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EFFICIENT SIXTH ORDER P-STABLE METHODS WITH MINIMAL LOCAL TRUNCATION ERROR FOR $y'' = f(x, y)$ *1)

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Abstract

A family of symmetric (hybrid) two step sixth P-stable methods for the accurate numerical integration of second order periodic initial value problems have been considered in this paper. These methods, which require only three (new) function evaluation per iteration and per step integration. These methods have minimal local truncation error (LTE) and smaller phase-lag of sixth order than some sixth orders P-stable methods in [1-3,10-11]. The theoretical and numerical results show that these methods in this paper are more accurate and efficient than some methods proposed in [1-3,10].

Key words: Second order periodic initial value problems, P-stable, Phase-lag, Local truncation error.

1. Introduction

We consider a class of direct hybrid methods proposed in [1] for solving the second order initial value problem

$$y'' = f(t, y), \qquad y(0), y'(0) \quad given \tag{1.1}$$

The basic method has the form

$$\begin{aligned} & y_{n+1} - 2y_n + y_{n-1} \\ =& h^2 \left\{ \beta_0 \left(f_{n+1} + f_{n-1} \right) + \gamma f_n + \beta_1 \left(f_{n+\alpha_1} + f_{n-\alpha_1} \right) + \beta_2 \left(f_{n+\alpha_2} + f_{n-\alpha_2} \right) \right\} \end{aligned} \tag{1.2a}$$

$$y_{n\pm\alpha_1} = A_\pm y_{n+1} + B_\pm y_n + C_\pm y_{n-1} + h^2 \left(S_\pm f_{n+1} + Q_\pm f_n + U_\pm f_{n-1} \right) \tag{1.2b}$$

$$\begin{aligned} y_{n\pm\alpha_2} =& R_\pm y_{n+1} + L_\pm y_n + T_\pm y_{n-1} \\ &+ h^2 \left(Y_\pm f_{n+1} + V_\pm f_n + W_\pm f_{n-1} + Z_\pm f_{n+\alpha_1} + X_\pm f_{n-\alpha_1} \right) \end{aligned} \tag{1.2c}$$

and

$$f_n = f(t_n, y_n), \quad f_{n\pm1} = f(t_n \pm h, y_{n\pm1}),$$
$$f_{n\pm\alpha_1} = f(t_n \pm \alpha_1 h, y_{n\pm\alpha_1}), \quad f_{n\pm\alpha_2} = f(t_n \pm \alpha_2 h, y_{n\pm\alpha_2}).$$

Here $t_n = nh$ and we define $t_{n\pm\alpha_i} = t_n \pm \alpha_i h, i = 1, 2$ and n=0,1,2,3,... Several authors (for example, Cash[1,2], Chawla and Rao[3]) have derived sixth order methods of the form(1.2) which are P-stable (see Lambert and Watson[8]). The methods proposed by Cash [1] require five function evaluations per iteration, in general. Cash [2] and Voss and Serbin[12] shown how the number of function evaluation may be reduced to four per iteration. The method proposed in [2] is obtained by choosing $\alpha_2 = 0$ and requiring the points $(t_n \pm \alpha_2 h, y_{n\pm\alpha_2})$ to be coincident. For the method proposed by Chawla and Rao [3], the number of function

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evaluations per iteration is reduced to three by requiring that $y_{n-\alpha_1}$ and $y_{n-\alpha_2}$are independent of y_{n+1}. This implies that$f(t_n-\alpha_1 h, y_{n-\alpha_1})$ and $f(t_n-\alpha_2 h, y_{n-\alpha_2})$ must be computed once per step rather than once per iteration. Finally, Thomas [10], Khiyal and Thomas [7], Khiyal [6] have derived sixth order P-stable, three function evaluation methods of the form (1.2), for which the iteration matrix is a true perfect cube. Here, in section 2, we have given the order condition of sixth order accuracy with form (1.2) and their local truncation error.

In section 3, we introduce two classes of sixth order accuracy, sixth order phase lag, P-stable methods which require only three function evaluation per iteration and per step integration. By choosing free parameters, so that these methods with minimal local truncation error. By computing the LTE of Cash [2], Chawla and Rao [3] and Thomas [10], we know that the efficient sixth order P-stable methods in this paper are smaller of LTE and smaller phase-lag than these methods of Cash [2], Chawla and Rao [3] and Thomas [10].

In section 4, we discuss the implementation of these methods in this paper and numerical illustration on one simple problem.

2. Basic Theory

Thomas [9] have shown that for methods of the form (1.2) applied to the scalar test equation

$$y'' + \lambda^2 y = \omega e^{ivt} \tag{2.1}$$

withλ, ω and v real, the numerical forced oscillation is in phase (Gladwell and Thomas [5]) with its analytical counterpart if and only if

$$\begin{gathered} C_+ + C_- = A_+ + A_-, S_+ + S_- = U_+ + U_-, T_+ + T_- = R_+ + R_-, \\ X_+ + X_- = Z_+ + Z_-, W_+ + W_- = Y_+ + Y_- \end{gathered} \tag{2.2}$$

and they are sixth order accurate if and only if

$$\beta_0 = \frac{1}{12} - \beta_1\alpha_1^2 - \beta_2\alpha_2^2, \qquad \gamma = 1 - 2\beta_0 - 2\beta_1 - 2\beta_2,$$

$$\beta_1\alpha_1^2 + \beta_2\alpha_2^2 = \frac{1}{20} + \beta_1\alpha_1^4 + \beta_2\alpha_2^4,$$

$$A_+ + B_+ + C_+ = 1, \qquad A_- + B_- + C_- = 1,$$

$$A_+ - C_+ = \alpha_1, A_- - C_- = -\alpha_1,$$

$$A_+ + C_+ + 2(S_+ + Q_+ + U_+) = \alpha_1^2,$$

$$A_- + C_- + 2(S_- + Q_- + U_-) = \alpha_1^2,$$

$$R_+ + L_+ + T_+ = 1, R_- + L_- + T_- = 1,$$

$$R_+ - T_+ = \alpha_2, R_- - T_- = -\alpha_2,$$

$$R_+ + T_+ + 2(Y_+ + W_+ + V_+ + Z_+ + X_+) = \alpha_2^2,$$

$$R_- + T_- + 2(Y_- + W_- + V_- + Z_- + X_-) = \alpha_2^2,$$

$$\begin{aligned} &\beta_1\alpha_1\left[\frac{1}{12}(A_- - C_+) + (S_- - U_+)\right] \\ &+\beta_2\alpha_2\left[\frac{1}{12}(R_- - T_+) + (Y_- - W_+) + (Z_- - X_+)\alpha_1^2\right] = 0, \end{aligned}$$

$$\beta_1\left[\frac{1}{12}\alpha_1^4 - \frac{1}{12}(A_+ + A_-) - (S_+ + S_-)\right]$$
$$+\beta_2\left[\frac{1}{12}\alpha_2^4 - \frac{(R_+ + R_-)}{12} - (Y_+ + Y_-) - (Z_+ + Z_-)\alpha_1^2\right] = 0,$$
$$\beta_1\alpha_1\left[\frac{1}{3}\alpha_1^3 + \frac{(A_- - C_-)}{3} + 2(S_- - U_-)\right]$$
$$+\beta_2\alpha_2\left[\frac{1}{3}\alpha_2^3 + \frac{(R_- - T_-)}{3} + 2(Y_- - W_-) + 2(Z_- - X_-)\alpha_1\right] = 0 \tag{2.3}$$

The local truncation error (LTE) is given by

$$LTE = \left\{\begin{array}{l} C_1 y_n^{(8)} + C_2 F_n y_n^{(6)} + C_3 F'_n y_n^{(5)} + C_4 F''_n y_n^{(4)} + C_5 F_n^{(3)} y_n^{(3)} \\ +C_6 F_n^2 y_n^{(4)} + C_7 F_n F'_n y_n^{(3)} \end{array}\right\} h^8 + O(h^{10}) \tag{2.4}$$

where

$$F = \frac{\partial f}{\partial y}, F' = \frac{dF}{dt}, F'' = \frac{d^2F}{dt^2}, F^{(3)} = \frac{d^3F}{dt^3},$$
$$C_1 = \frac{1}{20160} - \frac{\beta_0}{360} - \frac{\beta_1\alpha_1^6}{360} - \frac{\beta_2\alpha_2^6}{360},$$
$$C_2 = \beta_1\left[\frac{\alpha_1^6}{360} - \frac{(A_+ + A_-)}{360} - \frac{(S_+ + S_-)}{12}\right]$$
$$+ \beta_2\left[\frac{\alpha_2^6}{360} - \frac{(R_+ + R_-)}{360} - \frac{(Y_+ + Y_-)}{12} - \frac{(Z_+ + Z_-)\alpha_1^4}{12}\right],$$
$$C_3 = \beta_1\alpha_1\left[\frac{\alpha_1^5}{60} + \frac{(A_- - C_-)}{60} + \frac{(S_- - U_-)}{3}\right]$$
$$+ \beta_2\alpha_2\left[\frac{\alpha_2^5}{60} + \frac{(R_- - T_-)}{60} + \frac{(Y_- - W_-)}{3} + \frac{(Z_- - X_-)\alpha_1^4}{3}\right],$$
$$C_4 = \frac{\beta_1\alpha_1^2}{2}\left[\frac{\alpha_1^4}{12} - \frac{(A_+ + A_-)}{12} - (S_+ + S_-)\right]$$
$$+ \frac{\beta_2\alpha_2^2}{2}\left[\frac{\alpha_2^4}{12} - \frac{(R_+ + R_-)}{12} - (Y_+ + Y_-) - (Z_+ + Z_-)\alpha_1^2\right],$$
$$C_5 = \frac{\beta_1\alpha_1^3}{6}\left[\frac{\alpha_1^3}{3} + \frac{(A_- - C_-)}{3} + 2(S_- - U_-)\right]$$
$$+ \frac{\beta_2\alpha_2^3}{6}\left[\frac{\alpha_2^3}{3} + \frac{(R_- - T_-)}{3} + 2(Y_- - W_-) + 2(Z_- - X_-)\alpha_1\right],$$
$$C_6 = \beta_2(Z_+ + Z_-)\left[\frac{\alpha_1^4}{12} - \frac{(A_+ + A_-)}{12} - (S_+ + S_-)\right],$$
$$C_7 = \beta_2\alpha_1(Z_+ + Z_-)\left[\frac{\alpha_1^3}{3} + \frac{(A_- - C_-)}{3} + 2(S_- - U_-)\right]$$
$$+ \beta_2\alpha_2\left[(Z_+ + Z_-)\left(\frac{\alpha_1^3}{6} - \frac{(A_+ - C_+)}{6} - (S_+ - U_+)\right)\right. \tag{2.5}$$
$$\left.- (X_+ - X_-)\left(\frac{\alpha_1^3}{6} + \frac{(A_- - C_-)}{6} + (S_- - U_-)\right)\right]$$

By applying the family of sixth order methods (1.2) to test equation

$$y'' + \lambda^2 y = 0, \qquad \lambda > 0,$$

we derived the characteristic equation of the resulting recurrence relation with the form

$$\xi^2 - 2R_{33}(H^2)\xi + 1 = 0, \qquad H = \lambda h \tag{2.6}$$

and

$$R_{33}(H^2) = \frac{1 + a_1 H^2 + a_2 H^4 + a_3 H^6}{1 + b_1 H^2 + b_2 H^4 + b_3 H^6} \tag{2.7}$$

Here

$$a_1 = -\frac{1}{2}\left[\gamma + \beta_1 (B_+ + B_-) + \beta_2 (L_+ + L_-)\right],$$

$$a_2 = \frac{1}{2}\left[\beta_1 (Q_+ + Q_-) + \beta_2 (V_+ + V_-) + \beta_2 (Z_+ + Z_-)(B_+ + B_-)\right] \quad ,$$

$$\alpha_3 = -\beta_2 (Z_+ + Z_-)(Q_+ + Q_-).$$

3. Efficient Sixth Order P-Stable Methods with Minimal LTE

To reduce the number of function evaluation (1.2) to three (new) evaluations per iteration, we consider two possibilities.

Case (1)

We suppose that $y_{n-\alpha_1}$ and $y_{n-\alpha_2}$ are independent of y_{n+1} and suppose that the value of $f_{n+\alpha_1}$at one step can be used for$f_{n-\alpha_1}$ at the next. Then, based on order conditions (2.2) and (2.3), we obtain the following sixth order accurate methods in phase, with sixth order phase-lag and only require three function evaluations per iteration and per step integration, which is denoted as EM6-1.

$$\begin{aligned} &y_{n+1} - 2y_n + y_{n-1} \\ &= h^2 \left\{ \frac{f_{n+1} + f_{n-1}}{60} + \frac{4\left(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}}\right)}{15} + \beta_2 (f_{n+\alpha_2} + f_{n-\alpha_2}) + \left(\frac{13}{30} - 2\beta_2\right) f_n \right\}, \end{aligned} \tag{3.1a}$$

$$y_{n+\frac{1}{2}} = \frac{1}{2}(y_{n+1} + y_n) - \frac{h^2}{16}(f_{n+1} + f_n), \tag{3.1b}$$

$$y_{n-\frac{1}{2}} = \frac{1}{2}(y_n + y_{n-1}) - \frac{h^2}{16}(f_{n-1} + f_n), \tag{3.1c}$$

$$\begin{aligned} y_{n+\alpha_2} &= R_+ y_{n+1} + (1 - 2R_+) y_n + R_+ y_{n-1} \\ &\quad + h^2 \left(Y_+ f_{n+1} + V_+ f_n + Y_+ f_{n-1} + Z_+ \left(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}}\right)\right), \\ y_{n-\alpha_2} &= y_n. \end{aligned} \tag{3.1d}$$

where

$$\beta_2 Y_+ = \frac{1}{144} - \frac{\beta_2 R_+}{12} - \frac{\beta_2 Z_+}{4}, \quad \beta_2 V_+ = -\frac{1}{72} - \frac{5\beta_2 R_+}{6} - \frac{3\beta_2 Z_+}{2} \tag{3.2}$$

and $R_+, Z_+, \beta_2$are free parameters and $\beta_2 \neq 0$.

The local truncation error of the EM6-1 is

$$LTE = \left\{ \begin{array}{l} C_1 y_n^{(8)} + C_2 F_n y_n^{(6)} + C_3 F_n' y_n^{(5)} + C_4 F_n'' y_n^{(4)} + C_5 F_n^{(3)} y_n^{(3)} \\ + C_6 F_n^2 y_n^{(4)} + C_7 F_n F_n' y_n^{(3)} \end{array} \right\} h^8 + O(h^{10}) \tag{3.3}$$

where

$$C_1 = -\frac{1}{120960}, C_2 = \frac{39}{86400} + \frac{\beta_2 R_+}{240} + \frac{\beta_2 Z_+}{64},$$

$$C_3 = \frac{1}{576}, C_4 = \frac{1}{1152}, C_5 = 0, C_6 = \frac{5\beta_2 Z_+}{192}, C_7 = 0.$$

The phase-lag of EM6-1 is

$$\cos(H) - R_{33}(H^2) = d_4 H^8 + O(H^{10}), \tag{3.4}$$

where

$$\begin{aligned} d_4 &= -\frac{b_3}{2} + \frac{b_2}{24} - \frac{b_1}{720} + \frac{1}{40320} \\ b_1 &= \frac{3}{20} + \beta_2 R_+, \\ b_2 &= \frac{7}{720} + \frac{1}{12}\beta_2 R_+ - \frac{1}{4}\beta_2 Z_+, \\ b_3 &= -\frac{1}{16}\beta_2 Z_+, \end{aligned} \tag{3.5}$$

and

$$b_3 = \frac{b_2}{4} - \frac{b_1}{48} + \frac{1}{1440}. \tag{3.6}$$

P-stable condition became

$$q(H^2) = 1 + \left(\frac{1}{15} + \beta_2 R_+\right) H^2 - \frac{\beta_2 Z_+}{4} H^4 > 0, \tag{3.7}$$

and

$$p(H^2) = 1 + \left(-\frac{1}{10} + \beta_2 R_+\right) H^2 - \left(\frac{1}{144} + \frac{\beta_2 R_+}{6} + \frac{\beta_2 Z_+}{4}\right) H^4 > 0. \tag{3.8}$$

Based on Coleman [4] (Theorem 9 in [4]), the sixth order method EM6-1 (equation (3.1)), in which b_3 is given by (3.6), is P-stable if and only if one of the following three conditions holds

$$\begin{aligned} &b_1 \geq \frac{1}{4} \quad \textit{and} \quad b_2 \geq \frac{1}{4}\left(b_1 - \frac{1}{12}\right), \\ &\frac{1}{20} \leq b_1 < \frac{1}{4} \quad \textit{and} \quad b_2 > \frac{1}{4}\left[b_1 - \frac{1}{12} + \left(b_1 - \frac{1}{4}\right)^2\right], \\ &b_1 < \frac{1}{20} \quad \textit{and} \quad b_2 > \frac{1}{4}\left[\frac{1}{3}b_1 - \frac{1}{90} + \left(b_1 - \frac{1}{12}\right)^2\right]. \end{aligned} \tag{3.9}$$

That is

$$\begin{aligned} &\beta_2 R_+ \geq \frac{1}{10} \quad \textit{and} \quad \frac{\beta_2 Z_+}{4} \leq -\left(\frac{1}{144} + \frac{\beta_2 R_+}{6}\right), \\ &-\frac{1}{10} \leq \beta_2 R_+ < \frac{1}{10} \quad \textit{and} \quad \frac{\beta_2 Z_+}{4} < -\left[\frac{1}{144} + \frac{\beta_2 R_+}{6} + \frac{1}{4}\left(\beta_2 R_+ - \frac{1}{10}\right)^2\right], \\ &\beta_2 R_+ < -\frac{1}{10} \quad \textit{and} \quad \frac{\beta_2 Z_+}{4} < -\frac{1}{4}\left(\frac{1}{15} + \beta_2 R_+\right)^2. \end{aligned} \tag{3.10}$$

Now, based on P-stable conditions (3.10), choosing free parameters $\beta_2 R_+$ and $\beta_2 Z_+$, so that the LTE of the methods EM6-1 (equation (3.1)) is of minimal,that is

$$SLTE = C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 + C_7^2 = \min \tag{3.11}$$

by calculating, when taking

$\beta_2 R_+ = -\frac{1}{10}$ *and* $\beta_2 Z_+ < -\frac{1}{900}$ and limit to $-\frac{1}{900}$, the (3.11) is of minimal.

That is

$$\beta_2 R_+ = -0.1 \text{ and } \beta_2 Z_+ = -0.00111114 \tag{3.12}$$

$$SLTE = C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 + C_7^2 = 0.376880900 \times 10^{-5} \tag{3.13}$$

and the modular of phase-lag (equation (3.4)) is

$$|d_4| = 0.99211800 \times 10^{-5}, \tag{3.14}$$

and β_2 is free parameter and $\beta_2 \neq 0$.

For Thomas [10] sixth order P-stable efficient methods with the iterationmatrix is a true real perfect cube (equation (3) in Thomas [10]), taking

$$\beta_2 R_+ = Q^* - \tfrac{3}{20} = 1.819181$$
$$\beta_2 Z_+ = -\tfrac{16Q^*}{27} = -4.52494300$$

where Q^* is the largest root of

$$\frac{4Q^3}{27} - \frac{Q^2}{3} + \frac{Q}{12} - \frac{1}{360} = 0.$$

The SLTE of Thomas methods ((3) in [10]) is

$$SLTE = C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 + C_7^2 = 0.17816990 \times 10^{-1} \tag{3.15}$$

and the modular of phase-lag is

$$|d_4| = 0.90258120 \times 10^{-1}.$$

We consider the LTE and phase-lag of Cash [2] sixth order P-stable methods (equation (2.17)-(2.20) in [2]).

$$LTE = \left\{ C_1 y_n^{(8)} + C_2 F_n y_n^{(6)} + C_3 F_n' y_n^{(5)} \right\} h^8 + O\left(h^{10}\right),$$

where

$$C_1 = -0.00003196, C_2 = -0.00374470, C_3 = 0.00528971,$$

and

$$C_1^2 + C_2^2 + C_3^2 = 0.42004840 \times 10^{-4}. \tag{3.16}$$

The phase-lag

$$\cos(H) - R_{33}(H^2) = d_4 H^8 + O(H^{10})$$
$$d_4 = -0.18208110 \times 10^{-2}. \tag{3.17}$$

We consider again the LTE and the phase-lag of Chawla and Rao [3] (method $M_6(0)$ in [3]), the LTE is

$$LTE = \left\{ -\frac{1}{120960} y_n^{(8)} + \frac{1}{57600} F_n y_n^{(6)} - \frac{1}{36}\left(\frac{17}{192} - \alpha\right) F_n^2 y_n^{(6)} - \frac{1}{18}\left(\frac{1}{16} - \alpha\right) F_n F_n' y_n^{(3)} \right\} h^8 + O(H^{10})$$

and $\alpha = 0$.

Its SLTE is

$$SLTE = \left(-\frac{1}{120960}\right)^2 + \left(\frac{1}{57600}\right)^2 + \left(-\frac{17}{6912}\right)^2 + \left(-\frac{1}{288}\right)^2 = 0.18105790 \times 10^{-4},$$

and the phase-lag is

$$\cos(H) - R_{33}(H^2) = d_4 H^8 + O(H^{10}), \tag{3.19}$$

where

$$d_4 = -0.12251980 \times 10^{-2}.$$

From (3.13)-(3.19), it is easy to see that if we take free parameters of (3.1)

$$\beta_2 R_+ = -0.1, \beta_2 Z_+ = -0.00111114,$$

then the efficient sixth order P-stable method EM6-1 with approximate minimal LTE. The $SLET$ of LTE and the modular of phase-lag in EM6-1 are smaller than the sixth order P-stable methods of Cash [2], Chawla and Rao [3], Thomas [10].

Case (2)

Suppose that the points $(t_n \pm \alpha_2 h, y_{n\pm\alpha_2})$ are coincident and $y_{n-\alpha_1}$ does not depend on y_{n+1}, then

$$\begin{aligned} &\alpha_2 = 0, A_- = 0, S_- = 0, R_+ = R_-, L_+ = L_-, T_+ = T_-, \\ &Y_+ = Y_-, V_+ = V_-, W_+ = W_-, X_+ = X_-, Z_+ = Z_- \end{aligned} \tag{3.20}$$

Again it is possible to obtain a family of methods with the required properties. The number of function evaluations may be reduced further by requiring that the value of $f_{n+\alpha_1}$ at one step can be used for $f_{n-\alpha_1}$ at the next, then

$$\alpha_1 = \frac{1}{2}, C_+ = 0, C_- = A_+ = B_- = B_+ = \frac{1}{2}, S_+ = Q_- = U_- = Q_+ = -\frac{1}{16}, U_+ = 0. \tag{3.21}$$

Proceeding as in **Case (1)** above, we obtain a method given by equation (3.1a), (3.1b) and (3.1c) together with

$$\begin{aligned} y_{n\pm\alpha_2} = {} & R_+ y_{n+1} + (1 - 2R_+) y_n + R_+ y_{n-1} \\ & + h^2 \left\{ Y_+ f_{n+1} + V_+ f_n + Y_+ f_{n-1} + Z_+ \left(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}} \right) \right\} \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \beta_2 Y_+ &= \frac{1}{288} - \frac{\beta_2 R_+}{12} - \frac{\beta_2 Z_+}{4}, \\ \beta_2 V_+ &= -\frac{1}{144} - \frac{5\beta_2 R_+}{6} - \frac{3\beta_2 Z_+}{2}, \end{aligned}$$

and R_+, Z_+ are free parameters and β_2 is arbitrary, $\beta_2 \neq 0$.

The sixth order methods is denoted as EM6-2, the LET of EM6-2 is of

$$C_1 = -\frac{1}{120960}, C_2 = \frac{39}{86400} + \frac{\beta_2 R_+}{120} + \frac{\beta_2 Z_+}{32},$$

$$C_3 = \frac{1}{576}, C_4 = \frac{1}{1152}, C_5 = 0, C_6 = \frac{5\beta_2 Z_+}{96}, C_7 = 0, \tag{3.23}$$

and the phase-lag is

$$\cos(H) - R_{33}(H^2) = d_4 H^8 + O(H^{10}), \tag{3.24}$$

where

$$d_4 = -\frac{b_3}{2} + \frac{b_2}{24} - \frac{b_1}{720} + \frac{1}{40320},$$

$$b_1 = \frac{3}{20} + 2\beta_2 R_+,$$

$$b_2 = \frac{7}{720} + \frac{1}{6}\beta_2 R_+ - \frac{1}{2}\beta_2 Z_+,$$

$$b_3 = -\frac{1}{8}\beta_2 Z_+, \tag{3.25}$$

and

$$b_3 = \frac{b_2}{4} - \frac{b_1}{48} + \frac{1}{1440}.$$

The sixth order method EM6-2 is P-stable if and only if one of the following three conditions holds

$$\beta_2 R_+ \geq \frac{1}{20} \quad and \quad \beta_2 Z_+ \leq -\frac{1}{72} - \frac{2\beta_2 R_+}{3},$$

$$-\frac{1}{20} \leq \beta_2 R_+ < \frac{1}{20} \quad and \quad \beta_2 Z_+ < -\left[\frac{1}{72} + \frac{2\beta_2 R_+}{3} + \frac{1}{2}\left(2\beta_2 R_+ - \frac{1}{10}\right)^2\right],$$

$$\beta_2 R_+ < -\frac{1}{20} \; and \; \beta_2 Z_+ < -\frac{1}{2}\left(\frac{1}{15} + 2\beta_2 R_+\right)^2. \tag{3.26}$$

taking $\beta_2 R_+ = -0.05$ and $\beta_2 Z_+ < -\frac{1}{1800}$ and limit to $-\frac{1}{1800}$, the

$$C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 + C_7^2 = \min, \tag{3.27}$$

so, we take free parameters

$$\beta_2 R_+ = -0.05, \beta_2 Z_+ = -0.00055557. \tag{3.28}$$

The sixth order methods EM6-2 is P-stable and have approximate minimal

$$SLTE = C_1^2 + C_2^2 + C_3^2 + C_4^2 + C_5^2 + C_6^2 + C_7^2 = 0.3768809 \times 10^{-5}, \tag{3.29}$$

and phase-lag is

$$\cos(H) - R_{33}(H^2) = d_4 H^8 + O(H^{10}), \tag{3.30}$$

and

$$|d_4| = 0.99211800 \times 10^{-5}. \tag{3.31}$$

Therefor, in the sixth order methods EM6-2, taking parameters $\beta_2 R_+, \beta_2 Z_+$ is of (3.28), then this sixth order methods EM6-2 is P-stable with approximate minimal LTE and require only three (new) function evaluations per iteration and per step integration. The LTE and the modular of phase-lag of EM6-2 are smaller than the sixth order P-stable of Cash [2], Chawla and Rao [3] and Thomas [10].

The efficient sixth order methods EM6-1 and EM6-2 require only three function evaluations per iteration. On any step, it is not necessary to evaluate $f(t_n - \alpha_2 h, y_{n-\alpha_2})$ or, assuming the step size does not vary,$f(t_n - \alpha_1 h, y_{n-\alpha_1})$. Compared with the methods proposed by [1,2], Clawla[3]and Thomas[10],these methods have the advantage of a minimal LTE, smaller phase-lag and require slightly less function evaluations.

4. Numerical Illustration

We note that for non-linear $f(t, y)$ all these methods are implicit and shall need an iterative process for computing the solution at each step. We briefly consider application of modified Newton's method for the purpose.

The sixth order methods EM6-1 and EM6-2 in this paper written in the form

$$G(y_{n+1}) = y_{n+1} - \phi(y_{n+1}) = 0 \tag{4.1}$$

Let $y_{n+1}^{(0)}$denote an initial approximation for y_{n+1} defined by

$$y_{n+1}^{(0)} = 2y_n - y_{n-1} + h^2 f_n, \tag{4.2}$$

then, modified Newton's method for (4.1) is

$$\begin{aligned} G(y_{n+1}^{(i)}) + G'(y_{n+1}^{(0)})\Delta y_{n+1}^{(i)} &= 0, \\ y_{n+1}^{(i+1)} = y_{n+1}^{(i)} + \Delta y_{n+1}^{(i)}, &\qquad i = 0, 1, 2, \cdots. \end{aligned} \tag{4.3}$$

Modified Newton's method will converge for h sufficiently small, because $\left|y_{n+1}^{(0)} - y_{n+1}\right| = O(h^4)$ and $\left|1 - G'(y_{n+1}^{(0)})\right| = O(h^2)$, and $G'(y)$is a continuous function of y. We also note here that a still more accurate ($O(h^6)$-approximate) starting value $y_{n+1}^{(0)}$ for use with (4.3) is provided by Noumerov made explicit method

$$\begin{aligned} \bar{y}_{n+1} &= 2y_n - y_{n-1} + h^2 f_n, \\ y_{n+1}^{(0)} &= 2y_n - y_{n-1} + \frac{h^2}{12}\left(\bar{f}_{n+1} + 10 f_n + f_{n-1}\right) \end{aligned} \tag{4.4}$$

In this section, we discuss a numerical test for the methods introduced in this paper and for three sixth order direct hybrid methods derived in the literature [2],[3] and[10].

Example 4.1. (Lambert &Watson[8])

$$Z'' + Z = 0.001\exp(it), Z(0) = 1, Z'(0) = 0.9995i, \quad (i^2 = -1), \tag{4.5}$$

with the exact solution $Z(t) = \exp(it)(1 - 0.0005it)$. We computed $\gamma(t) = \sqrt{\mu(t)^2 + v(t)^2}$, where $Z(t) = \mu(t) + iv(t)$, at $t = 40\pi$,by the efficient sixth order methods EM6-1 with $\beta_2 = 1, \beta_2 R_+ = -0.1$ and $\beta_2 Z_+ = -0.00111114$, Cash's sixth order methods (equations (2.17)-(2.20) of [2]), Chawla and Rao's sixth order method $M_6(0)$and Thomas's sixth order P-stable methods (3)-(5) in [10] with perfect cube.

In Table 1, we given the absolute errors in $\gamma(40\pi)$ using $h = \frac{\pi}{4}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{9}, \frac{\pi}{12}$ together with the total number of function evaluations required.

As can be seen, the sixth order methods EM6-1 is more efficient than the sixth order formula of Cash [2], Chawla and Rao[3] and Thomas[10] for this problem.

Table 1. absolute errors and function evaluations for different values of h in $\gamma(40\pi)$ of the problem (4.5)

h	Cash's 6th order method		Chawla &Rao' method $M_6(0)$		Thomas's 6th order method		EM6-1($\beta_2 = 1$) (in this paper)	
	$\|error\|$	Fns	$\|error\|$	Fns	$\|error\|$	Fns	$\|error\|$	Fns
$\frac{\pi}{4}$	1.25×10^{-5}	640	1.08×10^{-3}	640	1.21×10^{-5}	480	1.22×10^{-4}	480
$\frac{\pi}{5}$	3.39×10^{-6}	800	2.74×10^{-4}	800	2.71×10^{-5}	600	1.68×10^{-6}	600
$\frac{\pi}{6}$	1.19×10^{-6}	960	1.84×10^{-5}	960	1.10×10^{-5}	720	7.29×10^{-7}	720
$\frac{\pi}{9}$	9.79×10^{-8}	1440	5.27×10^{-6}	1440	8.91×10^{-7}	1080	6.28×10^{-8}	1080
$\frac{\pi}{12}$	1.71×10^{-8}	1920	3.82×10^{-7}	1920	6.28×10^{-8}	1440	4.25×10^{-9}	1440

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