Journal of Computational Mathematics, Vol.20, No.2, 2002, 129–152.

MULTIGRID METHODS FOR THE GENERALIZED STOKES EQUATIONS BASED ON MIXED FINITE ELEMENT METHODS*

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Abstract

Multigrid methods are developed and analyzed for the generalized stationary Stokes equations which are discretized by various mixed finite element methods. In this paper, the multigrid algorithm, the criterion for prolongation operators and the convergence analysis are all established in an abstract and element–independent fashion. It is proven that the multigrid algorithm converges optimally if the prolongation operator satisfies the criterion. To utilize the abstract result, more than ten well–known mixed finite elements for the Stokes problems are discussed in detail and examples of prolongation operators are constructed explicitly. For nonconforming elements, it is shown that the usual local averaging technique for constructing prolongation operators can be replaced by a computationally cheaper alternative, random choice technique. Moreover, since the algorithm and analysis allows using of nonnested meshes, the abstract result also applies to low order mixed finite elements, which are usually stable only for some special mesh structures.

Key words: Generalized Stokes equations, Mixed methods, Multigrid methods.

1. Introduction

In this paper, we consider the following generalized stationary Stokes equations:

$$\begin{cases} -\Delta \underset{\sim}{u} + \nabla p = \underset{\sim}{F}, & \text{in } \Omega, \\ \operatorname{div} \underset{\sim}{u} = G, & \text{in } \Omega, \\ \underset{\sim}{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in R^2, $\underset{\sim}{u}$ represents the velocity of fluid, p its pressure; $\underset{\sim}{F}$ and G are external force and source terms. Note that the source must satisfy the compactability condition of having zero mean value, and (1.1) reduces to the stationary Stokes equations when $G \equiv 0$.

The mixed variational formulation of the generalized Stokes equations with arbitrary given force $\underset{\sim}{f}$ and source g is to find $[\underset{\sim}{u}, p] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\begin{cases} (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}) - (p, \operatorname{div} \underset{\sim}{v}) = (\underset{\sim}{f}, \underset{\sim}{v}), & \forall \underset{\sim}{v} \in (H_0^1(\Omega))^2, \\ (q, \operatorname{div} \underset{\sim}{u}) = (g, q), & \forall\, q \in L_0^2(\Omega), \end{cases} \tag{1.2}$$

or equivalently, find $[\underset{\sim}{u}, p] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\mathcal{L}([\underset{\sim}{u}, p], [\underset{\sim}{v}, q]) = (\underset{\sim}{f}, \underset{\sim}{v}) - (g, q), \quad \forall\, [\underset{\sim}{v}, q] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega), \tag{1.3.1}$$

* Received March 17, 1997; Final revised December 4, 2001.

where $(\cdot,\cdot) \equiv (\cdot,\cdot)_\Omega$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$, $L_0^2(\Omega)$ is the space of $L^2(\Omega)$–integrable functions which have zero mean value (cf. [7] for space notations) and

$$\mathcal{L}([\underset{\sim}{u},p],[\underset{\sim}{v},q]) = (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v}) - (p, \operatorname{div} \underset{\sim}{v}) - (q, \operatorname{div} \underset{\sim}{u}). \tag{1.3.2}$$

Note that when $\underset{\sim}{f} = \underset{\sim}{F}$ and $g = G$, (1.2) or (1.3) is the variational formulation of (1.1).

It is well–known (cf. [13] and [14]) that the problem (1.2) is uniquely solvable if $\underset{\sim}{f} \in (H^{-1}(\Omega))^2$, $g \in L_0^2(\Omega)$. Moreover, if $\underset{\sim}{f} \in (L^2(\Omega))^2$, $g \in L_0^2(\Omega) \cap H^1(\Omega)$, then the solution $[\underset{\sim}{\sigma}, \tau] \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$ and there holds

$$\|\underset{\sim}{\sigma}\|_{H^\ell(\Omega)} + \|\tau\|_{H^{\ell-1}(\Omega)} \le C[\|\underset{\sim}{f}\|_{H^{\ell-2}(\Omega)} + \|g\|_{H^{\ell-1}(\Omega)}], \quad \ell = 1,2. \tag{1.4}$$

To describe mixed finite element methods for the generalized Stokes equations, we begin with the triangulations of the domain Ω. Let $\mathcal{T}_k (k \ge 0)$ be a quasi–uniform triangular or rectangular partition of Ω with mesh size h_k, that is, there exists some constant $\alpha_0 > 0$, $\theta_0 > 0$ such that

$$h_K \ge \alpha_0 \rho_k, \qquad \theta_K \ge \theta_0, \quad \forall K \in \mathcal{T}_k,\ k \ge 0, \tag{A.0}$$

where h_K, θ_K and ρ_K denote, respectively, the diameter of K, the smallest angle of K and the the diameter of the largest ball contained in K. For simplicity, we also assume that $\overline{\Omega} = \cup_{K\in\mathcal{T}_k} \overline{K}$. Finally, in order to get optimal order algorithm we assume that the mesh sizes of two consecutive meshes are related as follows (cf. subsection 5.4 for the other restrictions):

$$\alpha_1^{-1} h_k \le h_{k+1} < h_k, \quad k \ge 0, \tag{1.5}$$

for some constant $\alpha_1 > 1$. Obviously, for a nested mesh family, namely, $\mathcal{T}_k$ is obtained by connecting the midpoints of the three edges of all triangles of $\mathcal{T}_{k-1}$ or by linking the midpoints of two opposite sides of all rectangles of $\mathcal{T}_{k-1}$, (1.5) holds with $\alpha_1 = 2$.

Let $\underset{\sim}{X}_k \subset (L^2(\Omega))^2, M_k \subset L_0^2(\Omega)$ be two finite element approximate spaces of $(H_0^1(\Omega))^2$ and $L_0^2(\Omega)$ associated with $\mathcal{T}_k$. The mixed finite element method for (1.2) at level k is to find $[\underset{\sim}{u}_k, p_k] \in \underset{\sim}{X}_k \times M_k$ such that

$$\begin{cases} (\nabla \underset{\sim}{u}_k, \nabla \underset{\sim}{v})_k - (p_k, \operatorname{div} \underset{\sim}{v})_k = (\underset{\sim}{f}, \underset{\sim}{v})_k, & \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_k, \\ (q, \operatorname{div} \underset{\sim}{u}_k)_k = (g,q)_k, & \forall\ q \in M_k, \end{cases} \tag{1.6}$$

or equivalently, find $[\underset{\sim}{u}_k, p_k] \in \underset{\sim}{X}_k \times M_k$ such that

$$\mathcal{L}_k([\underset{\sim}{u}_k, p_k], [\underset{\sim}{v}, q]) = (\underset{\sim}{f}, \underset{\sim}{v})_k - (g,q)_k, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k, \tag{1.7}$$

where

$$(\cdot,\cdot)_k = \sum_{K\in\mathcal{T}_k} (\cdot,\cdot)_K, \tag{1.8}$$

$$\mathcal{L}_k([\underset{\sim}{u},p],[\underset{\sim}{v},q]) = (\nabla \underset{\sim}{u}, \nabla \underset{\sim}{v})_k - (p, \operatorname{div} \underset{\sim}{v})_k - (q, \operatorname{div} \underset{\sim}{u})_k. \tag{1.9}$$

It is well–known that $\underset{\sim}{X}_k$ and M_k must satisfy the following Babuška–Brezzi condition in order to guarantee the existence and stability of the mixed finite element approximations:

$$\sup_{\underset{\sim}{v}_k \in \underset{\sim}{X}_k} \frac{|(q,\ \operatorname{div}\ \underset{\sim}{v}_k)|}{\|\underset{\sim}{v}_k\|_k} \ge \gamma_0 \|q\|_{L^2(\Omega)}, \quad \forall\ q \in M_k, \tag{1.10}$$

where $\|\underset{\sim}{v}_k\|_k^2 = (\nabla \underset{\sim}{v}_k, \nabla \underset{\sim}{v}_k)_k$, and γ_0 is some positive number independent of k and h_k.

In this paper, in addition to (1.10) we also assume that each finite element solution $[\underset{\sim}{u}_k, p_k]$ of (1.6) satisfies the following error estimate:

$$\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_{L_2(\Omega)} + h_k(\|\underset{\sim}{u} - \underset{\sim}{u}_k\|_k + \|p - p_k\|_{L_2(\Omega)}) \leq Ch_k^2(\|\underset{\sim}{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}). \tag{1.11}$$

Multigrid methods are optimal order methods for solving systems of equations arising from finite element discretizations of elliptic boundary value problems since the error reduction per iteration cycle of such a method is independent of the mesh size h (cf. [3] and [12]). Most works in this direction have been devoted to solving the standard finite element equations. Only few are directed to solving the mixed finite element equations. This is mainly due to the difficulties caused by the indefiniteness of the mixed finite element equations and by having different orders of differentiability for the two unknown functions, also by the nonnestedness of mixed finite element spaces for most mixed elements.

To the best our knowledge, rigorous multigrid methods for mixed finite element equation (1.7) of the Stokes problems were only developed for nested elements (cf. [15]), mini element (cf. [16]) and the nonconforming Crouzeix–Raviart element (cf. [4]). The framework of multigrid methods for the mixed finite element equations of the Stokes problems was first developed in [15] for the class of nested elements, i.e., the multilevel finite element spaces are nested. The key ideas of [15] are to overcome the indefiniteness of the problem by applying the smoothing part of the algorithm to the squared system and to take care the difficulty caused by having different orders of differentiability for the velocity and the pressure by introducing a scaled mesh–dependent norm. Later, the result of [15] was extended to the nonnested mini element in [16], where the bubble functions were L^2 projected from coarser grid to finer grid and a strengthened Cauchy inequality played an important role in the convergence analysis. In [4] the multigrid method was analyzed for the nonconforming Crouzeix–Raviart element and a quadrature formula was used to define the prolongation operator. In addition, in [5] the multigrid method was developed for the Stokes equations discretized by a standard finite element method (i.e., a divergence–free finite element space was used for the velocity).

The main goal of this paper is to develop and to analyze multigrid methods for solving the mixed finite element equation of the generalized Stokes equations in an abstract fashion. The multigrid algorithm to be studied in this paper is similar to the one developed in [16]. But our objectives are to find an easy to verify criterion for "good" prolongation operators and to develop an element–independent convergence analysis which then is applied to various mixed finite elements for the Stokes problems. Indeed, we propose a criterion for "good" prolongation operators, and show that the multigrid algorithm converges optimally if the prolongation operator satisfies the criterion. Unlike the convergence analyses of [16], [4] and [5] which strongly depended on the structure of each mixed element under discussion, the idea of our convergence analysis is to employ separated duality arguments for the velocity and pressure and to make full use of the mixed finite element error estimate (1.11). Moreover, nonnested meshes are allowed in the algorithm and analysis, consequently, the abstract result also applies to low order mixed elements which are usually stable only for very special mesh structures. For nonconforming elements, we show that the usual local averaging technique which have been used to construct prolongation operators can be replaced by a computationally cheaper alternative, random choice technique. Finally, as a byproduct, the analysis of this paper provides an alternative convergence proofs for above mentioned existing mixed multigrid methods for the Stokes equations.

The organization of the rest of the paper is as follows. In Section 2, the multigrid algorithm is defined for the mixed finite element equations (1.6). In Section 3, we establish some preliminary lemmas for proving convergence of the multigrid algorithm in Section 4, where the convergence of the algorithm is demonstrated under three abstract assumptions on the prolongation operator. Finally, in Section 5 the abstract framework developed in Sections 1–4 is applied to more than ten well–known mixed elements for the Stokes problems, and "good" prolongation operators are constructed in detail for all cited mixed elements.

2. A Multigrid Algorithm

In this section we are going to define a multigrid algorithm by following the basic idea of the multigrid method for symmetric indefinite problems which was first demonstrated in [15] by Verfürth. Precisely, the smoothing part of the algorithm is applied to the squared system of the discretized Stokes problems by mixed finite element methods, and a scaled mesh–dependent norm is used to overcome the difficulty caused by having different orders of differentiability for the velocity and the pressure.

In addition, since we allow the multilevel finite element spaces to be nonnested, which is caused either by the nature of a specific element or by the nonnested refining of the meshes, it is necessary for us to introduce a prolongation operator which is other than the natural injection for nonnested elements. The prolongation operator is introduced in an abstract fashion, which will be specified for each element in Section 5. To ensure convergence of the algorithm, three sufficient conditions are imposed for a qualified prolongation operator in the next section.

Suppose that we are given a family of finite elment spaces $\underset{\sim}{X}_k \times M_k,\ k \geq 0$ such that the Babuška–Brezzi condition (1.10) holds for each $k \geq 0$, and each resulted finite element solution satisfies the error estimate (1.11). In Sections 2–4, we always assume above assumptions unless stated otherwise.

Following [15], for each $k \geq 0$ we equipped $\underset{\sim}{X}_k \times M_k$ with the mesh–dependent norm

$$|||[\underset{\sim}{v}, q]|||_{0,k} = (\|\underset{\sim}{v}\|^2_{L^2(\Omega)} + h_k^2\|p\|^2_{L^2(\Omega)})^{\frac{1}{2}} = ((\underset{\sim}{v}, \underset{\sim}{v})_k + h_k^2(p,p)_k)^{\frac{1}{2}}. \tag{2.1}$$

In addition, we assume there exists a linear operator I_{k-1}^k such that

$$I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k] : \underset{\sim}{X}_{k-1} \times M_{k-1} \to \underset{\sim}{X}_k \times M_k. \tag{2.2}$$

The multigrid algorithm for solving the following mixed finite element equation of (1.1) at level k is defined as follows: Find $[\underset{\sim}{w}, \alpha] \in \underset{\sim}{X}_k \times M_k$, such that

$$\mathcal{L}_k([\underset{\sim}{w}, \alpha], [\underset{\sim}{v}, q]) = \mathcal{F}_k([\underset{\sim}{v}, q]), \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k, \tag{2.3}$$

where $\mathcal{F}_k$ is a linear functional on $\underset{\sim}{X}_k \times M_k$, in particular, it takes the following form on the finest grid:

$$\mathcal{F}_k([\underset{\sim}{v}, q]) = (\underset{\sim}{F}, \underset{\sim}{v})_k - (G, q)_k. \tag{2.4}$$

Multigrid Algorithm.

(i) If $k = 0$, (2.3) is solved directly.

(ii) If $k > 0$, let $[\underset{\sim}{w}^0, \alpha^0] \in \underset{\sim}{X}_k \times M_k$ be an initial guess, and define $[\underset{\sim}{w}^{m+1}, \alpha^{m+1}] \in \underset{\sim}{X}_k \times M_k$ as follows:

Smoothing step: For $1 \leq i \leq m,\ [w^i, \alpha^i]$ is defined by

$$(\underset{\sim}{z}^i, \underset{\sim}{v})_k + h_k^2(\beta^i, q)_k = \Lambda_k^{-2}\{\mathcal{F}_k([\underset{\sim}{v}, q]) - \mathcal{L}_k([\underset{\sim}{w}^{i-1}, \alpha^{i-1}], [\underset{\sim}{v}, q])\},$$
$$\forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k. \tag{2.5.i}$$

and

$$(\underset{\sim}{w}^i - \underset{\sim}{w}^{i-1}, \underset{\sim}{v})_k + h_k^2(\alpha^i - \alpha^{i-1}, q)_k = \mathcal{L}_k([\underset{\sim}{z}^i, \beta^i], [\underset{\sim}{v}, q]). \tag{2.5.ii}$$

Correction Step: Set

$$[\underset{\sim}{w}^{m+1}, \alpha^{m+1}] = [\underset{\sim}{w}^m, \alpha^m] + I_{k-1}^k[\underset{\sim}{\psi}, \rho], \tag{2.6}$$

where $[\underset{\sim}{\psi}, \rho] \in \underset{\sim}{X}_{k-1} \times M_{k-1}$ is the approximation of $[\underset{\sim}{\psi}^*, \rho^*] \in \underset{\sim}{X}_{k-1} \times M_{k-1}$ defined by applying μ iterations with zero as the initial guess of the level $(k-1)$ algorithm to the residual equation

$$\mathcal{L}_{k-1}([\underset{\sim}{\psi}^*, \rho^*], [\underset{\sim}{v}, q]) = \mathcal{F}_{k-1}([\underset{\sim}{v}, q]), \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}, \tag{2.7}$$

where for any $[\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}$

$$\mathcal{F}_{k-1}([\underset{\sim}{v}, q]) = \mathcal{F}_k(I_{k-1}^k[\underset{\sim}{v}, q]) - \mathcal{L}_k([\underset{\sim}{w}^m, \alpha^m], I_{k-1}^k[\underset{\sim}{v}, q]). \tag{2.8}$$

In the algorithm, m is some positive integer to be determined and μ any positive integer constant is greater than or equal to two. In addition, $\Lambda_k = O(h_k^{-2})$ is chosen to be the maximal absolute value of the eigenvalue for the following eigenvalue problem: Find $[\underset{\sim}{\varphi}_k, \nu_k] \in$ $, \underset{\sim}{X}_k \times M_k,\ \lambda \in R \setminus \{0\}$, such that

$$\mathcal{L}_k([\underset{\sim}{\varphi}_k, \nu_k], [\underset{\sim}{v}, q]) = \lambda((\underset{\sim}{\varphi}_k, \underset{\sim}{v})_k + h_k^2(\nu_k, q)_k), \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k. \tag{2.9}$$

Since $\mathcal{L}_k(\cdot, \cdot)$ is symmetric, eigenvalue problem (2.9) has a complete set of eigenfunctions. Let $\{\lambda_j\},\ \{\underset{\sim}{\varphi}_k^j, \nu_k^j\}$ be the eigenvalues and corresponding eigenfunctions, i.e., for $j = 1, 2, \cdots, N_k$,

$$\mathcal{L}_k([\underset{\sim}{\varphi}_k^j, \nu_k^j], [\underset{\sim}{v}, q]) = \lambda_j((\underset{\sim}{\varphi}_k^j, \underset{\sim}{v})_k + h_k^2(\nu_k^j, q)_k), \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k. \tag{2.10}$$

We also assume that

$$0 < |\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_{N_k}|, \tag{2.11}$$

$$(\underset{\sim}{\varphi}_k^i, \underset{\sim}{\varphi}_k^j)_k + h_k^2(\nu_k^j, \nu_k^j)_k = \delta_{ij}, \quad 1 \le i, j \le N_k, \tag{2.12}$$

where δ_{ij} is Kronecker delta.

3. Preliminaries of Convergence

This section is devoted to establish some preliminary lemmas about the prolongation operator $I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k]$ under three abstract assumptions. Throughout this paper (unless stated otherwise) C denotes a generic positive constant which is independent of the grid level k and mesh size h_k.

For any $[\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k$, it follows from (2.7)–(2.9) that there exists $c_j,\ j = 1, 2, \cdots, N_k$, such that

$$[\underset{\sim}{v}_k, q_k] = \sum_{j=1}^{N_k} c_j[\underset{\sim}{\varphi}_k^j, \nu_k^j]. \tag{3.1}$$

Define the mesh–dependent norm

$$|||[\underset{\sim}{v}_k, q_k]|||_{s,k} = \left\{\sum_{j=1}^{N_k} c_j^2|\lambda_j|^s\right\}^{\frac{1}{2}}. \tag{3.2}$$

By (2.1), (2.9)–(2.12), it is easy to verify the following inequalities:

$$|||[\underset{\sim}{v}_k, q_k]|||_{0,k} = \|[\underset{\sim}{v}_k, q_k]\|_{0,k}, \quad \forall\ [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k. \tag{3.3}$$

$$|||[\underset{\sim}{v}_k, q_k]|||_{s,k} \le Ch_k^{t-s}|||[\underset{\sim}{v}_k, q_k]|||_{t,k}, \quad \forall [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k, \quad t < s. \tag{3.4}$$

$$|\mathcal{L}_k([\underset{\sim}{u}_k, p_k], [\underset{\sim}{v}_k, q_k])| \le |||[\underset{\sim}{u}_k, p_k]|||_{2,k}|||[\underset{\sim}{v}_k, q_k]|||_{0,k}, \quad \forall\ [\underset{\sim}{u}_k, p_k],\ [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k. \tag{3.5}$$

We introduce the following criterion for the prolongation operator $I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k]$:

$$\|\underset{\sim}{v} - H_{k-1}^k\underset{\sim}{v}\|_{L^2(\Omega)} \le Ch_k\|\underset{\sim}{v}\|_{k-1}, \quad \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}. \tag{A.1}$$

$$\|J_{k-1}^k q\|_{L^2(\Omega)} \le C\|q\|_{L^2(\Omega)}, \quad \forall\ q \in M_{k-1}. \tag{A.2}$$

$$|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \le Ch_k^2(\|\underset{\sim}{\sigma}_f\|_{H^2(\Omega)} + \|\tau_f\|_{H^1(\Omega)}), \tag{A.3}$$

where $[\underset{\sim}{\sigma}_f, \tau_f]$ is the solution of (1.3) with the force term $\underset{\sim}{f} \in (L^2(\Omega))^2$ and the source term $g \equiv 0$, and $[\underset{\sim}{\sigma}_j, \tau_j]$ $(j = k-1, k)$ is the mixed finite element approximation of $[\underset{\sim}{\sigma}_f, \tau_f]$ at level j.

It is easy to see that (A.1), (A.2) and the inverse inequality imply that

$$|||I_{k-1}^k[\underset{\sim}{v}, q]|||_{0,k} \le C|||[\underset{\sim}{v}, q]|||_{0,k-1}, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}. \tag{3.6}$$

In the rest of this section, we are going to establish several lemmas, which will be used in next section to prove the convergence theorem.

Lemma 3.1. *Let $(I_{k-1}^k)^* : \underset{\sim}{X}_k \times M_k \longrightarrow \underset{\sim}{X}_{k-1} \times M_{k-1} (k \ge 1)$ be defined as follows:*

$$\mathcal{L}_{k-1}((I_{k-1}^k)^*[\underset{\sim}{v}_k, q_k], [\underset{\sim}{v}_{k-1}, q_{k-1}]) = \mathcal{L}_k([\underset{\sim}{v}_k, q_k], I_{k-1}^k[\underset{\sim}{v}_{k-1}, q_{k-1}]), \quad \forall\ [\underset{\sim}{v}_{k-1}, q_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1},\ [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k. \tag{3.7}$$

Then we have

$$|||(I_{k-1}^k)^*[\underset{\sim}{v}_k, q_k]|||_{2,k-1} \le C|||[\underset{\sim}{v}_k, q_k]|||_{2,k}, \quad \forall\ [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k. \tag{3.8}$$

Proof. It follows from (3.2), (3.5), (3.6) and (3.7) that

$$\begin{aligned}
&|||(I_{k-1}^k)^*[\underset{\sim}{v}_k, q_k]|||_{2,k-1} \\
&= \sup_{0 \ne [v_{k-1}, q_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1}} \frac{|\mathcal{L}_{k-1}((I_{k-1}^k)^*[\underset{\sim}{v}_k, q_k], [\underset{\sim}{v}_{k-1}, q_{k-1}])|}{|||[\underset{\sim}{v}_{k-1}, q_{k-1}]|||_{0,k-1}} \\
&= \sup_{0 \ne [\underset{\sim}{v}_{k-1}, p_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1}} \frac{|\mathcal{L}_k([\underset{\sim}{v}_k, q_k], I_{k-1}^k[\underset{\sim}{v}_{k-1}, q_{k-1}])|}{|||[\underset{\sim}{v}_{k-1}, q_{k-1}]|||_{0,k-1}} \\
&\le \sup_{0 \ne [\underset{\sim}{v}_{k-1}, q_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1}} \frac{|||[\underset{\sim}{v}_k, q_k]|||_{2,k}|||I_{k-1}^k[\underset{\sim}{v}_{k-1}, q_{k-1}]|||_{0,k}}{|||[\underset{\sim}{v}_{k-1}, q_{k-1}]|||_{0,k-1}} \\
&\le C|||[\underset{\sim}{v}_k, q_k]|||_{2,k}, \qquad \forall\ [\underset{\sim}{v}_k, q_k] \in \underset{\sim}{X}_k \times M_k.
\end{aligned}$$

Lemma 3.2. *Let $[\underset{\sim}{\sigma}_k, \tau_k]$ be the solution of (1.7) with $\underset{\sim}{f} \equiv 0$, then*

$$\|\underset{\sim}{\sigma}_k\|_k + \|\tau_k\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}. \tag{3.9}$$

Proof. Choosing $[\underset{\sim}{v}, q] = [\underset{\sim}{\sigma}_k, \tau_k]$ in (1.7), then we have

$$\|\sigma_k\|_k^2 \le \|g\|_{L^2(\Omega)}\|\tau_k\|_{L^2(\Omega)}. \tag{3.10}$$

It follows from (1.10) that

$$\|\tau_k\|_{L^2(\Omega)} \le \gamma_0^{-1} \sup_{\underset{\sim}{v}_k \in \underset{\sim}{X}_k} \frac{|(\tau_k, \operatorname{div} \underset{\sim}{v}_k)_k|}{\|\underset{\sim}{v}_k\|_k} = \gamma_0^{-1} \sup_{\underset{\sim}{v}_k \in \underset{\sim}{X}_k} \frac{|(\nabla \underset{\sim}{\sigma}_k, \nabla \underset{\sim}{v}_k)_k|}{\|\underset{\sim}{v}_k\|_k} \le \gamma_0^{-1}\|\sigma_k\|_k. \tag{3.11}$$

So (3.9) follows from (3.10) and (3.11).

Lemma 3.3. *Let $[\underset{\sim}{\sigma}, \tau]$, $[\underset{\sim}{\sigma}_j, \tau_j]$ $(j = k-1, k)$ be the solutions of (1.3) and (1.7) with $g \equiv 0$, respectively. Then we have*

$$|||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1} \le Ch_k^2\|\underset{\sim}{f}\|_{L^2(\Omega)}. \tag{3.12}$$

Proof. Let

$$[\underset{\sim}{\xi}, \lambda] = [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k] \in \underset{\sim}{X}_{k-1} \times M_{k-1}. \tag{3.13}$$

Then

$$|||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1}^2 = \|\underset{\sim}{\xi}\|_{L^2(\Omega)}^2 + h_{k-1}^2\|\lambda\|_{L^2(\Omega)}^2. \tag{3.14}$$

We now consider the following auxiliary problem and its finite element approximation: Find $[\underset{\sim}{\eta}, \delta] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\mathcal{L}([\underset{\sim}{\eta}, \delta], [\underset{\sim}{v}, q]) = (\underset{\sim}{\xi}, \underset{\sim}{v}), \quad \forall\ [\underset{\sim}{v}, q] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega); \tag{3.15}$$

Find $[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1}$ such that

$$\mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1}, \delta_{k-1}, [\underset{\sim}{v}, q])_{k-1} = (\underset{\sim}{\xi}, \underset{\sim}{v})_{k-1}, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}. \tag{3.16}$$

By (1.4) with $\ell = 2$ and (A.3) we have

$$\|\underset{\sim}{\eta}\|_{H^2(\Omega)} + \|\delta\|_{H^1(\Omega)} \le C\|\underset{\sim}{\xi}\|_{L^2(\Omega)}. \tag{3.17}$$

$$|||[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}] - I_{k-1}^k[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}]|||_{0,k} \le Ch_k^2(\|\underset{\sim}{\eta}\|_{H^2(\Omega)} + \|\delta\|_{H^1(\Omega)}). \tag{3.18}$$

So from (1.5), (3.7) and (3.16)–(3.18) we have

$$\begin{aligned}
\|\underset{\sim}{\xi}\|_{L^2(\Omega)}^2 &= (\underset{\sim}{\xi}, \underset{\sim}{\xi})_{k-1} = \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1}, \delta_{k-1}], [\underset{\sim}{\xi}, \lambda]) \\
&= \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1}, \delta_{k-1}], [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}) - \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1}, \delta_{k-1}], (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]) \\
&= \mathcal{L}_{k-1}([\sigma_{k-1}, \tau_{k-1}], [\underset{\sim}{\eta}_{k-1}, \delta_{k-1}]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], I_{k-1}^k[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}]) \\
&= (\underset{\sim}{f}, \underset{\sim}{\eta}_{k-1})_{k-1} - (\underset{\sim}{f}, H_{k-1}^k\underset{\sim}{\eta}_{k-1})_k \\
&\le \|\underset{\sim}{f}\|_{L^2(\Omega)}\|\underset{\sim}{\eta}_{k-1} - H_{k-1}^k\underset{\sim}{\eta}_{k-1}\|_{L^2(\Omega)} \\
&\le Ch_k^2\|\underset{\sim}{f}\|_{L^2(\Omega)}\|\underset{\sim}{\xi}\|_{L^2(\Omega)}.
\end{aligned}$$

Hence

$$\|\underset{\sim}{\xi}\|_{L^2(\Omega)} \le Ch_k^2\|\underset{\sim}{f}\|_{L^2(\Omega)}. \tag{3.19}$$

To estimate $\|\lambda\|_{L^2(\Omega)}$, we consider another auxiliary problem and its finite element approximations which are defined as follows: Find $[\underset{\sim}{\eta}', \delta'] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\mathcal{L}([\underset{\sim}{\eta}', \delta'], [\underset{\sim}{v}, q]) = (\lambda, q), \quad \forall\, [\underset{\sim}{v}, q] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega); \tag{3.20}$$

Find $[\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}] \in \underset{\sim}{X}_{k-1} \times M_{k-1}$ such that

$$\mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}], [\underset{\sim}{v}, q]) = (\lambda, q)_{k-1}, \quad \forall\, [\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}. \tag{3.21}$$

It follows from Lemma 3.2 that

$$\|\eta'_{k-1}\|_{k-1} + \|\delta'_{k-1}\|_{L^2(\Omega)} \le C\|\lambda\|_{L^2(\Omega)}. \tag{3.22}$$

Moreover, by using (3.7), (3.13), (3.21), (3.22), (1.5) and (A.1) we have

$$\begin{aligned}
\|\lambda\|^2_{L^2(\Omega)} &= (\lambda,\lambda)_{k-1} = \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}], [\underset{\sim}{\xi}, \lambda])\\
&= \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}], [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]) - \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}], (I^k_{k-1})^*[\underset{\sim}{\sigma}_k, \tau_k])\\
&= \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}, [\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], I^k_{k-1}[\underset{\sim}{\eta}'_{k-1}, \delta'_{k-1}])\\
&= (\underset{\sim}{f}, \underset{\sim}{\eta}'_{k-1})_{k-1} - (\underset{\sim}{f}, H^k_{k-1}\underset{\sim}{\eta}'_{k-1})_k\\
&\le \|\underset{\sim}{f}\|_{L^2(\Omega)}\|\underset{\sim}{\eta}'_{k-1} - H^k_{k-1}\underset{\sim}{\eta}'_{k-1}\|_{L^2(\Omega)}\\
&\le Ch_k\|\underset{\sim}{f}\|_{L^2(\Omega)}\|\underset{\sim}{\eta}'_{k-1}\|_{k-1}\\
&\le Ch_k\|\underset{\sim}{f}\|_{L^2(\Omega)}\|\lambda\|_{L^2(\Omega)},
\end{aligned}$$

hence

$$\|\lambda\|_{L^2(\Omega)} \le Ch_k\|\underset{\sim}{f}\|_{L^2(\Omega)}. \tag{3.23}$$

Finally, (3.12) follows from (3.14), (3.19) and (3.23).

Lemma 3.4. *Let $[\underset{\sim}{\sigma}, \tau], [\underset{\sim}{\sigma}_j, \tau_j]$ $(j = k-1, k)$ be the solutions of (1.3) and (1.7) with $\underset{\sim}{f} \equiv 0$, respectively. Then we have*

$$|||[\underset{\sim}{\sigma}_k, \tau_k] - I^k_{k-1}[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \le Ch_k\|g\|_{L^2(\Omega)}. \tag{3.24}$$

Proof. Notice that

$$|||[\underset{\sim}{\sigma}_k, \tau_k] - I^k_{k-1}[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||^2_{0,k} = \|\underset{\sim}{\sigma}_k - H^k_{k-1}\underset{\sim}{\sigma}_{k-1}\|^2_{k-1} + h_k^2\|\tau_k - J^k_{k-1}\tau_{k-1}\|^2_{L^2(\Omega)}, \tag{3.25}$$

by (A.2) and Lemma 3.2 we get

$$\begin{aligned}
\|\tau_k - J^k_{k-1}\tau_{k-1}\|_{L^2(\Omega)} &\le \|\tau_k\|_{L^2(\Omega)} + \|J^k_{k-1}\tau_{k-1}\|_{L^2(\Omega)}\\
&\le \|\tau_k\|_{L^2(\Omega)} + C\|\tau_{k-1}\|_{L^2(\Omega)}\\
&\le C\|g\|_{L^2(\Omega)}.
\end{aligned} \tag{3.26}$$

To estimate $\|\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}\|_{L^2(\Omega)}$, we consider the following auxiliary problem and its finite element approximations: Find $[\underset{\sim}{\eta}, \delta] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$\mathcal{L}([\underset{\sim}{\eta}, \delta], [\underset{\sim}{v}, q]) = (\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}, \underset{\sim}{v}), \quad \forall\ [\underset{\sim}{v}, q] \in (H_0^1(\Omega))^2 \times L_0^2(\Omega); \tag{3.27}$$

and find $[\underset{\sim}{\eta}_j, \delta_j] \in \underset{\sim}{X}_j \times M_j$ $(j = k-1, k)$ such that

$$\mathcal{L}_j([\underset{\sim}{\eta}_j, \delta_j], [\underset{\sim}{v}, q]) = (\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}, \underset{\sim}{v})_j, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_j \times M_j. \tag{3.28}$$

Then, from (1.5), (1.11) and (1.4) with $\ell = 2$,

$$\|\underset{\sim}{\eta}_k - \underset{\sim}{\eta}_{k-1}\|_k + \|\delta_k - \delta_{k-1}\|_{L^2(\Omega)} \le Ch_k \|\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}\|_{L^2(\Omega)}. \tag{3.29}$$

and by using Lemma 3.3 for $[\underset{\sim}{\eta}_k, \delta_k]$, and $[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}]$

$$|||[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\eta}_k, \delta_k]|||_{0,k-1} \le Ch_k^2 \|\underset{\sim}{\sigma}_k - H_{k-1}^k \sigma_{k-1}\|_{L^2(\Omega)}. \tag{3.30}$$

Furthermore, by (3.7), (3.28)–(3.30) we get

$$\begin{aligned}
&\|\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}\|_{L^2(\Omega)}^2 \\
&\quad = (\underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1}, \underset{\sim}{\sigma}_k - H_{k-1}^k \underset{\sim}{\sigma}_{k-1})_k \\
&\quad = \mathcal{L}_k([\underset{\sim}{\eta}_k, \delta_k], [\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]) \\
&\quad = \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{\eta}_k, \delta_k]) - \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}], (I_{k-1}^k)^*[\underset{\sim}{\eta}_k, \delta_k]) \\
&\quad = (g, \delta_k)_k - (g, \theta')_{k-1} \\
&\quad \le \|g\|_{L^2(\Omega)} \|\delta_k - \theta'\|_{L^2(\Omega)},
\end{aligned} \tag{3.31}$$

where we have defined

$$[\underset{\sim}{\xi}', \theta'] = (I_{k-1}^k)^*[\underset{\sim}{\eta}_k, \delta_k]. \tag{3.32}$$

Hence, it follows from (1.4), (1.5), (1.11), (3.30) and (3.32) that

$$\begin{aligned}
&\|\delta_k - \theta'\|_{L^2(\Omega)} \\
&\quad \le \|\delta_k - \delta_{k-1}\|_{L^2(\Omega)} + \|\delta_{k-1} - \theta'\|_{L^2(\Omega)} \\
&\quad \le \|\delta_k - \delta_{k-1}\|_{L^2(\Omega)} + h_k^{-1} |||[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\eta}_k, \delta_k]|||_{0,k-1} \\
&\quad \le Ch_k \|\underset{\sim}{\sigma}_k - H_{k-1}^k \sigma_{k-1}\|_{L^2(\Omega)}.
\end{aligned} \tag{3.33}$$

Finally, (3.24) follows from (3.25), (3.26), (3.31) and (3.33).

Lemma 3.5. *Under the assumptions of Lemma 3.4, we also have*

$$|||[\underset{\sim}{\sigma}_{k-1} \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1} \le Ch_k \|g\|_{L^2(\Omega)}. \tag{3.34}$$

Proof. Similar to the proof of Lemma 3.3, we introduce the notations (3.13), and the auxiliary problems (3.15), (3.16), (3.20), (3.21). Then by similar arguments as used in (3.17), (3.18) and (3.22), we can show

$$|||[\underset{\sim}{\eta}_{k-1}, \delta_{k-1}] - I_{k-1}^k [\underset{\sim}{\eta}_{k-1}, \delta_{k-1}]|||_{0,k} \le Ch_k^2 \|\underset{\sim}{\xi}\|_{L^2(\Omega)}. \tag{3.35}$$

$$\|\delta'_{k-1}\|_{L^2(\Omega)} \le C\|\lambda\|_{L^2(\Omega)}. \tag{3.36}$$

Moreover, by using (3.7) and (3.13)–(3.16) we get

$$
\begin{aligned}
\|\underset{\sim}{\xi}\|^2_{L^2(\Omega)} &= (\underset{\sim}{\xi},\underset{\sim}{\xi})_{k-1} = \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1},\delta_{k-1}],[\underset{\sim}{\xi},\lambda]) \\
&= \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1},\delta_{k-1}],[\underset{\sim}{\sigma}_{k-1},\tau_{k-1}]) - \mathcal{L}_{k-1}([\underset{\sim}{\eta}_{k-1},\sigma_{k-1}],(I^k_{k-1})^*[\underset{\sim}{\sigma}_k,\tau_k]) \\
&= \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1},\tau_{k-1}],[\underset{\sim}{\eta}_{k-1},\delta_{k-1}]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k,\tau_k],I^k_{k-1}[\underset{\sim}{\eta}_{k-1},\delta_{k-1}]) \\
&= (g,\delta_{k-1})_{k-1} - (g,J^k_{k-1}\delta_{k-1})_k \\
&\le \|g\|_{L^2(\Omega)}\|\sigma_{k-1} - J^k_{k-1}\sigma_{k-1}\|_{L^2(\Omega)} \\
&\le h_k^{-1}\|g\|_{L^2(\Omega)}|||[\underset{\sim}{\eta}_{k-1},\delta_{k-1}] - I^k_{k-1}[\underset{\sim}{\eta}_{k-1},\delta_{k-1}]|||_{0,k} \\
&\le Ch_k\|g\|_{L^2(\Omega)}\|\underset{\sim}{\xi}\|_{L^2(\Omega)},
\end{aligned}
$$

hence

$$\|\underset{\sim}{\xi}\|_{L^2(\Omega)} \le Ch_k\|g\|_{L^2(\Omega)}. \tag{3.37}$$

On the other hand, by using (3.7), (3.13), (3.20), (3.21) and (A.2) we have

$$
\begin{aligned}
\|\lambda\|^2_{L^2(\Omega)} &= (\lambda,\lambda)_{k-1} = \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1},\delta'_{k-1}],[\underset{\sim}{\xi},\lambda]) \\
&= \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1},\delta'_{k-1}],[\underset{\sim}{\sigma}_{k-1},\tau_{k-1}]) - \mathcal{L}_{k-1}([\underset{\sim}{\eta}'_{k-1},\delta'_{k-1}],(I^k_{k-1})^*[\underset{\sim}{\sigma}_k,\tau_k]) \\
&= \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1},\tau_{k-1}],[\underset{\sim}{\eta}'_{k-1},\delta'_{k-1}]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k,\tau_k],I^k_{k-1}[\underset{\sim}{\eta}'_{k-1},\delta'_{k-1}]) \\
&= (g,\delta'_{k-1})_{k-1} - (g,J^k_{k-1}\delta'_{k-1})_k \\
&\le \|g\|_{L^2(\Omega)}\|\delta'_{k-1} - J^k_{k-1}\delta'_{k-1}\|_{L^2(\Omega)} \\
&\le C\|g\|_{L^2(\Omega)}\|\delta'_{k-1}\|_{L^2(\Omega)} \\
&\le C\|g\|_{L^2(\Omega)}\|\lambda\|_{L^2(\Omega)},
\end{aligned}
$$

hence

$$\|\lambda\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}. \tag{3.38}$$

Finally, (3.34) follows from (3.14), (3.37) and (3.38).

4. The Proof of Convergence

In this section we will prove the convergence of the multigrid algorithm in Section 2 by induction under abstract assumptions (A.0)–(A.3). A uniform error reduction rate bounded away form one is first proved in the two–grid case provided that sufficiently many smoothing steps are performed. By the standard perturbation technique, the result is then extended to the multilevel algorithm. Since the smoothing property was shown in [15], it will not be repeated here. To prove the approximation property, the duality techniques which have been used in the proofs of Lemma 3.3–3.5 will be utilized again here.

Let

$$[\underset{\sim}{e}^i,\varepsilon^i] = [\underset{\sim}{w} - \underset{\sim}{w}^i,\alpha - \alpha^i], \quad j = 0,1,2,\cdots,m+1, \tag{4.1}$$

be error functions of the ith iteration of the multigrid algorithm defined in Section 2 with m smoothing steps at level k. The following smoothing property was proven by Verfürth in [15].

Lemma 4.1. (smoothing property) *For any initial guess, there holds*

$$|||[\underset{\sim}{e}^m,\varepsilon^m]|||_{2,k} \le Ch_k^{-2}m^{\frac{1}{2}}|||[\underset{\sim}{e}^0,\varepsilon^0]|||_{0,k}. \tag{4.2}$$

Next, we are going to establish the approximation property under the assumptions (A.0)–(A.3) on the prolongation operator I_{k-1}^k.

Lemma 4.2. (approximation property) *There holds the following inequality:*

$$|||[\underset{\sim}{v}, q] - I_{k-1}^k (I_{k-1}^k)^* [\underset{\sim}{v}, q]|||_{0,k} \leq C h_k^2 |||[\underset{\sim}{v}, q]|||_{2,k}, \quad \forall\ [\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k. \tag{4.3}$$

Proof. For any $[\underset{\sim}{v}, q] \in \underset{\sim}{X}_k \times M_k$, let

$$[\underset{\sim}{\zeta}, \theta] = (I_{k-1}^k)^* [\underset{\sim}{v}, q] \in \underset{\sim}{X}_{k-1} \times M_{k-1}. \tag{4.4}$$

It follows form (2.3), (3.2), (4.4) and Lemma 3.1 that

$$|||[\underset{\sim}{v}, q] - I_{k-1}^k (I_{k-1}^k)^* [\underset{\sim}{v}, q]|||_{0,k}^2 = \|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)}^2 + h_k^2 \|q - J_{k-1}^k \theta\|_{L^2(\Omega)}^2, \tag{4.5}$$

and

$$|||[\underset{\sim}{\zeta}, \theta]|||_{2,k-1} \leq C |||[\underset{\sim}{v}, q]|||_{2,k}. \tag{4.6}$$

To estimate $\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)}$, we consider the problem (1.3) with $\underset{\sim}{f} = \underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}$ and $g \equiv 0$. By (1.4) with $\ell = 2$, (A.3) and Lemma 3.3 we obtain

$$\mathcal{L}_j([\underset{\sim}{\sigma}_j, \tau_j], [\underset{\sim}{w}, r]) = (\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}, \underset{\sim}{w})_j, \quad \forall\ [\underset{\sim}{w}, r] \in \underset{\sim}{X}_j \times M_j,\ j = k, k-1. \tag{4.7}$$

$$|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \leq C h_k^2 \|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)}. \tag{4.8}$$

$$|||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^* [\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1} \leq C h_k^2 \|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)}. \tag{4.9}$$

Therefore, from (1.5), (3.5), (3.7), (4.4)–(4.9) we have

$$\begin{aligned}
\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)}^2 =& (\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}, v - H_{k-1}^k \underset{\sim}{\zeta})_k \\
=& \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{v}, q]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], I_{k-1}^k [\underset{\sim}{\zeta}, \theta]) \\
=& \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{v}, q]) - \mathcal{L}_{k-1}((I_{k-1}^k)^* [\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{\zeta}, \theta]) \\
=& \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{v}, q]) - \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}], [\underset{\sim}{\zeta}, \theta]) \\
& + \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^* [\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{\zeta}, \theta]) \\
=& \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}], [\underset{\sim}{v}, q]) \\
& + \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^* [\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{\zeta}, \theta]) \\
\leq& |||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} |||[\underset{\sim}{v}, q]|||_{2,k} \\
& + |||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^* [\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1} |||[\underset{\sim}{\zeta}, \theta]|||_{2,k-1} \\
\leq& C h_k^2 \|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)} |||[\underset{\sim}{v}, q]|||_{2,k}.
\end{aligned} \tag{4.10}$$

Hence,

$$\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{\zeta}\|_{L^2(\Omega)} \leq C h_k^2 |||[\underset{\sim}{v}, q]|||_{2,k}. \tag{4.11}$$

Similarly, to estimate $\|q - J_{k-1}^k\theta\|_{L^2(\Omega)}$, we consider the problem (1.3) with $\underset{\sim}{f} = 0$ and $g = q - J_{k-1}^k\theta \in L_0^2(\Omega)$. Then it follows from Lemmas 3.4 and 3.5 that

$$\mathcal{L}_j([\underset{\sim}{\sigma}_j, \tau_j), [\underset{\sim}{w}, r]) = (q - J_{k-1}^k\theta, r)_j, \quad \forall\, [\underset{\sim}{w}, r] \in \underset{\sim}{X}_j \times M_j,\ j = k-1, k. \tag{4.12}$$

$$|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \le Ch_k\|q - J_{k-1}^k\theta\|_{L^2(\Omega)}. \tag{4.13}$$

$$|||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k-1} \le Ch_k\|q - J_{k-1}^k\theta\|_{L^2(\Omega)}. \tag{4.14}$$

By (1.5), (3.5), (3.7), (4.4), (4.12)–(4.14) we have

$$\begin{aligned}
\|q - J_{k-1}^k\theta\|_{L^2(\Omega)}^2 &= (q - J_{k-1}^k\theta, q - J_{k-1}^k\theta)_k \\
&= \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{v}, q]) - \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k], I_{k-1}^k[\underset{\sim}{\zeta}, \theta]) \\
&= \mathcal{L}_k([\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}], [\underset{\sim}{v}, q]) \\
&\quad + \mathcal{L}_{k-1}([\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k], [\underset{\sim}{\zeta}, \theta]) \\
&\le |||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \underset{\sim}{\tau}_{k-1}]|||_{0,k}|||[\underset{\sim}{v}, q]|||_{2,k} \\
&\quad + |||[\underset{\sim}{\sigma}_{k-1}\tau_{k-1}] - (I_{k-1}^k)^*[\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k}|||[\underset{\sim}{\zeta}, \theta]|||_{2,k} \\
&\le Ch_k\|q - J_{k-1}^k\theta\|_{L^2(\Omega)}|||[\underset{\sim}{v}, q]|||_{2,k}.
\end{aligned}$$

Hence,

$$\|q - J_{k-1}^k\theta\|_{L^2(\Omega)} \le Ch_k|||[\underset{\sim}{v}, q]|||_{2,k}. \tag{4.15}$$

Finally, (4.3) follows from (4.5), (4.11) and (4.15).

Since we have shown both smoothing and approximation properties, by the standard perturbation argument for showing convergence of a W–cycle multigrid algorithm (cf. [3], [4] and [15]), we then get the following convergence theorem for the multigrid algorithm defined in Section 2.

Theorem I. (Convergence Theorem) *Let $\mu > 1$ in the multigrid algorithm. Then, under the assumptions (A.0)–(A.3) there exists a constant $0 < \gamma < 1$ and a positive integer m, all independent of the level number k, such that if*

$$|||[\underset{\sim}{\psi}^*, \rho^*] - [\underset{\sim}{\psi}, \rho]|||_{0,k} \le \gamma|||[\underset{\sim}{\psi}^*, \rho^*|||_{0,k}, \tag{4.16}$$

then

$$|||[\underset{\sim}{w}, \alpha] - [\underset{\sim}{w}^{m+1}, \alpha^{m+1}]|||_{0,k} \le \gamma|||[\underset{\sim}{w}, \alpha] - [\underset{\sim}{w}^0, \alpha^0]|||_{0,k}. \tag{4.17}$$

5. Applications

The objective of this section is to apply our general multigrid algorithm developed and analyzed in previous sections to some well–known mixed finite elements for the Stokes problems. To accomplish the goal, we only need to explicitly define the prolongation operator I_{k-1}^k for each specific element and to verify that I_{k-1}^k does satisfy the assumptions (A.1)–(A.3). We divide all cited elements into four groups, which are nested conforming mixed elements, nonnested conforming mixed elements, nonconforming mixed elements, and nonnested mixed elements caused by nonnested mesh refinings which are either necessary or artificial. We show that our abstract framework provides an alternative convergence analysis for the multigrid algorithms proposed early in [4], [15] and [16], moreover, it also applies to various other mixed elements for the Stokes problems. In addition, we prove that the usual local averaging technique which have been used to construct prolongation operators for nonconforming elements can be replaced by a computationally cheaper alternative, random choice technique.

In subsections 5.1–5.3, the nested mesh refining is assumed, that is, $\mathcal{T}_k$ is obtained by connecting the midpoints of the three sides of all triangles of $\mathcal{T}_{k-1}$ or by linking the midpoints of two opposite sides of all rectangles of $\mathcal{T}_{k-1}$. So the assumptions (A.0) and (1.5) hold naturally. In subsection 5.4, elements which result in nonnested mesh refinings in a multilevel algorithm are considered. It is interesting to point out that by looking from a different point of view, i.e., to think the elements as composite elements, then these elements can also be treated as nonnested elements but with the nested mesh refinings.

5.1. Nested Conforming Mixed Finite Elements

We first consider the nested conforming mixed elements for the Stokes problems. The multigrid method for those elements has been studied in [15]. In this subsection, we will provide an alternative convergence analysis to the one given in [15] by showing that our abstract framework applies to those elements. In the following, only three such a kind elements are listed as examples, the argument obviously applies to other nested conforming elements.

Example 1. *$P_2 - P_0$ element*

In this element, $\mathcal{T}_k$ is a triangulation of Ω for each $k \geq 0$, and $\underset{\sim}{X}_k, M_k$ are defined as follows:

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (P_2(K))^2,\ K \in \mathcal{T}_k\}. \tag{5.1}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_0(K),\ K \in \mathcal{T}_k\}. \tag{5.2}$$

Example 2. *Composite $P_1 - P_0$ element*

In this example, for each $k \geq 0, \mathcal{T}_k$ also is a triangulation of Ω, and $\underset{\sim}{X}_k, M_k$ are defined as follows (cf. [11]):

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (P_1(K))^2,\ K \in \mathcal{T}_{k+1}\}. \tag{5.3}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_0(K),\ K \in \mathcal{T}_k\}. \tag{5.4}$$

(5.3) shows for any $v \in \underset{\sim}{X}_k, K \in \mathcal{T}_k, v|_K$ is a four–piece linear function.

Example 3. *$Q_2 - P_1$ element*

In this example, for each $k \geq 0, \mathcal{T}_k$ is a rectangular partition of Ω, and $\underset{\sim}{X}_k, M_k$ are defined as follows

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (Q_2(K))^2,\ \forall\ K \in \mathcal{T}_k\}. \tag{5.5}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}. \tag{5.6}$$

It is well–known that above three elements are stable (cf. [6], [11]), and it is easy to see that they are nested elements, namely,

$$\underset{\sim}{X}_{k-1} \times M_{k-1} \subseteq \underset{\sim}{X}_k \times M_k,$$

so for these elements we can choose the prolongation operator to be

$$I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k] = [i_{\underset{\sim}{X}}, i_M] : \underset{\sim}{X}_{k-1} \times M_{k-1} \longrightarrow \underset{\sim}{X}_k \times M_k, \tag{5.7}$$

where and later $i_{\underset{\sim}{X}}$ and i_M stand for the natural injections in $\underset{\sim}{X}_k$ and M_k, respectively. So $I_{k-1}^k = [i_{\underset{\sim}{X}}, i_M]$ is the natural injection in $\underset{\sim}{X}_k \times M_k$. Therefore, it is natural that (A.1)–(A.3) hold. Thus, we have the following theorem

Theorem II. *For any nested stable conforming mixed finite element for the Stokes problems, Theorem I holds with I_{k-1}^k defined by (5.7).*

5.2. Nonnested Conforming Mixed Finite Elements

We now consider the conforming mixed elements whose multilevel finite element spaces are not nested. This class includes two kinds of elements, elements enriched by bubble functions (e.g. mini element) and composite elements. The multigrid method for mini element was first developed in [15], where the bubble functions were L^2 projected from coarser grid to finer grid. In the following, we will show that the bubble function part of a coarse level correction can be ignored in the prolongation step for all elements enriched by bubble functions, this then results in using the natural injection of the principal part as a prolongation operator.

Among nonnested conforming elements, the following four typical elements will be discussed in detail.

Example 4. *Mini element*

This element was introduced by Arnold–Brezzi–Fortin [1] as a remedy for the unstable $P_1 - P_1$ element. Let $\mathcal{T}_k$ be a triangulation of Ω for each $k \geq 0$, define $\underset{\sim}{X}_k$ and M_k to be

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_k \in [P_1(K) \oplus \text{span}\{\lambda_1\lambda_2\lambda_3\}]^2,\ \forall\ K \in \mathcal{T}_k\}. \tag{5.8}$$

$$M_k = \{q \in C(\overline{\Omega}) \cap L_0^2(\Omega),\ q|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}, \tag{5.9}$$

where λ_j $(j = 1, 2, 3)$ are the barycentric coordinates. It is easy to see that

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

Example 5. *Bernardi–Raugel element*

This element was presented by Bernardi and Raugel (cf. [6]) as a simplification of $P_2 - P_0$ element. For each $k \geq 0$ we define $\underset{\sim}{X}_k$ and M_k as follows

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (P_1(K))^2 \oplus \text{span}\{\underset{\sim}{p}_1, \underset{\sim}{p}_2, \underset{\sim}{p}_3\},\ \forall\ K \in \mathcal{T}_k\}. \tag{5.10}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_0(K),\ \forall\ K \in \mathcal{T}_k\}, \tag{5.11}$$

where $\mathcal{T}_k$ is a triangulation of Ω, p_j is given by

$$\underset{\sim}{p}_1 = \lambda_2\lambda_3\underset{\sim}{n}_1, \quad \underset{\sim}{p}_2 = \lambda_1\lambda_3\underset{\sim}{n}_2, \quad \underset{\sim}{p}_3 = \lambda_1\lambda_2\underset{\sim}{n}_3,$$

where λ_j $(j = 1, 2, 3)$ are the barycentric coordinates and $\underset{\sim}{n}_j$ $(j = 1, 2, 3)$ are the unit normal vectors to the opposite edges to the vertices p_j $(j = 1, 2, 3)$. Clearly, we have

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

Example 6. *Crouzeix–Raviart $P_2^+ - P_1$ element*

For each $k \geq 0$, let $\mathcal{T}_k$ be a triangulation of Ω and $\underset{\sim}{X}_k$ and M_k be defined by

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in [P_2(K) \oplus \text{span}\{\lambda_1\lambda_2\lambda_3\}]^2,\ \forall\ K \in \mathcal{T}_k\}. \tag{5.12}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}. \tag{5.13}$$

Again, λ_j $(j = 1, 2, 3)$ are the barycentric coordinates. It is easy to see that

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

Example 7. *Composite $P_1 - P_1$ element*

For the triangulations $\mathcal{T}_k,\ k \geq 0$, $\underset{\sim}{X}_k$ and M_k are defined as follows

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_{K_i} \in (P_1(K_i))^2,\ \overline{K} = \cup_{i=1}^3 \overline{K_i},\ \forall\ K \in \mathcal{T}_k\}. \tag{5.14}$$

$$M_k = \{q \in L_0^2(\Omega) \cap C(\overline{\Omega}),\ q|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}, \tag{5.15}$$

where K_i $(i = 1, 2, 3)$ are obtained by connecting the three vertices of K with its barycenter. It is not hard to check that this element is stable and satisfies (1.11). Obviously,

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

To apply our abstract framework to above four elements, we define the prolongation operator $I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k] : \underset{\sim}{X}_{k-1} \times M_{k-1} \longrightarrow \underset{\sim}{X}_k \times M_k$ as follows:

$$J_{k-1}^k = i_M :\ M_{k-1} \longrightarrow M_k, \text{ the natural injection,} \tag{5.16}$$

$$H_{k-1}^k = \Pi_k^p \times \Pi_k^p :\ \underset{\sim}{X}_{k-1} \longrightarrow \underset{\sim}{X}_k, \tag{5.17}$$

where Π_k^p denotes the standard linear interpolation operator associated with $\mathcal{T}_k$ for the elements in Examples 4, 5 and 7, and the quadratic interpolation operator associated with $\mathcal{T}_{k-1}$ for the element in Example 6. In other words, the contribution of the bubble functions is ignored in the prolongation step.

Obviously, H_{k-1}^k in (5.17) is well–defined, and J_{k-1}^k defined by (5.16) satisfies (A.2). Moreover, by using the standard scaling argument, we can prove the following lemma.

Lemma 5.1. *For H_{k-1}^k defined by (5.17), (A.1) holds, i.e.,*

$$\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{v}\|_{L^2(\Omega)} \leq C h_k \|\underset{\sim}{v}\|_{k-1}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}, \quad k \geq 1, \tag{5.18}$$

for each $\underset{\sim}{X}_{k-1}$ which is defined in Examples 4–7.

Proof. For any $K \in \mathcal{T}_{k-1}$, let

$$H_K = H_{k-1}^k|_K :\ \underset{\sim}{X}_{k-1}|_K \longrightarrow \underset{\sim}{X}_k^p|_K,$$

where $\underset{\sim}{X}_k^p$ is the conforming linear finite element space with respect to $\mathcal{T}_k$ for the elements in Examples 4, 5 and 7, and the conforming quadratic finite element space with respect to $\mathcal{T}_{k-1}$ for the element in Example 6.

Obviously, H_K is a linear operator from the finite dimension space $\underset{\sim}{X}_{k-1}|_K$ to the finite dimension space $\underset{\sim}{X}_k^p|_K$ and satisfies

$$H_K \underset{\sim}{p}_0 = \underset{\sim}{p}_0, \quad \forall \underset{\sim}{p}_0 \in (P_0(K))^2.$$

Furthermore, it is easy to check that $\|\nabla \underset{\sim}{v}\|_{L^2(\Omega)(K)}$ is a norm over $(\underset{\sim}{X}_{k-1}|_K)/(P_0(K))^2$. Therefore, by the scaling argument, we have that for the reference element $\widehat{K}$

$$\|\underset{\sim}{v} - H_{\widehat{K}} \underset{\sim}{v}\|_{L^2(\widehat{K})} \leq C \|\nabla \underset{\sim}{v}\|_{L^2(\widehat{K})}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_{\widehat{K}}.$$

This implies that

$$\|\underset{\sim}{v} - H_K \underset{\sim}{v}\|_{L^2(K)} \le C h_k \|\nabla \underset{\sim}{v}\|_{L^2(K)}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_K. \tag{5.19}$$

Then, summing up (5.19) over all $K \in \mathcal{T}_{k-1}$, we get (5.18).

It now remains to check (A.3). In fact, for $[\underset{\sim}{\sigma}_j, \tau_j] \in \underset{\sim}{X}_j \times M_j$ $(j = 1, 2)$, $[\underset{\sim}{\sigma}, \tau] \in (H^2(\Omega) \cap H_0^1(\Omega))^2 \times (H^1(\Omega) \cap L_0^2(\Omega))$ in (A.3) defined in Section 3, from (1.11), (3.2) and (3.3) we get

$$|||[\underset{\sim}{\sigma}, \tau] - [\underset{\sim}{\sigma}_k, \tau_k]|||_{0,k} \le C h_k^2 (\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}). \tag{5.20}$$

Let

$$\underset{\sim}{\sigma}' = (\Pi_{k-1}^p \times \Pi_{k-1}^p)\underset{\sim}{\sigma}, \quad \tau' = \Pi_{k-1}' \tau,$$

where Π_{k-1}^p is defined in (5.17). Here and later Π_{k-1}' denotes the standard finite element interpolation operator of M_{k-1}. Then

$$[\underset{\sim}{\sigma}', \tau'] \in (\underset{\sim}{X}_{k-1} \times M_{k-1}) \cap (\underset{\sim}{X}_k \times M_k).$$

Moreover, it follows from (5.16), (5.17) and the interpolation theory that

$$I_{k-1}^k [\underset{\sim}{\sigma}', \tau'] = [\underset{\sim}{\sigma}', \tau'], \tag{5.21}$$

$$|||[\underset{\sim}{\sigma}, \tau] - [\underset{\sim}{\sigma}', \tau']|||_{0,k-1} \le C h_{k-1}^2 (\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}). \tag{5.22}$$

Therefore, by (A.1), (A.2), (3.6), (5.20)–(5.22), we have

$$\begin{aligned}
&|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \\
&\le |||[\underset{\sim}{\sigma}_k, \tau_k] - [\underset{\sim}{\sigma}', \tau']|||_{0,k} + |||I_{k-1}^k [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - [\underset{\sim}{\sigma}', \tau']|||_{0,k} \\
&\le C(|||[\underset{\sim}{\sigma}_k, \tau_k] - [\underset{\sim}{\sigma}', \tau']|||_{0,k} + |||[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}] - [\underset{\sim}{\sigma}', \tau']|||_{0,k-1}) \\
&\le C h_k^2 (\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}).
\end{aligned}$$

This shows (A.3) holds with I_{k-1}^k being defined by (5.16) and (5.17) for Example 4–7.

To sum up, we have

Theorem III. *For the nonnested conforming mixed finite elements for the Stokes problems, namely mini element, Bernardi–Raugel element. Crouzeix–Raviart $P_2^+ - P_1$ element and the composite $P_1 - P_1$ element, Theorem I holds with I_{k-1}^k defined by (5.16) and (5.17).*

Remark 5.1. For mini element. if we choose I_{k-1}^k be one constructed in [15], clearly, this I_{k-1}^k satisfies the assumptions (A.1)–(A.3). Hence, our abstract framework gives an alternative convergence analysis for the algorithm in [15].

Remark 5.2. For Crouzeix–Raviart $P_2^+ - P_1$ element, if we choose $H_{k-1}^k = \Pi_k^p \times \Pi_k^p$, where Π_k^p is the quadratic interpolation operator associated with $\mathcal{T}_k$, then I_{k-1}^k also satisfies (A.1)–(A.3), but it is effected by the bubble function now.

Remark 5.3. It is not difficult to see that our abstract framework with the prolongation operator being chosen as above also applies to other conforming mixed elements constructed by adding bubble functions, e.g. enriched Taylor–Hood element and other composite elements (cf. [6], [11]).

5.3. Nonconforming Mixed Finite Elements

Nonconforming mixed finite elements for the Stokes problems are also used very often in practice. Crouzeix–Raviart nonconforming element is the simplest and most popular one in this class. Some early multigrid result was given in [4]. In this section, we introduce a simple and different prolongation operator I_{k-1}^k for two typical elements of this class, our prolongation is defined as a simple modification of the standard interpolation operator either by the usual local averaging technique or, in particular, by a computationally cheaper alternative, the random choice technique. The two elements are considered in detail are Crouzeix–Raviart element and Fortin–Soulie element.

Example 8. *Crouzeix–Raviart nonconforming element*

It is the most well–known nonconforming finite element. Now for each $k \geq 0, \mathcal{T}_k$ is a triangle partition; $\underset{\sim}{X}_k, M_k$ are defined as follows

$$\begin{aligned} \underset{\sim}{X}_k =&\{\underset{\sim}{v} \in (L^2(\Omega))^2,\ \underset{\sim}{v}|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k,\ \text{ and } \underset{\sim}{v} \text{ is continuous} \qquad (5.23)\\ &\text{at the midpoints of the interelement boundaries and vanishes at}\\ &\text{the midpoints of edges along } \partial\Omega\}.\\ M_k =&\{q \in L_0^2(\Omega),\ q|_K \in P_0(K),\ \forall\ K \in \mathcal{T}_k\}. \qquad (5.24) \end{aligned}$$

Obviously, we have

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

So this element is nonnested. To apply our abstract framework, we define the prolongation operator $I_{k-1}^k = [H_{k-1}^k, J_{k-1}^h] : \underset{\sim}{X}_{k-1} \times M_{k-1} \longrightarrow \underset{\sim}{X}_k \times M_k$ as follows:

$H_{k-1}^k\ :\ \underset{\sim}{X}_{k-1} \longrightarrow \underset{\sim}{X}_k$ is the local averaging modification of the standard interpolation operator associated with Crouzeix–Raviart nonconforming element. Specifically, for any $\underset{\sim}{v} \in \underset{\sim}{X}_{k-1}$, $H_{k-1}^k\underset{\sim}{v}$ satisfies

$$H_{k-1}^k\underset{\sim}{v}(m_i) = \begin{cases} \underset{\sim}{v}(m_i), \quad m_i \in K, \quad K \in \mathcal{T}_{k-1}, \\ R[\underset{\sim}{v}(m_i)] \text{ or } \dfrac{1}{2}(\underset{\sim}{v}|_{K_1}(m_i) + \underset{\sim}{v}|_{K_2}(m_i)), \quad m_i \in \partial K_1 \cap \partial K_2,\ K_1,\ K_2 \in \mathcal{T}_k, \end{cases} \tag{5.25}$$

where m_i stands for a midpoint of the internal edges of the elements in $\mathcal{T}_k$, and

$$R[\underset{\sim}{v}(m_i)] \in \mathcal{S}(m_i) \equiv \{\underset{\sim}{v}|_{K_1}(m_i),\ \ \underset{\sim}{v}|_{K_2}(m_i)),\ \ m_i \in \partial K_1 \cap \partial K_2\}, \tag{5.26}$$

that is, $R[\underset{\sim}{v}(m_i)]$ randomly takes one of the two values in the set $\mathcal{S}(m_i)$.

$$J_{k-1}^k = i_M :\ M_{k-1} \longrightarrow M_k,\ \text{the natural injection operator.} \tag{5.27}$$

Lemma 5.2. *The operator H_{k-1}^k defined by (5.25) satisfies (A.1), that is,*

$$\|\underset{\sim}{v} - H_{k-1}^k\underset{\sim}{v}\|_{L^2(\Omega)} \leq Ch_k\|\underset{\sim}{v}\|_{k-1}, \quad \forall\underset{\sim}{v} \in \underset{\sim}{X}_{k-1},\ k \geq 1. \tag{5.28}$$

Proof. For any $K \in \mathcal{T}_{k-1}$, let $\tau(K) = \cup\{K' \in \mathcal{T}_{k-1},\ \text{meas}(\partial K' \cap \partial K) > 0\}$ and

$$H_\tau = H_{k-1}^k|_{\tau(K)} : \quad \underset{\sim}{X}_{k-1}|_{\tau(K)} \longrightarrow \underset{\sim}{X}_k|_K.$$

If $\text{meas}(\partial\tau(K) \cap \partial\Omega) > 0$, it is easy to check that $[P_0(\tau(K))]^2 \subset \underset{\sim}{X}_{k-1}|_{\tau(K)}$ and

$$H_\tau \underset{\sim}{p}_0 = \underset{\sim}{p}_0, \qquad \forall\underset{\sim}{p}_0 \in [P_0(\tau(K))]^2.$$

Moreover, we can show that $\|\nabla \underset{\sim}{v}\|_{L^2(\tau(K))}$ is a norm over $\underset{\sim}{X}_{k-1}|_{\tau(K)}/[P_0(\tau(K))]^2$, hence by the standard scaling argument we have that for the reference element $\widehat{K}$ and corresponding $\widehat{\tau}(\widehat{K})$

$$\|\underset{\sim}{\hat{v}} - \widehat{H}_{\widehat{\tau}}\underset{\sim}{\hat{v}}\|_{L^2(\widehat{K})} \leq C\|\nabla \underset{\sim}{\hat{v}}\|_{L^2(\widehat{\tau}(\widehat{K}))}, \quad \forall \underset{\sim}{\hat{v}} \in \underset{\sim}{\widehat{X}}_{k-1}|_{\widehat{\tau}(\widehat{K}))},$$

this implies that

$$\|\underset{\sim}{v} - \widehat{H}_{\tau}\underset{\sim}{v}\|_{L^2(K)} \leq Ch_k\|\nabla \underset{\sim}{v}\|_{L^2(\tau(K))}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_{\tau(K))}. \tag{5.29}$$

If $\mathrm{meas}(\partial\tau(K) \cap \partial\Omega) = 0$, clearly, $\|\nabla \underset{\sim}{v}\|_{L^2(\tau(K))}$ is a norm over $\underset{\sim}{X}_{k-1}|_{\tau(K)}$, so (5.29) still holds. Finally, (5.28) follows from summing up (2.29) over all $K \in \mathcal{T}_{k-1}$.

Now, since (A.2) holds trivially. it remains to check (A.3), by repeating the argument used in subsection 5.2 we can show that (5.20) still holds for Crouzeix–Raviart nonconforming element. Moreover, let

$$\underset{\sim}{\sigma}' = \underset{\sim}{\sigma}^I, \quad \tau' = \Pi'_{k-1}\tau,$$

where $\underset{\sim}{\sigma}^I$ is the continuous linear interpolation of $\underset{\sim}{\sigma}$ with respect to $\mathcal{T}_{k-1}$, we can show that (5.21) and (5.22) also hold for Crouzeix–Raviart nonconforming element. Hence, (A.3) holds for the above I_{k-1}^k. Thus, our analysis applies to Crouzeix–Raviart nonconforming finite element.

Example 9. *Fortin–Soulie nonconforming element*

This is a P_2–P_1 type triangular mixed element. Let $\mathcal{T}_k$ be the family of triangle partition, and define $\underset{\sim}{X}_k, M_k$ as follows (cf. [10])

$$\begin{aligned}\underset{\sim}{X}_k =&\{\underset{\sim}{v} \in (L^2(\Omega))^2,\ \underset{\sim}{v}|_K \in (P_2(K))^2,\ \forall\ K \in \mathcal{T}_k,\ \text{and } \underset{\sim}{v} \text{ is continuous at} \\ &\text{the Gauss–Legendre points of the interelement boundaries and} \\ &\text{vanishes at the Gauss–Legendre points of edges along } \partial\Omega\}.\end{aligned} \tag{5.30}$$

$$M_k =\{q \in L_0^2(\Omega),\ q|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}. \tag{5.31}$$

It is easy to check that

$$M_{k-1} \subset M_k, \quad \underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad k \geq 1.$$

Similar to Example 8, we define $I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k] : \underset{\sim}{X}_{k-1} \times M_k \longrightarrow \underset{\sim}{X}_k \times M_k$ as follows:

$H_{k-1}^k :\ \underset{\sim}{X}_{k-1} \longrightarrow \underset{\sim}{X}_k$ such that for any $\underset{\sim}{v} \in \underset{\sim}{X}_{k-1},\ H_{k-1}^k\underset{\sim}{v}$ satisfies

$$H_{k-1}^k\underset{\sim}{v}(t_i) = \begin{cases} \underset{\sim}{v}(t_i), \quad t_i \in K,\ K \in \mathcal{T}_{k-1}, \\ R[\underset{\sim}{v}(t_i)],\ \text{or}\ \dfrac{1}{2}(\underset{\sim}{v}|_{K_1}(t_i) + \underset{\sim}{v}|_{K_2}(t_i)), \quad t_i \in \partial K_1 \cap \partial K_2,\ K_1,\ K_2 \in \mathcal{T}_k, \end{cases} \tag{5.32}$$

where t_i is a Gauss–Legendre point of the internal edges of the elements in $\mathcal{T}_k$ and $R[\underset{\sim}{v}(t_i)]$ is defined by (5.26) with t_i in the place of m_i.

$$J_{k-1}^k = i_M :\ M_{k-1} \longrightarrow M_k, \text{the natural injection operator.} \tag{5.33}$$

Clearly, (A.2) holds. By repeating the proof of Lemma 5.2 we can show (A.1) holds. Then by a similar argument as the one for Example 8 except replacing $\underset{\sim}{\sigma}', \tau'$ in (5.37) by

$$\underset{\sim}{\sigma}' = \underset{\sim}{\sigma}^{II}, \quad \tau' = \Pi'_{k-1}\tau,$$

where $\underset{\sim}{\sigma}^{II}$ denotes the conforming quadratic interpolation of $\underset{\sim}{\sigma}$ with respect to $\mathcal{T}_{k-1}$, we show that (A.3) is valid. Thus, the abstract result applies to Fortin–Soulie nonconforming element.

Theorem IV. *For the nonconforming mixed elements for the Stokes problems, namely, Crouzeix–Raviart element and Fortin–Soulie element, Theorem I holds with I_{k-1}^k being defined by (5.25) and (5.27) and (5.32) and (5.33).*

5.4. Multigrid Methods with Nonnested Meshes

In this section we will show that our abstract framework also applies to some other nonnested mixed elements where the nonnestedness is caused by using nonnested meshes, which is either due to artificial arrangement or to the nature of the element in question. For early results on multigrid methods with nonnested meshes for second order elliptic equations, we refer to [17] and the reference therein.

Example 10. *Composite $P_1 - P_0$ element with nonnested meshes*

We first consider the composite $P_1 - P_0$ in Example 2, where the multigrid method is studied in the case that the mesh family is nested. In the following we will show that our abstract framework also applies to this element in the case that the mesh family is not nested. We remark that for the composite $P_1 - P_0$ element, the use of the nonnested meshes is not required by the element. For simplicity, we adopt the nonnested mesh refining techniques of [17] in this subsection.

Recall that for the triangulation $\mathcal{T}_k$ $(k \geq 0)$, the finite element spaces corresponding to the composite $P_1 - P_0$ element are defined by

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (P_1(K))^2,\ K \in \mathcal{T}_k^{\frac{1}{2}}\}. \tag{5.34}$$

$$M_k = \{q \in L_0^2(\Omega),\ q|_K \in P_0(K),\ K \in \mathcal{T}_k\}, \tag{5.35}$$

where $\mathcal{T}_k^{\frac{1}{2}}$ is obtained by connecting the midpoints of the three sides of all triangles of $\mathcal{T}_k$.

Here we do not assume the nestedness of the mesh, i.e., $\mathcal{T}_{k-1}$ may not be a subpartition of $\mathcal{T}_k$. Thus we have

$$\underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad M_{k-1} \not\subset M_k, \quad k \geq 0.$$

That is, the multilevel finite spaces are not nested for both velocity field and pressure field. To ensure convergence of the multigrid algorithm, in addition to the assumption (A.0), we also impose the following restrictions on the meshes which were suggested in [17]:

$$\sup_{K \in \mathcal{T}_k} \{\text{cardinality}(\{K' \in \mathcal{T}_{k-1} |\ \overline{K'} \cap \overline{K} \neq \emptyset\})\} \leq \beta_0, \quad k \geq 1, \tag{5.36}$$

$$\sup_{K \in \mathcal{T}_k} \{\text{cardinality}(\{K' \in \mathcal{T}_{k+1} |\ \overline{K'} \cap \overline{K} \neq \emptyset\})\} \leq \beta_0, \quad k \geq 0, \tag{5.37}$$

$$\alpha_1^{-1} h_k \leq h_{k+1} < h_k, \quad k \geq 0, \tag{5.38}$$

$$\alpha_2 N_k \leq N_{k+1}, \quad \alpha_2 N_k' \leq N_{k+1}', \quad k \geq 0, \tag{5.39}$$

for some constants $\beta_0 \geq 1$, $\alpha_1 > 1$ and $\alpha_2 > 2$. Where

$$N_k = \dim(\underset{\sim}{X}_k) = O(h_k^{-2}), \quad N_k' = \dim(M_k) = O(h_k^{-2}).$$

Obviously, for any nested mesh family, $\beta_0 = 1$ or 4, $\alpha_1 = 2$ and $\alpha_2 = 4$.

Next, we define the prolongation operator

$$I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k]: \ \underset{\sim}{X}_{k-1} \times M_{k-1} \longrightarrow \underset{\sim}{X}_k \times M_k$$

as follows:

$$H_{k-1}^k = [E_k, E_k] : \; X_{k-1} \longrightarrow X_k,$$

where E_k is the standard interpolation operator of the conforming linear element with respect to the triangulation $\mathcal{T}_k^{\frac{1}{2}}$, i.e.,

$$(E_k v)(x) = \sum_{p_i \in N_k^{\frac{1}{2}}} v(p_i)\psi_{k,i}(x), \qquad \forall v \in C_0(\overline{\Omega}), \tag{5.40}$$

where $N_k^{\frac{1}{2}}$ is the set of vertices of the triangulation $\mathcal{T}_k^{\frac{1}{2}}$ and $\psi_{k,i}(x)$ is the nodal basis function of the conforming linear element at the vertex p_i of the triangulation $\mathcal{T}_k^{\frac{1}{2}}$.

$J_{k-1}^k : \; M_{k-1} \longrightarrow M_k$ such that for any $q \in M_{k-1}$, $J_{k-1}^k q$ satisfies

$$J_{k-1}^k q = \frac{1}{|K|}\int_K q dx, \quad \forall K \in \mathcal{T}_k. \tag{5.41}$$

Clearly, H_{k-1}^k and J_{k-1}^k are well–defined. Moreover, we have

Lemma 5.3. *Suppose $\{\mathcal{T}_k\}$ $(k \geq 0)$ satisfies (1.6) and (5.36)–(5.39), then the operators H_{k-1}^k and J_{k-1}^k defined above satisfy the assumptions (A.1) and (A.2) respectively, namely,*

$$\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{v}\|_{L^2(\Omega)} \leq C h_k \|\underset{\sim}{v}\|_{k-1}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}. \tag{5.42}$$

$$\|J_{k-1}^k q\|_{L^2(\Omega)} \leq C\|q\|_{L^2(\Omega)}, \quad \forall q \in M_{k-1}. \tag{5.43}$$

Proof. To show (5.42), it suffices to show that

$$\|v_j - E_k v_j\|_{L^2(\Omega)} \leq C h_k \|v_j\|_{H^1(\Omega)}, \quad j = 1,2; \quad \forall \underset{\sim}{v} = (v_1, v_2) \in \underset{\sim}{X}_{k-1}. \tag{5.44}$$

By Proposition 2.2 of [17] we know that (5.44) does hold. Thus, so does (5.42). Finally, (5.43) can be easily checked by direct computations and making use of (5.35), (5.41) and Cauchy–Schwarz inequality.

Lemma 5.4. *Under the same assumptions of Lemma 5.3, we have*

$$\|\underset{\sim}{w} - H_{k-1}^k \underset{\sim}{v}\|_{L^2(\Omega)} \leq \rho_0 \|\underset{\sim}{w} - \underset{\sim}{v}\|_{L^2(\Omega)}, \quad \forall \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}, \; \underset{\sim}{w} \in \underset{\sim}{X}_k, \tag{5.45}$$

$$\|r - J_{k-1}^k q\|_{L^2(\Omega)} \leq \|r - q\|_{L^2(\Omega)}, \quad \forall q \in M_{k-1}, \; r \in M_k, \tag{5.46}$$

for some positive number ρ_0 which is independent of $\underset{\sim}{w}$, $\underset{\sim}{v}$ and k.

Proof. Since (5.46) can be obtained by direct computations and making use of (5.35), (5.41) and Cauchy–Schwarz inequality, in the following we only provide a proof for (5.5).

To show (5.45), it suffices to show the following inequalities:

$$\begin{aligned} \|w_j - E_k v_j\|_{L^2(\Omega)} \leq & \rho_0 \|w_j - v_j\|_{L^2(\Omega)}, \quad j = 1,2; \\ & \forall \underset{\sim}{v} = (v_1, v_2) \in \underset{\sim}{X}_{k-1}, \; \forall \underset{\sim}{w} = (w_1, w_2) \in \underset{\sim}{X}_k. \end{aligned} \tag{5.47}$$

We now show (5.47) for $j = 1$. For any $K \in \mathcal{T}_k^{\frac{1}{2}}$, let

$$S_K = \cup\{\, \overline{K'} \mid K' \in \mathcal{T}_{k-1}^{\frac{1}{2}}, \; \overline{K'} \cap \overline{K} \neq \emptyset\}.$$

Thus, (5.36) implies that there would be no more than $4\beta_0$ triangles in S_K. Noticing that $w_1 - E_k v_1$ is a linear function on K, we have

$$\begin{aligned}\|w_1 - E_k v_1\|^2_{L^2(\Omega)} =& \frac{|K|}{12}\{[(w_1 - E_k v_1)(p_1) + (w_1 - E_k v_1)(p_2)]^2 \\ &+[(w_1 - E_k v_1)(p_2) + (w_1 - E_k v_1)(p_3)]^2 \\ &+[(w_1 - E_k v_1)(p_3) + (w_1 - E_k v_1)(p_1)]^2\} \\ \le& Ch_k^2 \sum_{j=1}^{3}[(w_1 - E_k v_1)(p_j) + (w_1 - E_k v_1)(p_{j+1})]^2,\end{aligned} \tag{5.48}$$

where $p_j,\ j = 1, 2, 3$ are the vertices of K, p_4 and p_1 stand for the same vertex. Also noticing that (5.36) and (5.37) imply

$$\sup_{K\in\mathcal{T}_k^{\frac{1}{2}}} \{\text{cardinality}(\{K' \in \mathcal{T}_{k-1}^{\frac{1}{2}} \mid \overline{K'} \cap \overline{K} \neq \emptyset\})\} \le 4\beta_0, \quad k \ge 0, \tag{5.49}$$

$$\sup_{K\in\mathcal{T}_k^{\frac{1}{2}}} \{\text{cardinality}(\{K' \in \mathcal{T}_{k+1}^{\frac{1}{2}} \mid \overline{K'} \cap \overline{K} \neq \emptyset\})\} \le 4\beta_0, \quad k \ge 1, \tag{5.50}$$

by an argument similar to the one used in [17] to derive (2.2) of the reference, we can show the following estimate holds:

$$[(w_1 - E_k v_1)(p_j) + (w_1 - E_k v_1)(p_{j+1})]^2 \le C\|\underset{\sim}{w} - \underset{\sim}{v}\|^2_{L^2(S_K)}, \quad j = 1, 2, 3. \tag{5.51}$$

Finally, combining (5.48)–(5.51) we get

$$\begin{aligned}\|w_1 - E_k v_1\|^2_{L^2(\Omega)} &\le \sum_{K\in\mathcal{T}_k^{\frac{1}{2}}} \|w_1 - E_k v_1\|^2_{L^2(K)} \le C \sum_{K\in\mathcal{T}_k^{\frac{1}{2}}} \|w_1 - v_1\|^2_{L^2(S_K)} \\ &\le 4\beta_0 \sum_{K'\in\mathcal{T}_{k-1}^{\frac{1}{2}}} \|w_1 - v_1\|^2_{L^2(K')} \le \rho_0 \|w_1 - v_1\|^2_{L^2(\Omega)},\end{aligned}$$

where ρ_0 is a positive constant which is independent of $w_1,\ v_1,\ h_{k-1}$ and h_k.

So we have shown (5.44) holds for $j = 1$. Similarly, we can show (5.44) is also true for $j = 2$. Therefore, the proof of Lemma 5.4 is completed.

By using Lemma 5.4, we can easily show the assumption (A.3) holds. In fact, by (5.45), (5.46), (1.11), (5.38) and the triangle inequality, we have

$$\begin{aligned}&|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \\ &\quad\le \max(\rho_0, 1)|||[\underset{\sim}{\sigma}_k, \tau_k] - [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \\ &\quad\le C\{|||[\underset{\sim}{\sigma}_k, \tau_k] - [\underset{\sim}{\sigma}, \tau]|||_{0,k} + |||[\underset{\sim}{\sigma}, \tau] - [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k-1}\} \\ &\quad\le Ch_k^2[\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}],\end{aligned}$$

where $[\underset{\sim}{\sigma}, \tau]$ and $[\underset{\sim}{\sigma}_j, \tau_j]$ $(j = k-1, k)$ are the solutions of (1.3) and (1.7) with $g \equiv 0$, respectively.

Therefore, we conclude

Theorem V. *For the composite $P_1 - P_0$ element with nonnested mesh refinings, under the assumptions (A.0) and (5.36)–(5.39), Theorem I holds with I_{k-1}^k being defined by (5.40) and (5.41).*

Example 11. *A stable $P_2 - P_1$ element*

Let $\{\widehat{\mathcal{T}}_k\}_{k\geq 0}$ be a family of nested quasi–uniform triangulations of Ω. Let $\{\mathcal{T}_k\}_{k\geq 0}$ be the family of barycentric trisected triangulations which is subordinated to $\{\widehat{\mathcal{T}}_k\}_{k\geq 0}$ in the sense of that each $\mathcal{T}_k$ is obtained by connecting the three vertices of every triangle of $\widehat{\mathcal{T}}_k$ to its barycenter. It is easy to see that $\{\mathcal{T}_k\}_{k\geq 0}$ is a family of nonnested meshes which satisfies (A.0). We remark that to use the nonnested meshes $\{\mathcal{T}_k\}_{k\geq 0}$ is to guarantee the stability of the following $P_2 - P_1$ element.

Let $\underset{\sim}{X}_k$ and M_k be defined as follows:

$$\underset{\sim}{X}_k = \{\underset{\sim}{v} \in (C_0(\overline{\Omega}))^2,\ \underset{\sim}{v}|_K \in (P_2(K))^2,\ \forall\ K \in \mathcal{T}_k\}. \tag{5.52}$$

$$M_k = \{p \in L_0^2(\Omega),\ p|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_k\}. \tag{5.53}$$

It is shown in [2] that this is a stable element for the Stokes problems, i.e., $\underset{\sim}{X}_k$ and M_k satisfies the Babuška–Brezzi condition (1.10), and the error estimate (1.11) holds. Since the family $\{\mathcal{T}_k\}_{k\geq 0}$ is nonnested, we have

$$\underset{\sim}{X}_{k-1} \not\subset \underset{\sim}{X}_k, \quad M_{k-1} \not\subset M_k, \quad k \geq 1.$$

To apply our abstract framework, we define the prolongation operator

$$I_{k-1}^k = [H_{k-1}^k, J_{k-1}^k] :\ \underset{\sim}{X}_{k-1} \times M_{k-1} \longrightarrow \underset{\sim}{X}_k \times M_k$$

as follows:

$$H_{k-1}^k = \widehat{\Pi}_k \times \widehat{\Pi}_k \text{ or } \Pi_k \times \Pi_k :\ \underset{\sim}{X}_{k-1} \longrightarrow \underset{\sim}{X}_k, \tag{5.54}$$

where $\widehat{\Pi}_k$ and Π_k stand for the standard conforming linear or quadratic finite element interpolation operators associated with $\widehat{\mathcal{T}}_k$ and $\mathcal{T}_k$, respectively.

$$J_{k-1}^k :\ M_{k-1} \longrightarrow M_k \text{ such that for any } K \in \mathcal{T}_k,$$

$$\int_K J_{k-1}^k q dx = \int_K q dx, \quad \forall\ q \in M_{k-1},\ K \in \mathcal{T}_k. \tag{5.55}$$

The following example shows that such J_{k-1}^k does exist:

$$J_{k-1}^k q|_K(x) = \begin{cases} q|_K(x), & \text{if } x \in K \subset K' \in \mathcal{T}_{k-1}, \\ \frac{1}{|K|}\int_K q dx, & \text{otherwise.} \end{cases} \tag{5.56}$$

Next, we are going to show that the prolongation operator I_{k-1}^k defined above satisfies the assumptions (A.1)–(A.3).

Lemma 5.5. *H_{k-1}^k defined by (5.54) satisfies (A.1), i.e.,*

$$\|\underset{\sim}{v} - H_{k-1}^k \underset{\sim}{v}\|_{L^2(\Omega)} \leq C h_k \|\underset{\sim}{v}\|_{k-1}, \quad \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}. \tag{5.57}$$

Proof. We only give a proof for $H_{k-1}^k = \widehat{\Pi}_k \times \widehat{\Pi}_k$, a proof for $H_{k-1}^k = \Pi_k \times \Pi_k$ can be written up similarly.

Noticing that for any $K \in \widehat{\mathcal{T}}_{k-1}$,

$$H_K \equiv H_{k-1}^k|_K = \widehat{\Pi}_k \times \widehat{\Pi}_k|_K :\ \underset{\sim}{X}_{k-1}|_K \longrightarrow \underset{\sim}{X}_k|_K$$

is a linear operator a finite dimensional space to another finite dimensional space and satisfies

$$H_K \underset{\sim}{p}_0 = \underset{\sim}{p}_0, \quad \forall\ \underset{\sim}{p}_0 \in (P_0(K))^2.$$

On the other hand, for any $\underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_K$, if $\underset{\sim}{v}|_{K_i} \in (P_0(K_i))^2,\ i=1,2,3$, then $\underset{\sim}{v} \in (P_0(K))^2$, where $\overline{K} = \cup_{i=1}^3 \overline{K_i},\ K_i \in \mathcal{T}_{k-1}$. This implies that $\|\nabla \underset{\sim}{v}\|_{L^2(K)}$ is a norm on $\underset{\sim}{X}_{k-1}|_K/(P_0(K))^2$. Thus, on the reference element $\widehat{K}$, we have

$$\|\underset{\sim}{v} - H_{\widehat{K}}\underset{\sim}{v}\|_{L^2(\widehat{K})} \le C\|\nabla \underset{\sim}{v}\|_{L^2(\widehat{K})}, \quad \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_{\widehat{K}}.$$

Now by the standard scaling argument we conclude that

$$\|\underset{\sim}{v} - H_K\underset{\sim}{v}\|_{L^2(K)} \le Ch_k\|\nabla \underset{\sim}{v}\|_{L^2(K)}, \quad \forall\ \underset{\sim}{v} \in \underset{\sim}{X}_{k-1}|_K. \tag{5.58}$$

Finally, (5.57) follows from summing up (5.58) over all $K \in \widehat{\mathcal{T}}_{k-1}$.

By an argument similar to the one in the proof of Lemma 5.3, we can show the following lemma holds.

Lemma 5.6. *J_{k-1}^k defined by (5.56) satisfies (A.2), i.e.,*

$$\|J_{k-1}^k q\|_{L^2(\Omega)} \le C\|q\|_{L^2(\Omega)}, \quad \forall q \in M_{k-1}. \tag{5.59}$$

To check (A.3), let $[\underset{\sim}{\sigma}_j, \tau_j]$ $(j = k-1, k)$, $[\underset{\sim}{\sigma}, \tau] \in (H^2(\Omega)\cap H_0^1(\Omega))^2 \times (H^1(\Omega)\cap L_0^2(\Omega))$ be the solutions of (1.3) and (1.7) with $g \equiv 0$, respectively. Then by (3.1), (3.3) and (1.11) we get

$$|||[\underset{\sim}{\sigma}, \tau] - [\underset{\sim}{\sigma}_j, \tau_j]|||_{0,j} \le Ch_j^2(\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}), \quad j = k-1, k. \tag{5.60}$$

Let $\widehat{\underset{\sim}{\sigma}}$ be the linear conforming finite element interpolation of $\underset{\sim}{\sigma}$ and $\widehat{\tau}$ be the piecewise constant interpolation of τ both associated with $\widehat{\mathcal{T}}_{k-1}$. Then $[\widehat{\underset{\sim}{\sigma}}, \widehat{\tau}] \in (\underset{\sim}{X}_{k-1}\times M_{k-1})\cap(\underset{\sim}{X}_k \times M_k)$ and

$$H_{k-1}^k \widehat{\underset{\sim}{\sigma}} = \widehat{\underset{\sim}{\sigma}}, \quad J_{k-1}^k\widehat{\tau} = \widehat{\tau}. \tag{5.61}$$

Finally, from (5.60), (5.61), (5.57) and (5.59) we have

$$\begin{aligned}
&|||[\underset{\sim}{\sigma}_k, \tau_k] - I_{k-1}^k[\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k} \\
&\quad \le |||[\underset{\sim}{\sigma}_k, \tau_k] - [\widehat{\underset{\sim}{\sigma}}, \widehat{\tau}]|||_{0,k} + |||I_{k-1}^k([\widehat{\underset{\sim}{\sigma}}, \widehat{\tau}] - [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}])|||_{0,k} \\
&\quad \le Ch_k^2(\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}) + C|||[\widehat{\underset{\sim}{\sigma}}, \widehat{\tau}] - [\underset{\sim}{\sigma}_{k-1}, \tau_{k-1}]|||_{0,k-1} \\
&\quad \le Ch_k^2(\|\underset{\sim}{\sigma}\|_{H^2(\Omega)} + \|\tau\|_{H^1(\Omega)}).
\end{aligned}$$

Thus, (A.3) is verified. Therefore we have

Theorem VI. *For $P_2 - P_1$ element with the barycentric trisected triangulations, Theorem I holds with I_{k-1}^k being defined by (5.54) and (5.56).*

Remark 5.4. Similarly, we can show that the abstract result of Sections 1–4 applies to other stable low order mixed finite elements for the Stokes problems (cf. [2]).

Remark 5.5. If we regard the above $P_2 - P_1$ element as a composite element built on the triangulations $\widehat{\mathcal{T}}_k,\ k \ge 0$, then the multilevel finite element spaces for both the velocity and the pressure are still nonnested, but the meshes become nested. Hence, this element also belongs to the class of elements in subsection 5.2 and can be treated accordingly.

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