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ON THE CONVERGENCE OF PROJECTOR-SPLINES FOR THE NUMERICAL EVALUATION OF CERTAIN TWO-DIMENSIONAL CPV INTEGRALS*1)

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Abstract

In this paper, product formulas based on projector-splines for the numerical evaluation of 2-D CPV integrals are proposed. Convergence results are proved, numerical examples and comparisons are given.

Key words: 2-D Cauchy principal value integral, Tensor product, projector-splines.

1. Introduction

We consider the numerical evaluation of Cauchy principal value integrals of the form

$$J(f;z,\vartheta)=\int_a^b\int_{\tilde a}^{\tilde b} w_1(x)w_2(\tilde x)\frac{f(x,\tilde x)}{(x-z)(\tilde x-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde x \tag{1.1}$$

where $z\in(a,b)$, $\vartheta\in(\tilde a,\tilde b)$, the weight functions $w_1(x)$, $w_2(\tilde x)$ and the function f are such that $J(f;z,\vartheta)$ exists.

The numerical evaluation of the integrals (1.1) are of two types: global and local. The global methods have generally to be used when f is differentiable with 'small' derivatives. However, one of the difficulties wich occur in the use of global methods usually based on orthogonal polynomials, lies in the fact that a greater accuracy in approximating (1.1) requires to increase the number of the nodes coinciding with the zeros of above polynomials. Therefore, when the weight functions w_1, w_2 are different from the classical Jacobi weights, the evaluation of the nodes requires a considerable computational effort.

Besides, global methods are generally not appropriate when f behave 'badly' in some subinterval of $[a,b]\times[\tilde a\times\tilde b]$, then for such integrals a local method with no restriction on the choice of the nodes would have to be preferred.

In this paper we will consider an approximation function of the form:

$$Q_{N\tilde N}f(x,\tilde x)=\sum_{i=1-k}^{N-1}\sum_{\tilde\imath=1-k}^{\tilde N-1}(\lambda_{i\tilde\imath}\tilde\lambda_{i\tilde\imath}f)B_{i\tilde\imath k}(x,\tilde x) \tag{1.2}$$

in which the operators $\lambda_{i\tilde\imath}$, $\tilde\lambda_{i\tilde\imath}$ are such that $Q_{N\tilde N}$ is the tensor product of two one-dimensional projector-splines and we will examine a cubature rule for (1.1), considering that it can be written in the form

$$\begin{aligned}J(f;z,\vartheta) &= \int_a^b\int_{\tilde a}^{\tilde b} w_1(x)w_2(\tilde x)\frac{f(x,\tilde x)-f(z,\vartheta)}{(x-z)(\tilde x-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde x\,+\\ &+\; f(z,\vartheta)\int_a^b\frac{w_1(x)}{x-z}\,\mathrm{d}x\int_{\tilde a}^{\tilde b}\frac{w_2(\tilde x)}{\tilde x-\vartheta}\,\mathrm{d}\tilde x,\end{aligned} \tag{1.3}$$

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and then, it can be approximated by

$$J_{N\tilde{N}}(f;z,\vartheta) = \int_a^b\!\!\int_{\tilde{a}}^{\tilde{b}} w_1(x)w_2(\tilde{x})\frac{Q_{N\tilde{N}}f(x,\tilde{x}) - Q_{N\tilde{N}}f(z,\vartheta)}{(x-z)(\tilde{x}-\vartheta)}\,\mathrm{d}x\,\mathrm{d}\tilde{x} + f(z,\vartheta)\int_a^b \frac{w_1(x)}{x-z}\,\mathrm{d}x \int_{\tilde{a}}^{\tilde{b}} \frac{w_2(\tilde{x})}{\tilde{x}-\vartheta}\,\mathrm{d}\tilde{x}. \quad (1.4)$$

This paper is organized as follows. In Section 2 we will present some preliminaries and summarize numerical thecniques to be used; in Section 3 we will prove the convergence of the integration rules here proposed and we give conditions for their uniform convergence for (ζ,ϑ) belonging to any closed interval contained in $(a,b)\times(\tilde{a},\tilde{b})$. Finally, in Section 4, some numerical results are presented and compared with those obtained by using the method proposed in [2].

2. Preliminaries

Given $\Omega := [a,b]\times[\tilde{a},\tilde{b}]$, let $\{Y_n\}$ and $\{\tilde{Y}_{\tilde{n}}\}$ be two sequences of partitions of $I := [a,b]$ and $\tilde{I} := [\tilde{a},\tilde{b}]$ respectively:

$$Y_n := \{a = y_{0n} < y_{1n} < ... < y_{nn} = b\}, \quad \tilde{Y}_{\tilde{n}} := \{\tilde{a} = \tilde{y}_{0\tilde{n}} < \tilde{y}_{1\tilde{n}} < ... < \tilde{y}_{\tilde{n}\tilde{n}} = \tilde{b}\}.$$

If $h_i = y_{i+1} - y_i$ and $\tilde{h}_{\tilde{\imath}} = \tilde{y}_{\tilde{\imath}+1} - \tilde{y}_{\tilde{\imath}}$, we define

$$\delta_1 = \min_{1\le i\le n} h_{i-1}, \quad \delta_2 = \min_{1\le\tilde{\imath}\le\tilde{n}} \tilde{h}_{\tilde{\imath}-1}. \quad (2.1)$$

Let $\overline{\Delta}_1$, $\overline{\Delta}_2$ be the norms of the partitions Y_n and $\tilde{Y}_{\tilde{n}}$ respectively, given by

$$\overline{\Delta}_1 = \max_{1\le i\le n} h_{i-1}, \quad \overline{\Delta}_2 = \max_{1\le\tilde{\imath}\le\tilde{n}} \tilde{h}_{\tilde{\imath}-1}. \quad (2.2)$$

We say that the collection of partitions $\{Y_n \times \tilde{Y}_{\tilde{n}} : n = n_1, n_2 ...; \tilde{n} = \tilde{n}_1, \tilde{n}_2, ...\}$ of Ω, is quasi-uniform $(q.u.)$ if there exists a positive constant A such that

$$\frac{\overline{\Delta}_i}{\delta_j} \le A, \quad 1 \le i, j \le 2 \quad (2.3)$$

and we assume that

$$\overline{\Delta}_1 \to 0 \quad \text{as} \quad n\to\infty, \quad \overline{\Delta}_2 \to 0 \quad \text{as} \quad \tilde{n}\to\infty. \quad (2.4)$$

Let $\{d_{in}\}_1^{n-1}$, $\{\tilde{d}_{\tilde{\imath}\tilde{n}}\}_1^{\tilde{n}-1}$ be two sequences of positive integers with $d_{in} \le k-1$, $\tilde{d}_{\tilde{\imath}\tilde{n}} \le \tilde{k}-1$, where $k, \tilde{k}$ are assigned integers greater than 1, and let π be the non-decreasing sequence $\{x_i\}_0^N$ obtained from Y_n by repeating y_{in} exactly d_i times (thus $N = \sum_i^{n-1} d_i + 1$); similarly, let $\tilde{\pi}$ be the non-decreasing sequence $\{\tilde{x}_{\tilde{\imath}}\}_0^{\tilde{N}}$ obtained from $\tilde{Y}_{\tilde{n}}$ (thus $\tilde{N} = \sum_{\tilde{\imath}}^{\tilde{n}-1} \tilde{d}_{\tilde{\imath}} + 1$). We denote with $S_{\pi k}$ and $\tilde{S}_{\tilde{\pi}\tilde{k}}$ the polynomial spline spaces of order k and $\tilde{k}$ respectively. We shall call a sequence of spline spaces $\{S_{\pi k} \times \tilde{S}_{\tilde{\pi}\tilde{k}}\}$ $q.u.$ if they are based on a sequence of $q.u.$ partitions.

We can suppose, without loss of generality, $k = \tilde{k}$.

It is well known that considering the extended partitions $\pi_e = \{x_i\}_{i=1-k}^{N+k-1}$ and $\widetilde{\pi}_e = \{\widetilde{x}_i\}_{\widetilde{i}=1-k}^{\widetilde{N}+k-1}$, the normalized B-splines $\{B_{ik}(x)\}_{i=1-k}^{N-1}$ and $\{\tilde{B}_{\tilde{\imath}k}(\tilde{x})\}_{\tilde{\imath}=1-k}^{\tilde{N}-1}$ constitue a basis compactly supported for $S_{\pi k}$ and $\tilde{S}_{\tilde{\pi}k}$ respectively. By the above univariate normalized B-splines we may generate a collection of bivariate B-splines, defined on $[x_{1-k}, x_{N+k-1}]\times[\tilde{x}_{1-k}, \tilde{x}_{\tilde{N}+k-1}]$,

$$B_{i\tilde{\imath}k}(x,\tilde{x}) = B_{ik}(x)\tilde{B}_{\tilde{\imath}k}(\tilde{x}).$$

Let F be a linear space of real valued functions on Ω. Then, for any $f \in F$, we may define the approximation operator

$$Q_{N\tilde{N}} f(x,\tilde{x}) = \sum_{i=1-k}^{N-1} \sum_{\tilde{\imath}=1-k}^{\tilde{N}-1} (\lambda_{i\tilde{\imath}} \tilde{\lambda}_{i\tilde{\imath}} f) B_{i\tilde{\imath}k}(x,\tilde{x}). \tag{2.5}$$

Considering that $\lambda_{i\tilde{\imath}} \tilde{\lambda}_{i\tilde{\imath}} B_{i\tilde{\imath}k} = (\lambda_{i\tilde{\imath}} B_{ik})(\tilde{\lambda}_{i\tilde{\imath}} \tilde{B}_{\tilde{\imath}k})$, we have:

Proposition 2.1. *Suppose that* $\{\lambda_{i\tilde{\imath}}\}_{1-k}^{N-1}$, $\{\tilde{\lambda}_{i\tilde{\imath}}\}_{1-k}^{\tilde{N}-1}$ *are linear functionals that constitue a dual basis for* $\{B_{ik}\}_{1-k}^{N-1}$ *and* $\{\tilde{B}_{\tilde{\imath}k}\}_{1-k}^{\tilde{N}-1}$ *respectively, then*

$$\xi_{i\tilde{\imath}} = \lambda_{i\tilde{\imath}} \tilde{\lambda}_{i\tilde{\imath}} \quad i = 1-k, ..., N-1, \quad \tilde{\imath} = 1-k, ..., \tilde{N}-1$$

is a dual base for the B-splines tensor product $\{B_{i\tilde{\imath}k}\}_{i=1-k,\ \tilde{\imath}=1-k}^{N-1,\ \tilde{N}-1}$.

If $\{\lambda_{i\tilde{\imath}}\}$ and $\{\tilde{\lambda}_{i\tilde{\imath}}\}$ are dual basis for $\{B_{ik}\}$ and $\{\tilde{B}_{\tilde{\imath}k}\}$ respectively, the operator defined in (2.5) is a tensor product of two one-dimensional projector-splines, then

Proposition 2.2. *Let* $\xi_{i\tilde{\imath}} = \lambda_{i\tilde{\imath}} \tilde{\lambda}_{i\tilde{\imath}}$ *be a dual basis for* $\{B_{i\tilde{\imath}k}\}_{i=1-k,\tilde{\imath}=1-k}^{N-1,\ \tilde{N}-1}$. *Then the operator* $Q_{N\tilde{N}}$ *defined in (2.5) is a projector i.e.:*

$$Q_{N\tilde{N}} s = s, \quad \forall s \in S_{\pi k} \times \tilde{S}_{\tilde{\pi} k}. \tag{2.6}$$

We will choose

$$\lambda_{i\tilde{\imath}} = \sum_{j=1}^{k} \alpha_{i\tilde{\imath}j} \lambda_{i\tilde{\imath}j}, \qquad \tilde{\lambda}_{i\tilde{\imath}} = \sum_{\tilde{\jmath}=1}^{k} \tilde{\alpha}_{i\tilde{\imath}\tilde{\jmath}} \tilde{\lambda}_{i\tilde{\imath}\tilde{\jmath}} \tag{2.7}$$

where: $\alpha_{i\tilde{\imath}1} = 1$, $\alpha_{i\tilde{\imath}r} = \frac{(k-r)!}{(k-1)!} \sum \prod_{\ell=1}^{r-1} (x_{\nu_\ell} - \tau_{i_\ell})$ $\quad r = 2, \ldots, k$, (the sum above is taken over all choices of distinct $\nu_1, ..., \nu_{r-1}$ from $1, ..., i+k-1$), the $\alpha_{i\tilde{\imath}\tilde{\jmath}}$ $\ \tilde{\jmath} = 1, ..., k$ are similarly defined. Denoting by $[\tau_{i\tilde{\imath}1}, \tau_{i\tilde{\imath}2}, ..., \tau_{i\tilde{\imath}j}]$ the $(j-1)th-$order divided difference functional, we assume

$$\lambda_{i\tilde{\imath}j} f = [\tau_{i\tilde{\imath}1}, \tau_{i\tilde{\imath}2}, ..., \tau_{i\tilde{\imath}j}] f, \quad \tilde{\lambda}_{i\tilde{\imath}\tilde{\jmath}} f = [\tilde{\tau}_{i\tilde{\imath}1}, \tilde{\tau}_{i\tilde{\imath}2}, ..., \tilde{\tau}_{i\tilde{\imath}\tilde{\jmath}}] f\,. \tag{2.8}$$

We give now a sufficient condition to assure that $Q_{N\tilde{N}} f$ is a projector.

Theorem 2.1. *For* $i = 1-k, ..., N-1$, $\tilde{\imath} = 1-k, ..., \tilde{N}-1$, *let* $\{\tau_{i\tilde{\imath}j}\}_{j=1}^{k}$, $\{\tilde{\tau}_{i\tilde{\imath}\tilde{\jmath}}\}_{\tilde{\jmath}=1}^{k}$ *belong to the subinterval* $[x_{\nu_i}, x_{\nu_{i+1}}] \subset [x_i, x_{i+k}]$ *and* $[\tilde{x}_{\tilde{\nu}_{\tilde{\imath}}}, \tilde{x}_{\tilde{\nu}_{\tilde{\imath}}+1}] \subset [\tilde{x}_{\tilde{\imath}}, \tilde{x}_{\tilde{\imath}+k}]$ *respectively. Then* $Q_{N\tilde{N}} f$ *is a projector.*

Proof. The proof is based on the definition of $\{\lambda_{i\tilde{\imath}}\}_{i=1-k}^{N-1}$ and $\{\tilde{\lambda}_{i\tilde{\imath}}\}_{\tilde{\imath}=1-k}^{\tilde{N}-1}$, on the Propositions 2.1, 2.2 and the results in [4].

The subinterval $[x_{\nu_i}, x_{\nu_{i+1}}][\tilde{x}_{\tilde{\nu}_{\tilde{\imath}}}, \tilde{x}_{\tilde{\nu}_{\tilde{\imath}}+1}]$ considered in Theorem 2.1 can be selected following [3].

We assume that the space sequences $\{S_{\pi k} \times \tilde{S}_{\tilde{\pi} k}\}$ are $q.u.$ For fixed $(t, \tilde{t}) \in \Omega$, let m, $\tilde{m}$, $0 \le m \le N-1$, $0 \le \tilde{m} \le \tilde{N}-1$ be such that $x_m \le t < x_{m+1}$, $\tilde{x}_{\tilde{m}} \le \tilde{t} < \tilde{x}_{\tilde{m}+1}$.

Let $U_{m\tilde{m}} = [x_{m-k+1}, x_{m+k-1}] \times [\tilde{x}_{\tilde{m}-k+1}, \tilde{x}_{\tilde{m}+k-1}]$, we denote

$$\Delta_{m\tilde{m}} = \Delta_m + \tilde{\Delta}_{\tilde{m}} \tag{2.9}$$

with $\Delta_m = \max_{m+1-k\le j\le m+k-1}(x_{j+1}-x_j)$, $\quad 0\le m\le N-1$ (similar definition for $\tilde{\Delta}_{\tilde m}$), and we define $\delta_{m,k-r} = \min_{m+1-k+r\le j\le m}(x_{j+k-r}-x_j)$, $\quad 0\le m\le N-1$ (similarly for $\tilde{\delta}_{\tilde m,k-\tilde r}$).

For any integer ℓ, $1\le \ell\le k$, we assume

$$\rho_m = \max_{m+1-k\le i\le m}\frac{x_{i+k}-x_i}{\sigma_{i\tilde{\imath}\ell}},\quad \tilde{\rho}_{\tilde m} = \max_{\tilde m+1-k\le \tilde{\imath}\le \tilde m}\frac{\tilde{x}_{\tilde{\imath}+k}-\tilde{x}_{\tilde{\imath}}}{\tilde{\sigma}_{i\tilde{\imath}\ell}}$$

where $\sigma_{i\tilde{\imath}\ell} = \min_{1\le j\le m}\sigma_{i\tilde{\imath}j\ell}$, $\sigma_{i\tilde{\imath}j\nu} = \min_{1\le\mu\le j-\nu}[\tau^{(j)}_{i\tilde{\imath},\mu+\nu}-\tau^{(j)}_{i\tilde{\imath}\mu}]$, with $\{\tau^{(j)}_{i\tilde{\imath}1},\dots,\tau^{(j)}_{i\tilde{\imath}j}\}$ the non decreasing rearrangement of $\{\tau_{i\tilde{\imath}1},\dots,\tau_{i\tilde{\imath}j}\}$In the same way we define $\tilde{\sigma}_{i\tilde{\imath}\ell}$.

For any $h>0$ and any region Θ, denoting by $D^{v,p-v}\varphi = \dfrac{\partial^p\varphi}{\partial x^v\partial\tilde{x}^{p-v}}$ and

$$\omega(\psi;h;\Theta) = \sup_{\substack{(x,\tilde x),(x+\theta,\tilde x+\tilde\theta)\in\Theta\\ |\theta|,|\tilde\theta|\le h}} |\psi(x+\theta,\tilde{x}+\tilde{\theta})-\psi(x,\tilde{x})|,$$

we define: $\omega(D^p\varphi;h;\Theta) = \max_{0\le v\le p}\omega(D^{v,p-v}\varphi;h;\Theta)$.

Suppose $f\in C^p(U_{m\tilde m})$, $0\le p\le k-1$. From Theorem 9.2 in [4] we have the following

Theorem 2.2. *Let $0\le p\le k-1$, $f\in C^p(U_{m\tilde m})$. Denoting $H_{m\tilde m} = [x_m,x_{m+1})\times[\tilde{x}_{\tilde m},\tilde{x}_{\tilde m+1})$, $\quad 0\le r+\tilde r\le p$, then:*

$$\max_{(t,\tilde t)\in H_{m\tilde m}}|D^{r,\tilde r}(f-Q_{N\tilde N}f)(t,\tilde t)| \le K_{m\tilde m}\Delta^{p-r-\tilde r}_{m\tilde m}\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m}) \tag{2.10}$$

$$\max_{(t,\tilde t)\in H_{m\tilde m}}|D^{r+1,\tilde r}Q_{N\tilde N}f(t,\tilde t)| \le K_{m\tilde m}\Delta^{p-r-\tilde r-1}_{m\tilde m}\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m}) \tag{2.11}$$

$$\max_{(t,\tilde t)\in H_{m\tilde m}}|D^{r,\tilde r+1}Q_{N\tilde N}f(t,\tilde t)| \le K_{m\tilde m}\Delta^{p-r-\tilde r-1}_{m\tilde m}\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m}) \tag{2.12}$$

where $K_{m\tilde m}$ is a constant depending on k, m, $\tilde m$, p, ρ_m, $\tilde\rho_{\tilde m}$ and $\Delta_{m\tilde m}/\delta_{m,k-r}$, $\Delta_{m\tilde m}/\tilde\delta_{\tilde m,k-\tilde r}$.

Since the spline spaces are *q.u.*, under suitable choices [3] of the nodes $\{\tau_{i\tilde{\imath}j}\}_{j=1}^k$ and $\{\tilde\tau_{i\tilde{\imath}\tilde{\jmath}}\}_{\tilde{\jmath}=1}^k$, the quantities ρ_m, $\tilde\rho_{\tilde m}$, $\Delta_{m\tilde m}/\delta_{m,k-r}$, $\Delta_{m\tilde m}/\tilde\delta_{\tilde m,k-\tilde r}$ are uniformly bounded for all m, $\tilde m$ and for all N, $\tilde N$.

Let $\Delta = \overline{\Delta}_1+\overline{\Delta}_2$ and the nodes $\{\tau_{i\tilde{\imath}j}\}$, $\{\tilde\tau_{i\tilde{\imath}\tilde{\jmath}}\}$ such that Theorem 2.2 holds with $K_{m\tilde m}$ dependent only on k, m, $\tilde m$, p.

From the above local estimate (2.10) a global estimate can be deduced [4]:

Theorem 2.3. *Let $f\in C^p(\Omega)$, $0\le p\le k-1$, then*

$$\|f-Q_{N\tilde N}f\|_\infty \le C\Delta^p\omega(D^pf;\Delta;\Omega). \tag{2.13}$$

Theorem 2.4. *Let $f\in C^p(\Omega)$, $0\le p<k-1$ and consider any sequence of q.u. spline spaces $\{S_{\pi k}\times\tilde S_{\tilde\pi k}\}$. If any $S\in S_{\pi k}\times\tilde S_{\tilde\pi k}$ satisfies for $0\le r+\tilde r\le p$*

- $S\in C^p(\Omega)$,
- $|D^{r,\tilde r}(f-S)(t,\tilde t)|\le C_1\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m})$, $\quad (t,\tilde t)\in H_{m\tilde m}$,
- $|D^{r+1,\tilde r}S(t,\tilde t)|\le C_2\Delta^{-1}_{m\tilde m}\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m})$, $\quad (t,\tilde t)\in H_{m\tilde m}$,
- $|D^{r,\tilde r+1}S(t,\tilde t)|\le C_2\Delta^{-1}_{m\tilde m}\omega(D^pf;\Delta_{m\tilde m};U_{m\tilde m})$, $\quad (t,\tilde t)\in H_{m\tilde m}$,

then, [2]

$$\omega(D^pS,\Delta,\Omega)\le C_3\omega(D^pf,\Delta,\Omega) \tag{2.14}$$

holds.

We say $f \in H_p(\mu,\mu)$, if f is a continuous function with all partial derivatives of f of order $j = 0, 1, ..., p$, $p \geq 0$, continuous and each derivative of order p satisfying a Hölder condition of order $0 < \mu \leq 1$.

Since the spline operator reproducing splines and defined in (2.5) satisfies the hypotheses of Theorem 2.4, we can prove the following

Theorem 2.5. *Suppose $f \in H_p(\mu,\mu)$ in Ω, $0 \leq p < k-1$ then, for any q.u. projector-spline space $\{Q_{N\tilde{N}} f\}$ we have*

$$\|f - Q_{N\tilde{N}} f\|_\infty \leq C\Delta^{p+\mu}, \tag{2.15}$$

$$|Q_{N\tilde{N}} f(x,\tilde{x}) - Q_{N\tilde{N}} f(x,\tilde{x}_0)| \leq C_1 |\tilde{x} - \tilde{x}_0|^{\overline{\mu}} \tag{2.16}$$

and

$$|Q_{N\tilde{N}} f(x,\tilde{x}) - Q_{N\tilde{N}} f(x_0,\tilde{x})| = C_2 |x - x_0|^{\overline{\mu}} \tag{2.17}$$

where $\overline{\mu} = \mu$ if $p = 0$, $\overline{\mu} = 1$ if $p > 0$ and the constants C, C_1, C_2 are independent of $x, \tilde{x}$.

Proof. Formula (2.15) is a consequence of (2.13) and of the hypothesis $f \in H_p(\mu,\mu)$. If $p = 0$, by Theorem 2.4 we can deduce (2.16) and (2.17) with $\overline{\mu} = \mu$.

Suppose now $p > 0$. We can write

$$|Q_{N\tilde{N}} f(x,\tilde{x}) - Q_{N\tilde{N}} f(x,\tilde{x}_0)| = |Q^{(0,1)}_{N\tilde{N}} f(x,\eta)(\tilde{x} - \tilde{x}_0)| \leq \overline{C} |\tilde{x} - \tilde{x}_0|,$$

$\eta \in (\tilde{x}, \tilde{x}_0)$ and (2.16) holds with $\overline{\mu} = 1$. The same for proving (2.17).

3. Convergence of Quadrature Rules Using Projector-Splines

In this section we consider the numerical evaluation of (1.1) by

$$J(f; z, \vartheta) = J_{N\tilde{N}}(f; z, \vartheta) + E_{N\tilde{N}}(f; z, \vartheta) \tag{3.1}$$

where $J_{N\tilde{N}}(f; z, \vartheta)$ is defined in (1.4).

We assume

i) $f \in H_p(\mu,\mu)$ in Ω, $0 \leq p < k-1$,

ii) $w_1 \in L_1[a,b] \cap DT(N_{\overline{\delta}}(z))$, $w_2 \in L_1[\tilde{a},\tilde{b}] \cap DT(N_{\overline{\delta}}(\vartheta))$, where DT is the set of Dini type functions i.e., denoting by $\ell(\overline{A})$ the length of interval $\overline{A}$,

$$DT(\overline{A}) = \left\{ g \in C^0(\overline{A}) : \int_0^{\ell(\overline{A})} \omega(g; u) u^{-1} \mathrm{d}u < +\infty \right\},$$

and $N_{\overline{\delta}}(\lambda) := [\lambda - \overline{\delta}, \lambda + \overline{\delta}]$, $\overline{\delta} > 0$. In order to study the convergence of cubatures (3.1) we need some definitions and Lemmas. We define

$$r_{N\tilde{N}}(x,\tilde{x}) = f(x,\tilde{x}) - Q_{N\tilde{N}} f(x,\tilde{x}), \tag{3.2}$$

$$S_{N\tilde{N}}(x) = \int_{\tilde{a}}^{\tilde{b}} w_2(\tilde{x}) \frac{r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(x,\vartheta)}{\tilde{x} - \vartheta} \mathrm{d}\tilde{x}, \tag{3.3}$$

$$T_{N\tilde{N}}(\tilde{x}) = \int_a^b w_1(x) \frac{r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(z,\tilde{x})}{x - z} \mathrm{d}x. \tag{3.4}$$

Following [5] and [6], we have:

Lemma 3.1. *Let $f \in H_p(\mu,\mu)$, $0 \le p < k-1$, $0 < \mu \le 1$. For any sequence of q.u. projector-spline space $\{Q_{N\tilde{N}}f\}$ and for any $0 < \nu < \mu_1 = \min(p+\mu, 1)$, we have:*

$$\frac{|r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(x,\tilde{x}_0)|}{|\tilde{x}-\tilde{x}_0|^\nu} \le L_1 \Delta^{p+\mu-\delta}, \quad \frac{|r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(x_0,\tilde{x})|}{|x-x_0|^\nu} \le L_2 \Delta^{p+\mu-\delta}$$

where $\tilde{x} \ne \tilde{x}_0$ in the first inequality, $x \ne x_0$ in the second, and $\delta = \nu \frac{p+\mu}{\mu_1}$.

Lemma 3.2. *Let $f \in H_p(\mu,\mu)$, $0 \le p < k-1$, $0 < \mu \le 1$. For any q.u. projector-spline space $\{Q_{N\tilde{N}}f\}$ and for any $0 < \nu < \mu_1 = \min(p+\mu, 1)$, then:*

$$|S_{N\tilde{N}}(x)| \le \overline{C}_1 \Delta^{p+\mu-\delta}, \quad |S_{N\tilde{N}}(x) - S_{N\tilde{N}}(z)| \le \overline{L}_1 |x-z|^{\mu_1-\nu}, \tag{3.5}$$
$$|T_{N\tilde{N}}(\tilde{x})| \le \overline{C}_2 \Delta^{p+\mu-\delta}, \quad |T_{N\tilde{N}}(\tilde{x}) - T_{N\tilde{N}}(\vartheta)| \le \overline{L}_2 |\tilde{x}-\vartheta|^{\mu_1-\nu} \tag{3.6}$$

where $\delta = \nu \frac{p+\mu}{\mu_1}$.

We state now the following theorem that gives an error bound.

Theorem 3.1. *Let $\Delta = \overline{\Delta}_1 + \overline{\Delta}_2$ and $f \in H_p(\mu,\mu)$ in Ω, $0 \le p < k-1$. Assume that $\{Q_{N\tilde{N}}f\}$ is a q.u. sequence of projector-spline spaces. Then*

$$|E_{N\tilde{N}}(f;z;\vartheta)| \le C\Delta^{p+\mu-\gamma} \tag{3.7}$$

where γ is a real number with $0 < \gamma < \mu_1$, as small as we like.

Proof. The approximation $J_{N\tilde{N}}(f;z,\vartheta)$ is a tensor product of two formulas of the form

$$⨍_a^b w(x)\frac{f(x)}{x-u}dx = \sum_{i=1-k}^{N-1} (\lambda_i f)\mu_i(u) + f(u) ⨍_a^b \frac{w(x)}{x-u}dx + R(f) \tag{3.8}$$

applied considering firstly the function f for a fixed value of the variable $\tilde{x}$ and operating as a function of x, and then considering f for a fixed x and affecting the variable $\tilde{x}$. It is straightforward to notice that the remainder terms $R_1(f;\tilde{x})$ and $R_2(f;x)$ in the above formulas concide with $T_{N\tilde{N}}(\tilde{x})$ and $S_{N\tilde{N}}(x)$ respectively.

Therefore, following [9], we can write the remainder term of (3.1) in the form

$$\begin{aligned} E_{N\tilde{N}}(f;z,\vartheta) &= ⨍_a^b \frac{w_1(x)}{x-z}(R_2(f;x))\mathrm{d}x + \\ &+ ⨍_{\tilde{a}}^{\tilde{b}} \frac{w_2(\tilde{x})}{\tilde{x}-\vartheta}(R_1(f;\tilde{x}))\mathrm{d}\tilde{x} - R_2(R_1(f;x);\tilde{x}), \end{aligned} \tag{3.9}$$

and then

$$\begin{aligned} E_{N\tilde{N}}(f;z,\vartheta) &= \int_a^b w_1(x)\frac{S_{N\tilde{N}}(x) - S_{N\tilde{N}}(z)}{x-z}\mathrm{d}x + \\ &+ W_2(\vartheta)T_{N\tilde{N}}(\vartheta) + W_1(z)S_{N\tilde{N}}(z) \end{aligned} \tag{3.10}$$

where $W_1(z) = ⨍_a^b \frac{w_1(x)}{x-z}\,\mathrm{d}x$, $W_2(\vartheta) = ⨍_{\tilde{a}}^{\tilde{b}} \frac{w_2(\tilde{x})}{\tilde{x}-\vartheta}\,\mathrm{d}\tilde{x}$.

By the properties (3.5), for $0 < \varepsilon < \mu_1 - \nu$, we have:

$$\frac{|S_{N\tilde{N}}(x) - S_{N\tilde{N}}(z)|}{|x-z|^\varepsilon} \le \overline{L}_3 \Delta^{p+\mu-\delta_1} \tag{3.11}$$

where $\delta_1 = \delta + (p+\mu-\delta)(1 - \frac{\varepsilon}{\mu_1 - \nu})$ and $\overline{L}_3$ is a real constant.

From (3.5), (3.6), (3.11) and the hypotheses (ii), the thesis follows with $\gamma = \delta_1$.

Theorem 3.2. *Suppose that* $w_1 \in L_1[a,b] \cap C(a,b)$, $w_2 \in L_1[\tilde{a},\tilde{b}] \cap C(\tilde{a},\tilde{b})$ *and* $f \in H_p(\mu,\mu)$ *in* Ω. *Assume that* $\{Q_{N\tilde{N}}f\}$ *is a q.u. sequence of projector-spline spaces. If* D, $\tilde{D}$ *are any closed intervals in* $\mathring{I}$, $\mathring{\tilde{I}}$ *respectively, then*

$$E_{N\tilde{N}}(f;z,\vartheta) \to 0 \quad as \quad N \to \infty,\ \tilde{N} \to \infty, \quad uniformly\ in\ (z,\vartheta) \in D \times \tilde{D}. \tag{3.12}$$

Proof. In the above hypotheses, as $N \to \infty$, $\tilde{N} \to \infty$, we have that

$$S_{N\tilde{N}}(x) \to 0 \text{ uniformly } \forall\, \vartheta \in \mathring{\tilde{I}},\ \ T_{N\tilde{N}}(\tilde{x}) \to 0 \text{ uniformly } \forall\, z \in \mathring{I}, \tag{3.13}$$

$$\int_a^b w_1(x) \frac{S_{N\tilde{N}}(x) - S_{N\tilde{N}}(z)}{x-z} \mathrm{d}x \to 0 \text{ uniformly } \forall (z,\vartheta) \in \mathring{I} \times \mathring{\tilde{I}}. \tag{3.14}$$

In fact, by (3.2), since $|r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(x,\vartheta)| \le B|\tilde{x}-\vartheta|^{\mu_1}$ and w_2 is bounded in $N_{\overline{\delta}}(\vartheta)$, we have

$$\left| \int_{N_{\overline{\delta}}(\vartheta)} w_2(\tilde{x}) \frac{r_{N\tilde{N}}(x,\tilde{x}) - r_{N\tilde{N}}(x,\vartheta)}{\tilde{x}-\vartheta} \mathrm{d}\tilde{x} \right| \le \overline{c} \int_{N_{\overline{\delta}}(\vartheta)} |\tilde{x}-\vartheta|^{\mu_1 - 1} \mathrm{d}\tilde{x} < \varepsilon$$

if $\overline{\delta} < \overline{\delta}_1(\varepsilon)$. Hence $|S_{N\tilde{N}}(x)| \le \varepsilon + 2\frac{\|r_{N\tilde{N}}\|}{\overline{\delta}} \int_{\tilde{a}}^{\tilde{b}} w_2(\tilde{x})\mathrm{d}\tilde{x}$ and the first limit in (3.13) holds. The same for the other limits.

If $z \in D$ and $\vartheta \in \tilde{D}$, $W_1(z)$ and $W_2(\vartheta)$ are finite and continuous as function of z and ϑ respectively, then (3.12) holds.

4. Numerical Results

We consider now the evaluation of (1.1) by (1.4) for some integrand and weight functions.

In the tables 1-3 below, we denote: $E^{(1)}_{N\tilde{N}}$ the absolute error obtained by using the cubature rules here presented, $E^{(2)}_{N\tilde{N}}$ the absolute error obtained by using the cubature rules considered in [2], n, $\tilde{n}$ the knots number of the partitions Y_n and $\tilde{Y}_{\tilde{n}}$.

We performed our examples considering uniform partitions on $[a,b] := [-1,1]$, $[\tilde{a},\tilde{b}] := [-1,1]$, using simple knots and choosing

$$\tau_{i\tilde{i}j} = \begin{cases} x_{\nu_i} + j\dfrac{x_{\nu_i+1} - x_{\nu_i}}{k} & \text{if } \nu_i \ne N-1, \quad j = 1,2,...,k \\ x_{N-1} + j\dfrac{x_N - x_{N-1}}{k+1} & \text{if } \nu_i = N-1 \end{cases}$$

and, in similar way, $\tilde{\tau}_{i\tilde{i}\tilde{j}}$.

Table 1

$w_1(x) = 1,\quad w_2(\tilde{x}) = \left(1-\tilde{x}^2\right)^{-\frac{1}{2}},\quad f(x,\tilde{x}) = \left(25-x^2\right)^{-\frac{1}{2}}\left(25-\tilde{x}^2\right)^{-1}$							
$J(f;z;\vartheta)$	z	ϑ	k	n	$\tilde{n}$	$E^{(1)}_{N\tilde{N}}$	$E^{(2)}_{N\tilde{N}}$
0.0001232465	0.25	0.25	4	3	3	$8.91\ 10^{-6}$	$2.68\ 10^{-6}$
				10	10	$9.69\ 10^{-10}$	$4.57\ 10^{-8}$
				20	20	$8.92\ 10^{-11}$	$1.80\ 10^{-9}$

Table 2

$w_1(x) = \left(1-x^2\right)^{-\frac{1}{2}},\ w_2(\tilde{x}) = \left(1-\tilde{x}^2\right)^{-\frac{1}{2}},\ f(x,\tilde{x}) = \left(x^2+10^{-2}\right)^{-1}\left(25+\tilde{x}^2\right)^{-1}$							
$J(f;z;\vartheta)$	z	ϑ	k	n	$\tilde{n}$	$E^{(1)}_{N\tilde{N}}$	$E^{(2)}_{N\tilde{N}}$
0.5061494369	0.25	0.99	4	30	10	$5.46\ 10^{-4}$	$9.56\ 10^{-3}$
				60	20	$3.29\ 10^{-6}$	$2.27\ 10^{-4}$
				90	30	$1.72\ 10^{-7}$	$3.57\ 10^{-5}$

Table 3

$w_1(x)=1,\quad w_2(\widetilde{x})=1,\quad f(x,\widetilde{x})=\sin(x+\widetilde{x})$							
$J(f;z;\vartheta)$	z	ϑ	k	n	$\tilde{n}$	$E^{(1)}_{N\,\widetilde{N}}$	$E^{(2)}_{N\,\widetilde{N}}$
				10	10	$1.01\ 10^{-5}$	$6.49\ 10^{-5}$
-3.717033709	0.25	0.6	5	20	20	$5.27\ 10^{-8}$	$7.59\ 10^{-7}$
				30	30	$3.12\ 10^{-10}$	$4.83\ 10^{-8}$

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