Omega_i},\ |u|^2_{\frac{1}{2},\partial\Omega_i} = \int_{\partial\Omega_i}\int_{\partial\Omega_i}\frac{(u(x)-u(y))^2}{(x-y)^2}dxdy$$

and

$$||u||_{-\frac{1}{2},\partial\Omega_i} = \sup_{\substack{v\in H^{\frac{1}{2}}(\partial\Omega_i)\\ v\neq 0}} \frac{|\int_{\partial\Omega_i} uv|}{||v||_{\frac{1}{2},\partial\Omega_i}}.$$

For further notations we refer to Necás [21], Ciarlet and Lions [7], and Kikuchi and Oden [13].

We construct the domain decomposition variational formulation of the problem (1) as follows: Find $(u,\lambda)\in H(\Omega)\times\hat{H}(\Sigma)$ such that

$$\sum_{i=1}^{n_d}\{a_i(u,v)-[\lambda,v]_{\partial\Omega_i}\} = \sum_{i=1}^{n_d}(f,v)_{\Omega_i},\ \forall v\in H(\Omega), \tag{4}$$

$$\sum_{i=1}^{n_d}[\mu,u]_{\Sigma^i_{int}} = 0,\ \forall\mu\in H(\Sigma), \tag{5}$$

$$\sum_{j=1}^{m_d}[\mu-\lambda,u-g]_{\Gamma_j} \geq 0,\ \forall\mu\in\hat{H}(\Sigma), \tag{6}$$

where $[\cdot,\cdot]$ denotes duality pairing on $H^{-\frac{1}{2}}\times H^{\frac{1}{2}}$ and

$$a_i(u,v)=\int_{\Omega_i}(\nabla u\cdot\nabla v+uv),$$

$$\hat{H}(\Sigma)=\{\mu\in H(\Sigma)|\ \ \mu\geq 0 \quad \text{a.e. on } \Gamma\}.$$

Theorem 1. *Let $f\in L^2(\Omega)$ and $g\in H^1(\Omega)$. Then the problem (4)-(6) has a unique solution $(u,\lambda)\in H(\Omega)\times\hat{H}(\Sigma)$. Moreover, $u\in K$ is the solution of the problem (2) and satisfies*

$$\lambda=\frac{\partial u}{\partial n}\quad on\ \ \partial\Omega_i,\ \ 1\leq i\leq n_d. \tag{7}$$

Proof. Let $\mathcal{D}(\Omega)$ denote the space of test functions defined on Ω and (w,λ) be a solution of the problem (4)-(6). Then by the Green's formula in each Ω_i choosing $v\in\mathcal{D}(\Omega_i)$ in (4), we obtain

$$-\triangle w+w=f,\quad \text{in}\ \ \Omega_i.$$

For any $\mathbf{q}\in(\mathcal{D}(\Omega))^n$, we have by (5) and $\mathbf{n}\cdot\mathbf{q}\in H(\Sigma)$

$$\begin{aligned}\int_\Omega w\text{div}\mathbf{q} &= -\sum_{i=1}^{n_d}\int_{\Omega_i}\text{grad}w\cdot\mathbf{q}+\sum_{i=1}^{n_d}\int_{\partial\Omega_i}w\mathbf{n}\cdot\mathbf{q}\\ &= -\sum_{i=1}^{n_d}\int_{\Omega_i}\text{grad}w\cdot\mathbf{q},\end{aligned}$$

which deduces

$$|\int_\Omega w\text{div}\mathbf{q}|\leq(\sum_{i=1}^{n_d}||\text{grad}w||^2_{0,\Omega_i})^{\frac{1}{2}}||\mathbf{q}||_{0,\Omega},$$

i.e.: $w \in H^1(\Omega)$. By (6), it is easily deduced that $w - g \geq 0$ on Γ. Hence $w \in K$. Furthermore for any $v \in K$, we have by (4)-(6), $v - g \geq 0$ and $\lambda \geq 0$ on Γ,

$$\begin{aligned} a(w, v-w) &= (f, v-w) + \sum_{i=1}^{n_d} [\lambda, v-w]_{\partial\Omega_i} \\ &= (f, v-w) + \sum_{j=1}^{m_d} [\lambda, v-g]_{\Gamma_j} + \sum_{j=1}^{m_d} [\lambda, g-w]_{\Gamma_j} \\ &\geq (f, v-w). \end{aligned} \tag{8}$$

Thus from (2),(8) and the uniqueness of the solution of the problem (2), implies that w is a unique solution of the problem (2) and $w = u$.

Now since $w \in K$ is the solution of the problem (2), we get (7) by using the Green's formula in (4).

Conversely, let $u \in K$ be the solution of the problem (2). Then it is easily to check that $(u, \frac{\partial u}{\partial n}) \in H(\Omega) \times \hat{H}(\Sigma)$ satisfy (4). As far as (5) and (6), it is a direct verification by $u \in H^1(\Omega)$ and $u - g \geq 0$ on Γ. This completes our proof by the uniqueness of the solution of the problem (2).

Let $S_{h_i}(\Omega_i)$ denote the finite element space of piecewise m-th polynomial functions which are defined on meshes $T_{h_i} = \{e\}$ and

$$S_{H_i}(\partial\Omega_i) = \{\mu |\ \mu|_\tau \in P_n(\tau),\ \forall \tau \in T_{H_i}\}, i = 1, \cdots, n_d,$$

where $P_n(\tau)$ is the space of n-th polynomial functions defined on meshes T_{H_i}, $m \geq 1$ and $n \geq 0$. Let

$$S_H(\Sigma) = \prod_{i=1}^{n_d} S_{H_i}(\partial\Omega_i), \quad S_h(\Omega) = \prod_{i=1}^{n_d} S_{h_i}(\Omega_i),$$

$$N_H(\Sigma) = \{\mu \in S_H(\Sigma) |\ \mu|_{\Gamma_j} \geq 0 \text{ at nodes on } \Gamma_j, j = 1, \cdots, m_d\}.$$

Then the finite element approximation of the problem (4)-(6) is as follows: Find $(u^h, \lambda^H) \in S_h(\Omega) \times N_H(\Sigma)$ such that

$$\sum_{i=1}^{n_d} \{a_i(u^h, v^h) - [\lambda^H, v^h]_{\partial\Omega_i}\} = \sum_{i=1}^{n_d} (f, v^h)_{\Omega_i}, \ \forall v^h \in S_h(\Omega), \tag{9}$$

$$\sum_{i=1}^{n_d} [\mu, u^h]_{\Sigma^i_{int}} = 0, \ \forall \mu \in S_H(\Sigma), \tag{10}$$

$$\sum_{j=1}^{m_d} [\mu - \lambda^H, u^h - g]_{\Gamma_j} \geq 0, \ \forall \mu \in N_H(\Sigma). \tag{11}$$

Lemma 1. *We suppose that $E = \frac{h}{H}$ is a sufficiently small number. Then there exists a constant $\beta > 0$ independent of h and H such that*

$$\sup_{\substack{v \in S_h(\Omega) \\ v \neq 0}} \frac{\sum_{i=1}^{n_d} [\lambda, v]_{\partial\Omega_i}}{|||v|||_{1,\Omega}} \geq \beta |||\lambda|||_{-\frac{1}{2},\Sigma}, \quad \forall \lambda \in S_H(\Sigma). \tag{12}$$

Proof. Let $\lambda \in S_H(\Sigma)$ and w_i be the solution of the following Neumann problem:

$$\begin{cases} -\triangle w_i + w_i &= 0, \quad \text{in} \quad \Omega_i, \\ \dfrac{\partial w_i}{\partial n} &= \lambda, \quad \text{on} \quad \partial\Omega_i. \end{cases} \tag{13}$$

Let $w \in H(\Omega)$ such that $w|_{\Omega_i} = w_i, i = 1, \cdots, n_d$. Then we know that the function $w_i \in H^1(\Omega_i)$ exists and

$$\int_{\Omega_i} (\bigtriangledown w \bigtriangledown v + wv) = \int_{\partial\Omega_i} \lambda v ds, \quad \forall v \in H^1(\Omega_i). \tag{14}$$

Furthermore from the regularity results [2,pp.183] and inverse estimation [7,pp.140], we obtain

$$||w||_{\frac{3}{2},\Omega_i} \le c||\lambda||_{0,\partial\Omega_i} \le cH^{-\frac{1}{2}}||\lambda||_{-\frac{1}{2},\partial\Omega_i} \tag{15}$$

and exists $z \in S_h(\Omega)$ (cf., [2,pp.185], [8,pp.105]) such that

$$||w - z||_{1,\Omega_i} \le ch^{\frac{1}{2}}||w||_{\frac{3}{2},\Omega_i}, \quad ||z||_{1,\Omega_i} \le ||w||_{1,\Omega_i}. \tag{16}$$

Using (14) and Theorem 2.7 in [2], we have

$$||w||^2_{1,\Omega_i} = \int_{\partial\Omega_i} \lambda w ds \ge ||\lambda||^2_{-\frac{1}{2},\partial\Omega_i}, \tag{17}$$

$$\int_{\partial\Omega_i} z\lambda = \int_{\partial\Omega_i} w\lambda ds - \int_{\partial\Omega_i} \lambda(w - z)ds \ge \tilde{c}||w||^2_{1,\Omega_i} - \eta, \tag{18}$$

where $\eta = |\int_{\partial\Omega_i} \lambda(w - z)ds|$. Using (15)–(17), we get

$$\eta \le c(\frac{h}{H})^{\frac{1}{2}}||\lambda||^2_{-\frac{1}{2},\partial\Omega_i} \le c(\frac{h}{H})^{\frac{1}{2}}||w||^2_{1,\Omega_i},$$

which, combination (16)–(18) with $E = \frac{h}{H}$ sufficiently small, implies

$$\begin{aligned} \sum_{i=1}^{n_d} \int_{\partial\Omega_i} z\lambda &\ge (\tilde{c} - c(\frac{h}{H})^{\frac{1}{2}}) \sum_{i=1}^{n_d} ||w||^2_{1,\Omega_i} \\ &\ge c(\sum_{i=1}^{n_d} ||w||^2_{1,\Omega_i})^{\frac{1}{2}} (\sum_{i=1}^{n_d} ||w||^2_{1,\Omega_i})^{\frac{1}{2}} \\ &\ge c(\sum_{i=1}^{n_d} ||z||^2_{1,\Omega_i})^{\frac{1}{2}} (\sum_{i=1}^{n_d} ||\lambda||^2_{-\frac{1}{2},\partial\Omega_i})^{\frac{1}{2}}. \end{aligned}$$

This completes our proof.

Theorem 2. *Under the assumptions of the Lemma 1 and Theorem 1, the finite element approximation of the problem (9)–(11) has a unique solution* $(u^h, \lambda^H) \in S_h(\Omega) \times N_H(\Sigma)$.

Proof. This follows from the Lemma 1 and the v-ellipticity of $a(\cdot, \cdot)$.

3. Error Estimates

In this section, we will give error estimates and assume that $m = 1$ and $n = 0$ in $S_h(\Omega)$ and $S_H(\Sigma)$, respectively, i.e.: $S_h(\Omega)$ and $S_H(\Sigma)$ are the finite element spaces of piecewise linear and piecewise constant respectively. First we prove an auxiliary lemma.

Lemma 2. *Let (u,λ) and (u^h,λ^H) be the solutions of the problem (4)–(6) and (9)–(11), respectively. Then*

$$|||u-u^h|||_{1,\Omega}^2 \leq c|||u-v|||_{1,\Omega}^2 + c|||\lambda-\mu|||_{-\frac{1}{2},\Sigma}^2 + c\sum_{j=1}^{m_d}[\mu-\lambda, u-g]_{\Gamma_j}, \tag{19}$$

$\forall v \in S_h(\Omega), \mu \in N_H(\Sigma)$ *and*

$$|||\lambda-\lambda^H|||_{-\frac{1}{2},\Sigma} \leq c|||u-u^h|||_{1,\Omega} + c|||\lambda-\mu|||_{-\frac{1}{2},\Sigma}, \tag{20}$$

$\forall \mu \in N_H(\Sigma)$.

Proof. From (5), (6) and (11), we see that

$$\sum_{j=1}^{m_d}[\lambda-\lambda^H, u-g]_{\Gamma_j} \leq 0, \qquad \sum_{i=1}^{n_d}[\mu-\lambda^H, u-u^h]_{\Sigma^i_{int}} = 0,$$

$$\sum_{j=1}^{m_d}[\mu-\lambda^H, g-u^h]_{\Gamma_j} \leq 0, \qquad \forall \mu \in N_H(\Sigma).$$

Hence for any $v \in S_h(\Omega)$ and $\mu \in N_H(\Sigma)$, let $w = v-u^h$, we have by (4), (9) and above relations

$$\begin{aligned}\sum_{i=1}^{n_d} a_i(u-u^h, w) &= \sum_{i=1}^{n_d}[\lambda-\lambda^H, w]_{\partial\Omega_i} \\ &= \sum_{i=1}^{n_d}[\lambda-\mu, w]_{\partial\Omega_i} + \sum_{i=1}^{n_d}[\mu-\lambda^H, w]_{\partial\Omega_i} \\ &\leq c\sum_{i=1}^{n_d}||\lambda-\mu||_{-\frac{1}{2},\partial\Omega_i}||w||_{1,\Omega_i} \\ &\quad + c\sum_{i=1}^{n_d}||\mu-\lambda^H||_{-\frac{1}{2},\partial\Omega_i}||v-u||_{\frac{1}{2},,\partial\Omega_i} \\ &\quad + c\sum_{j=1}^{m_d}[\mu-\lambda, u-g]_{\Gamma_j},\end{aligned} \tag{21}$$

which implies (19) with $ab \leq \epsilon^{-1}a^2 + \epsilon b^2 (a, b \geq 0, \epsilon > 0)$, triangle inequality and the v-ellipticity of $a(\cdot,\cdot)$. From (12) and (3), we get

$$\begin{aligned}|||\mu-\lambda^H|||_{-\frac{1}{2},\Sigma} &\leq c \sup_{\substack{v\in S_h(\Omega)\\ v\neq 0}} \frac{\sum_{i=1}^{n_d}[\mu-\lambda^H, v]_{\partial\Omega_i}}{|||v|||_{1,\Omega}} \\ &\leq c|||\mu-\lambda|||_{-\frac{1}{2},\Sigma} + c \sup_{\substack{v\in S_h(\Omega)\\ v\neq 0}} \frac{\sum_{i=1}^{n_d} a_i(u-u^h, v)}{|||v|||_{1,\Omega}} \\ &\leq c|||\mu-\lambda|||_{-\frac{1}{2},\Sigma} + c|||u-v|||_{1,\Omega} + c|||v-u_h|||_{1,\Omega}\end{aligned} \tag{22}$$

and combining the triangle inequality we get (20) immediately.

Let I^h denote the interpolation operator in $S_h(\Omega)$ and R_{H_i} be the usual L^2- projection operator satisfying, $R_{H_i}\lambda \in S_{H_i}(\partial\Omega_i)$,

$$\int_\tau (\lambda - R_{H_i}\lambda) = 0, \quad \forall \tau \in T_{H_i}. \tag{23}$$

Let $R^H\lambda \in S_H(\Sigma)$ such that $R^H\lambda|_{\partial\Omega_i} = R_{H_i}\lambda$. Then we have following results (cf., [7,pp.124] and [24,pp.23-24]):

Lemma 3. *If $\lambda \in H^r(\Sigma), r \geq 0, u \in H^t(\Omega)$, $t \geq 1$, then we have*

$$||\lambda - R^H\lambda||_{-s,\tau} \leq cH^{\mu_1}||\lambda||_{r,\tau}, \quad \forall \tau \in T_{H_i},$$

$$|||u - I^h u|||_{s,\Omega} \leq ch^{\mu_2}|||u|||_{t,\Omega},$$

where $\mu_1 = min(r+s, 1+s), \mu_2 = min(t-s, 2+s)$ and $0 \leq s \leq 1$.

Theorem 3. *Let $(u,\lambda) \in H(\Omega) \times \hat{H}(\Sigma)$ and $(u^h, \lambda^H) \in S_h(\Omega) \times N_H(\Sigma)$ be the solutions of the problems (4)–(6) and (9)–(11), respectively. We suppose that $u \in H^2(\Omega_i)(i = 1, \cdots, n_d), g \in W^{1,q}(\Gamma_j)(j = 1, \cdots, m_d), q \geq 2$. Moreover, let us assume that the number of points where $u - g$ changes from $u - g > 0$ to $u - g = 0$ is finite. Then*

$$|||u - u^h|||_{1,\Omega} \leq c(h + H^{1-\epsilon}), \tag{24}$$

$$|||\lambda - \lambda^H|||_{0,\Sigma} \leq c(H^{-\frac{1}{2}}h + H^{\frac{1}{2}-\epsilon}), \tag{25}$$

$$|||\lambda - \lambda^H|||_{-\frac{1}{2},\Sigma} \leq c(h + H^{1-\epsilon}), \tag{26}$$

where $\epsilon = \frac{1}{2q}$.

Proof. By choosing $v = I^h u \in S_h(\Omega)$ and $\mu = R^H\lambda \in N_H(\Sigma)$ (since $\lambda \in \hat{H}(\Sigma)$) in (19), we have with the virtue of Lemma 3

$$|||u - u^h|||^2_{1,\Omega} \leq ch^2|||u|||^2_{2,\Omega} + cH^2|||\lambda|||^2_{\frac{1}{2},\Sigma} + c\sum_{j=1}^{m_d}[R^H\lambda - \lambda, u - g]_{\Gamma_j}. \tag{27}$$

Let $\Gamma^+ = \{x \in \Gamma | u - g > 0\}$ and $\Gamma^0 = \{x \in \Gamma | u - g = 0\}$. If $\tau \subseteq \Gamma^+$, then $\lambda = 0$ and $R^H\lambda = 0$ on τ. If $\tau \subseteq \Gamma^0$, then $u - g = 0$. Hence

$$[R^H\lambda - \lambda, u - g]_\tau = 0, \quad \text{for all} \quad \tau \subseteq \Gamma^+ \cup \Gamma^0.$$

Let $w = u - g$. If $\tau \not\subseteq \Gamma^+ \cup \Gamma^0$, then, using the assumptions of the Theorem and (23), for $q \geq 2$,

$$\begin{aligned} \sum_{\tau \not\subseteq \Gamma^+ \cup \Gamma^0}[R^H\lambda - \lambda, w]_\tau &= \sum_{\tau \not\subseteq \Gamma^+ \cup \Gamma^0}[R^H\lambda - \lambda, w - R^H w]_\tau \\ &\leq cH^{2-\frac{1}{q}}||\lambda||_{\frac{1}{2},\tau}(||g||_{1,q,\tau} + |||u|||_{2,\Omega}), \end{aligned} \tag{28}$$

where we have used the Sobolev imbedding theorem from $H^2(\Omega)$ to $W^{1,q}(\Gamma)$ and Lemma 3. This implies (24) by taking $\epsilon = \frac{1}{2q}$. From (20), (24), and Lemma 3 leads to (26) immediately. From inverse estimate(see [15]) and triangle inequality we have (25).

Remark 1. We only require that $g \in W^{1,q}(\Gamma_j)$ in Theorem 3. If $g \in W^{1,\infty}(\Gamma_j)$ and $u \in W^{1,\infty}(\Gamma_j)$, then from (28) we see easily that the ϵ in inequalities (24)–(26) may be omitted.

4. Global Superconvergence Estimates

In this section, we assume that $T_{h_i} = \{e\}$ is an almost uniform piecewise strongly regular mesh on the domain Ω_i (see [16]). $S_{h_i}(\Omega_i)(i = 1, \cdots, n_d)$ is the piecewise bilinear finite element space. Then we have the following Lemma (cf., [16,pp.131-132], [19,pp.368-370],[24,pp.101]):

Lemma 4. *Let $u \in H^3(\Omega_i)$. Then there exists a positive constant c, independent of h and H, such that*

$$|\int_{\Omega_i}(u - I^h u)_l v_k| \leq ch^{\frac{3}{2}}||u||_{3,\Omega_i}||v||_{1,\Omega_i}, \quad \forall v \in S_h(\Omega), \; k,l = x,y,$$

where v_k denotes the partial derivative of v with respect to x or y. We can derive the following results by the above Lemma:

Lemma 5. *Let $u \in H^3(\Omega_i)$. Then there exists a positive constant c independent of h and H such that*

$$\sum_{i=1}^{n_d} a_i(u - I^h u, v) \leq ch^{\frac{3}{2}}|||u|||_{3,\Omega}|||v|||_{1,\Omega}, \quad \forall v \in S_h(\Omega).$$

Theorem 4. *Let us assume that the number of points where $u-g$ changes from $u-g>0$ to $u-g=0$ is finite, $g \in W^{1,\infty}(\Gamma_j)(j = 1,\cdots,m_d)$ and the conditions of Lemma 4 hold. Then we have*

$$|||I^h u - u^h|||_{1,\Omega} \leq c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}}H^{\frac{3}{4}}), \tag{29}$$

$$|||R^H\lambda - \lambda^H|||_{-\frac{1}{2},\Sigma} \leq c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}}H^{\frac{3}{4}}). \tag{30}$$

Proof. Let $v = I^h u - u^h$. Then similar to (22) and (28), we have by Lemma 3 and Lemma 5

$$\sum_{i=1}^{n_d}[\lambda - R^H\lambda, v]_{\partial\Omega_i} \leq cH^{\frac{3}{2}}|||\lambda|||_{1,\Sigma}|||v|||_{1,\Omega}, \tag{31}$$

$$\sum_{i=1}^{n_d}[R^H\lambda - \lambda^H, I^h u - u]_{\partial\Omega_i} \leq ch^{\frac{3}{2}}|||R^H\lambda - \lambda^H|||_{-\frac{1}{2},\Sigma}|||u|||_{3,\Omega}, \tag{32}$$

$$\sum_{\tau\not\subseteq\Gamma_c^+\cup\Gamma_c^0}[R^H\lambda - \lambda, u - g]_\tau \leq cH^{\frac{5}{2}}|||\lambda|||_{1,\Gamma_j}(|||g|||_{1,\infty,\Gamma_j} + |||u|||_{3,\Omega}), \tag{33}$$

$$|||R^H\lambda - \lambda^H|||_{-\frac{1}{2},\Sigma} \leq cH^{\frac{3}{2}}|||\lambda|||_{1,\Sigma} + ch^{\frac{3}{2}}|||u|||_{3,\Omega} + c|||I^h u - u^h|||_{1,\Omega}. \tag{34}$$

Similar to (21) we get by Lemma 5

$$\begin{aligned}
|||I^h u - u^h|||_{1,\Omega}^2 \;&\leq\; c\sum_{i=1}^{n_d}[a_i(u - u^h, v) + a_i(I^h u - u, v)] \\
&\leq\; ch^{\frac{3}{2}}|||u|||_{3,\Omega}|||v|||_{1,\Omega} \\
&\quad + c\sum_{i=1}^{n_d}\{[\lambda - R^H\lambda, v]_{\partial\Omega_i} + [R^H\lambda - \lambda^H, I^h u - u]_{\partial\Omega_i}\} \\
&\quad + c\sum_{j=1}^{m_d}[R^H\lambda - \lambda, u - g]_{\Gamma_j}.
\end{aligned}$$

Hence (29) follows by (31)–(34) and the inequality (30) follows by (29) and (34).

We assume that T_{h_i} is obtaided from T_{2h_i} of mesh size $2h_i$ by uniformly subdividing each element in T_{2h_i} into 4 congruent elements of mesh size h_i. We define the high order interpolation operator π_{2h_i} on the finite element function space $S_{2h_i}(\Omega_i)$ defined on T_{2h_i} and satisfying(cf.,[17,pp.57-59]):

$$\begin{aligned} &\pi_{2h_i} I^h = \pi_{2h_i}, \qquad && \|\pi_{2h_i} v\|_{1,q,\Omega_i} \le \|v\|_{1,q,\Omega_i}, \quad \forall v \in S_{2h_i}(\Omega_i), \\ &\|\pi_{2h_i} u - u\|_{1,q,\Omega_i} \le ch^{\frac{3}{2}} \|u\|_{\frac{5}{2},q,\Omega_i}, && 2 \le q \le \infty. \end{aligned} \tag{35}$$

For $\forall v \in H(\Omega)$, we define $\pi^{2h} v|_{\Omega_i} = \pi_{2h_i} v$.

Theorem 5. *Under the assumptions of the Theorem 4, we obtain the following global superconvergence estimates and optimal estimates*

$$|||u - \pi^{2h} u^h|||_{1,\Omega} \le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}} H^{\frac{3}{4}}), \tag{36}$$

$$|||\lambda - \lambda^H|||_{-\frac{1}{2},\Sigma} \le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}} H^{\frac{3}{4}}), \tag{37}$$

$$|||\lambda - \lambda^H|||_{0,\Sigma} \le c(h^{\frac{3}{2}} H^{-\frac{1}{2}} + H^{\frac{3}{4}} + h^{\frac{3}{4}} H^{\frac{1}{4}}). \tag{38}$$

Proof. We have by (35), Theorem 4 and triangle inequality

$$\begin{aligned} |||u - \pi^{2h} u^h|||_{1,\Omega} &\le |||u - \pi^{2h} u|||_{1,\Omega} + |||\pi^{2h}(I^h u - u^h)|||_{1,\Omega} \\ &\le ch^{\frac{3}{2}} |||u|||_{\frac{5}{2},\Omega} + c|||I^h u - u^h|||_{1,\Omega} \\ &\le c(h^{\frac{3}{2}} + H^{\frac{5}{4}} + h^{\frac{3}{4}} H^{\frac{3}{4}}). \end{aligned}$$

The inequality (37) now follows by (30) and Lemma 3 and the inequality (38) follows by using inverse estimate and (30).

Remark 2. From (33) we see easily that if $\lambda \in W^{1,\infty}(\Gamma_j)(j = 1, \cdots, m_d)$, then the error bounds in inequality (29) and (30) is $\mathcal{O}(h^{\frac{3}{2}} + H^{\frac{3}{2}} + Hh^{\frac{3}{4}})$. Hence the error bounds in inequalities (36) and (37) are $\mathcal{O}(h^{\frac{3}{2}} + H^{\frac{3}{2}} + Hh^{\frac{3}{4}})$. Moreover we may extend the results of Theorem 1–Theorem 5 to bounded convex smooth domain as in [4], [5] and [13].

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