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A NEW STABILIZED FINITE ELEMENT METHOD FOR SOLVING THE ADVECTION–DIFFUSION EQUATIONS*[1)]

Huo-yuan Duan
(*Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This paper is devoted to the development of a new stabilized finite element method for solving the advection–diffusion equations having the form $-\kappa\,\Delta\,u+\underline{a}\bullet\underline{\nabla}\,u+\sigma\,u=f$ with a zero Dirichlet boundary condition. We show that this methodology is coercive and has a uniformly optimal convergence result for all mesh–Peclet number.

Key words: Advection–diffusion equation, Stabilized finite element method.

1. Introduction

Consider the advection–diffusion equation

$$\begin{cases} -\kappa\,\Delta\,u+\underline{a}\bullet\underline{\nabla}\,u+\sigma\,u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded polygonal domain $\Omega\subset I\!R^2$ with the boundary $\partial\Omega$, where $0<\kappa\ll 1$ is the diffusion parameter, $\sigma>0$ is a given positive constant, $\underline{a}(x)$ is a given vector field representing the flow with $\underline{\nabla}\bullet\underline{a}=0$ in Ω, and $f\in L^2(\Omega)$ is a given source function. The term $\sigma\,u$ is usually obtained by the time discretization of the nonstationary advection–diffusion equation arising from mathematical and engineering problems, so the item σ takes it form as $1/\triangle\,t$ with $\triangle\,t<1$ being the time step. Generally speaking, σ is comparatively large, and when $\triangle\,t$ or κ tends to zero, a boundary layer region may be present near the boundary.

It is now well known that the standard Galerkin method solving (1.1) often causes a bad numerical solution when the balance among the three parameters σ,κ and $\underline{a}$ is losing. The goal of stabilized finite element methods established in recent decade, e.g. see [5] [8] [10], etc., is to seek for some good approximating solutions of (1.1) on which the effects emanating from the disturbance among σ,κ and $\underline{a}$ can be cut down as much as possible. In [4] [5] [7] [8] [11], some stablized finite element methods with an additional mesh–dependent perturbation bilinear term were proposed, therein a good approximating result was obtained. [5] studied a stabilized method based on local bubble functions for (1.1) with $\sigma=0$ or $\underline{a}=0$, which deduced an optimal error estimation including higher order elements, independent of σ and κ for the case with $\underline{a}=0$, and independent of mesh–Peclet number for the case with $\sigma=0$.

In [1], a bubble–enriching method for advection–diffusion problem without the zeroth order term $\sigma\,u$ is analyzed in details. However, the bubble–enriching method is not fit for the advection dominated case. For this reason, some special local bubble functions are needed, e.g. see [2] [10], e.t., which are usually very difficult to construct. As for the fact that the bubble–enriching method often deduces a stabilized method associated with problem (1.1), it may be also referred to [11] for a heuristic observation.

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Now we consider a general model (1.1). Firstly by defining a stabilizing parameter and by adding a suitable mesh–dependent bilinear form, we design a finite element approximation for (1.1). Next, the coerciveness of the new formulation is shown, and finally the optimal error estimates for all mesh–Peclet number are obtained, including the L^2–norm, and the higher order elements for triangle and quadrilateral partitions of the domain. If introducing a mesh–Peclet number, it can be seen that our results may result in those of [5] [8].

The rest of this paper is outlined as follows. In section 2, the stabilized finite element formulation for (1.1) is described and the coerciveness of this method is investigated. The section 3 is devoted to a general error analysis. In the last section, every case of (1.1) is discussed, according to σ, κ and $\underline{a}$. Sharp error estimates are obtained.

In what follows, for simplicity we shall use C (or $C_i, i = 1, 2, \cdots$) to stand for different constant at different occurence, and they are all independent of $\sigma, \kappa, \underline{a}$ and the mesh size h.

2. Problem Fzormulation

For convenience, we rewrite (1.1)

$$\begin{cases} -\kappa\,\Delta\, u + \underline{a} \bullet \underline{\nabla}\, u + \sigma\, u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

The standard Galerkin variational problem is to find $u \in H_0^1(\Omega)$ such that

$$B(u, v) = (f, v)_0 \quad \forall\, v \in H_0^1(\Omega) \tag{2.2}$$

where

$$B(u, v) = (\sigma\, u, v)_0 + (\underline{a} \bullet \underline{\nabla}\, u, v)_0 + (\kappa \underline{\nabla}\, u, \underline{\nabla}\, v)_0 \tag{2.3}$$

The discrete version of problem (2.2) consists of finding $u_h \in U_h \subset H_0^1(\Omega)$ such that

$$B(u_h, v) = (f, v)_0 \quad \forall\, v \in U_h \tag{2.4}$$

where

$$U_h = \{ v \in H_0^1(\Omega) \cap C(\Omega) \,|\, v_K \in R_m(K), K \in \mathcal{E}_h \} \tag{2.5}$$

with $R_m(K) = P_m(K)$ or $Q_m(K)$ corresponding to the partition being triangle or quadrilateral, and $m \geq 1$, P_m, Q_m are the usual finite element subspaces depicted in [3]. $\mathcal{E}_h$ is the regular partition of the domain Ω, which is supposed to be a polygonal bounded region as usual. Also, $C(\Omega)$ is the space of continuous functions in Ω, and $H_0^1(\Omega)$ is the Hilbert space of functions, taking their values as zero along the boundary $\partial\Omega$, which, together with their first–order derivatives, are square–integrable.

For each $K \in \mathcal{E}_h$, define

$$\tau_{K,\alpha} = \frac{h_K^2}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \tag{2.6}$$

with $\alpha > 0$ to be determined later. Here h_K is the element parameter for $K \in \mathcal{E}_h$, and

$$[\underline{a}]_K = \sup_{x \in K} |\underline{a}(x)|_p \tag{2.7}$$

with $|\underline{a}(x)|_p = (\sum_{i=1,2} |a_i(x)|^p)^{\frac{1}{p}}$ for $1 \leq p \leq \infty$ and $|\underline{a}(x)|_\infty = \max_{i=1,2} |a_i(x)|$ for $p = \infty$ being a norm in the Euclidean two dimensional space IR^2. Now, a stabilizing bilinear form is introduced as follows.

$$T(u, v) = -\alpha \sum_{K \in \mathcal{E}_h} \tau_{K,\alpha}\, (\sigma\, u + \underline{a} \bullet \underline{\nabla}\, u - \kappa\, \Delta\, u, \sigma\, v - \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \tag{2.8}$$

and accordingly a linear form is introduced as follows:

$$R(v) = -\alpha \sum_{K \in \mathcal{E}_h} \tau_{K,\alpha}\,(f, \sigma\, v - \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \tag{2.9}$$

So we are able to formulate our stabilized finite element method which is to find $u_h \in U_h$ such that

$$B(u_h, v) + T(u_h, v) = (f, v)_0 + R(v) \quad \forall\, v \in U_h \tag{2.10}$$

In the following sections, for $s = 0, 1, 2$, $(\cdot,\cdot)_{s,D}$ and $|\cdot|_{s,D}$ are repsectively denoted the inner product and the associated norm on $H^s(D)$, and for $D = \Omega$ the subscript D is omitted. And some times the notation $|\underline{\nabla}\, v|_{0,D}$ and $(\underline{\nabla}\, v, \underline{\nabla}\, v)_{0,D}$ are also used only for $H^1(D)$.

With a view to giving the coerciveness of (2.10), a mesh–dependent norm is introduced by

$$\kappa\, |||v|||_h^2 = \sum_{K \in \mathcal{E}_h} \frac{\sigma\, |v|_{0,K}^2\,(\kappa + h_K\,[\underline{a}]_K)}{\alpha\,\sigma\, h_k^2 + \kappa + h_K\,[\underline{a}]_K} + \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2}{\alpha\,\sigma\, h_k^2 + \kappa + h_K\,[\underline{a}]_K} + \kappa\, |\underline{\nabla}\, v|_0^2 \tag{2.11}$$

Theorem 2.1. *There exists a constant $C > 0$ such that*

$$B(v, v) + T(v, v) \geq C\,\kappa\, |||v|||_h^2 \quad \forall v \in U_h \tag{2.12}$$

where C does not depend on $\sigma, \kappa, \underline{a}$ and h.

Proof. On the one hand, using the Green formulae and the assumption $\underline{\nabla} \bullet \underline{a} = 0$, we have $(\underline{a} \bullet \underline{\nabla}\, v, v)_0 = 0$ holds for all $v \in U_h$, since $v = 0$ on $\partial\Omega$. So,

$$B(v, v) = \sigma\, |v|_0^2 + \kappa\, |\underline{\nabla}\, v|_0^2 \tag{2.13}$$

On the other hand,

$$\begin{aligned} T(v, v) &= -\alpha \sum_{K \in \mathcal{E}_h} \tau_{K,\alpha}\,(\sigma\, v + \underline{a} \bullet \underline{\nabla}\, v - \kappa\,\Delta\, v, \sigma\, v - \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \\ &= \sum_{K \in \mathcal{E}_h} \frac{-\alpha\,\sigma^2\, h_K^2\, |v|_{0,K}^2 + \alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2 + w_K(v)}{\alpha\,\sigma\, h_K^2 + \kappa + h_K\,[\underline{a}]_K} \end{aligned} \tag{2.14}$$

with

$$w_K(v) = \alpha\,\kappa\,\sigma\, h_K^2\,(\Delta\, v, v)_{0,K} - \alpha\,\kappa\, h_K^2\,(\Delta\, v, \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \tag{2.15}$$

Hence,

$$B(v, v) + T(v, v) = \sum_{K \in \mathcal{E}_h} \frac{g_K(v) + w_K(v) + \alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2}{\alpha\,\sigma\, h_K^2 + \kappa + h_K\,[\underline{a}]_K} \tag{2.16}$$

with

$$g_K(v) = \sigma\, |v|_{0,K}^2\,(\kappa + h_K\,[\underline{a}]_K) + \kappa\, |v|_{1,K}^2\,(\alpha\,\sigma\, h_K^2 + \kappa + h_K\,[\underline{a}]_K) \tag{2.17}$$

Note that $h_K^2\, |\Delta\, v|_{0,K}^2 \leq C_I\, |v|_{1,K}^2$ where C_I is a constant independent of K, h, and the Young's inquality, i.e., for any two real number a, b $|ab| \leq \varepsilon\, a^2 + \frac{b^2}{4\,\varepsilon}$, for some small positive constant ε,

$$\alpha\,\kappa\,\sigma\, h_K^2\,(\Delta\, v, v)_{0,K} \geq -\alpha\,\kappa\,\sigma\,\sqrt{C_I}\,(\varepsilon\, |v|_{0,K}^2 + \frac{1}{4\,\varepsilon}\, h_K^2\, |v|_{1,K}^2) \tag{2.18}$$

$$-\alpha\,\kappa\, h_K^2\,(\Delta\, v, \underline{a} \bullet \underline{\nabla}\, v)_{0,K} \geq -\alpha\,\sqrt{C_I}\,(\varepsilon\,\kappa^2\, |v|_{1,K}^2 + \frac{1}{4\,\varepsilon}\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2) \tag{2.19}$$

and thus

$$\left\{\begin{array}{l} g_K(v) + w_K(v) + \alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2 \\ \geq (1 - \varepsilon\, \alpha\, \sqrt{C_I})\, \kappa\, (\sigma\, |v|_{0,K}^2 + \kappa\, |v|_{1,K}^2) \\ +(1 - \dfrac{\sqrt{C_I}}{4\,\varepsilon})\, \alpha\, h_K^2\, (|\underline{a} \bullet \underline{\nabla}\, v|_{0,K}^2 + \kappa\, \sigma\, |v|_{1,K}^2) \\ +h_K\, [\underline{a}]_K\, (\sigma\, |v|_{0,K}^2 + \kappa\, |v|_{1,K}^2) \end{array}\right. \tag{2.20}$$

with the following choices

$$0 < \alpha < \frac{4}{C_I}, \qquad \frac{\sqrt{C_I}}{4} < \varepsilon < \frac{1}{\alpha\, \sqrt{C_I}} \tag{2.21}$$

Therefore, let

$$C_1 = 1 - \varepsilon\, \alpha\, \sqrt{C_I}, \quad C_2 = 1 - \frac{\sqrt{C_I}}{4\,\varepsilon}$$

and let $C = \min(C_1, C_2)$, and the proof is completed with (2.12).

3. Error Analysis

In this section, we give a general estimates for the difference between the approximate solution $u_h \in U_h$ and the exact solution $u \in H_0^1(\Omega)$. Firstly, from (2.8) and (2.9) it follows that the usual consistency or error orthogonality is satisfied, that is

$$B(u - u_h, v) + T(u - u_h, v) = 0 \quad \forall v \in U_h \tag{3.1}$$

Next, let $\eta = u - I_h\, u, \quad e_h = u_h - I_h\, u$. Here $I_h\, u$ is the standard interpolation function belonging to U_h of $u \in H^{m+1}(\Omega) \cap H_0^1(\Omega)$ and has the property

$$|\eta|_{s,K} \leq C\, h_K^{m+1-s}\, |u|_{m+1,K} \quad (0 \leq s \leq m+1) \tag{3.2}$$

Note that (3.1) and (2.12), we have

$$C\, \kappa\, |||e_h|||_h^2 \leq B(e_h, e_h) + T(e_h, e_h) = B(\eta, e_h) + T(\eta, e_h) \tag{3.3}$$

In the following paragraph, $B(\eta, e_h)$ and $T(\eta, e_h)$ are respectively analyzed. During the analysis, the Young's inquality is being used repeatedly, so we no longer mention it later. In addition, it should be noticed that $\alpha\, \sigma^2\, h_K^2\, (\eta, e_h)_{0,K}$ is eliminated. In orther words,

$$\left\{\begin{array}{l} B(\eta, e_h) + T(\eta, e_h) \\ = (\underline{a} \bullet \underline{\nabla}\, \eta, e_h)_0 + \kappa\, (\underline{\nabla}\, \eta, \underline{\nabla}\, e_h)_0 \\ + \displaystyle\sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, \sigma\, (\eta, \underline{a} \bullet \underline{\nabla}\, e_h)_{0,K} + \sigma\, (\eta, e_h)_{0,K}\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ + \displaystyle\sum_{K \in \mathcal{E}_h} \frac{-\alpha\, h_K^2\, (\underline{a} \bullet \underline{\nabla}\, \eta - \kappa\, \Delta\, \eta, \sigma\, e_h - \underline{a} \bullet \underline{\nabla}\, e_h)_{0,K}}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \end{array}\right. \tag{3.4}$$

For convenience, we denote by $E_i(\eta, e_h), (1 \leq i \leq 3)$ the three items appearing in the righthand–side of (3.4), i.e.,

$$E_1(\eta, e_h) = (\underline{a} \bullet \underline{\nabla}\, \eta, e_h)_0 + \kappa\, (\underline{\nabla}\, \eta, \underline{\nabla}\, e_h)_0$$

$$E_2(\eta, e_h) = \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, \sigma\, (\eta, \underline{a} \bullet \underline{\nabla}\, e_h)_{0,K} + \sigma\, (\eta, e_h)_{0,K}\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K}$$

$$E_3(\eta, e_h) = \sum_{K \in \mathcal{E}_h} \frac{-\alpha\, h_K^2\, (\underline{a} \bullet \underline{\nabla}\, \eta - \kappa\, \Delta\, \eta, \sigma\, e_h - \underline{a} \bullet \underline{\nabla}\, e_h)_{0,K}}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K}$$

So, for some small constant $\varepsilon > 0$, first,

$$\left\{\begin{aligned} & E_1(\eta, e_h) = -(\underline{a} \bullet \underline{\nabla}\, e_h, \eta)_0 + \kappa\, (\underline{\nabla}\, \eta, \underline{\nabla}\, e_h)_0 \\ & \le \varepsilon \left(\kappa\, |\underline{\nabla}\, e_h|_0^2 + \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, e_h|_{0,K}^2}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \right) \\ & + \frac{1}{4\,\varepsilon} \max(1, \alpha^{-1}) \left(\kappa\, |\underline{\nabla}\, \eta|_0^2 + \sum_{K \in \mathcal{E}_h} h_K^{-2}\, |\eta|_{0,K}^2\, (\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K) \right) \end{aligned}\right. \tag{3.5}$$

Next,

$$\left\{\begin{aligned} & E_2(\eta, e_h) \\ & \le \varepsilon \max(1, \alpha^{-1}) \left(\sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, e_h|_{0,K}^2}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \right. \\ & \qquad\qquad \left. + \sum_{K \in \mathcal{E}_h} \frac{\sigma\, |e_h|_{0,K}^2\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \right) \\ & + \frac{1}{2\,\varepsilon} \sum_{K \in \mathcal{E}_h} \alpha\, \sigma\, h_K^2\, h_K^{-2}\, |\eta|_{0,K}^2 \end{aligned}\right. \tag{3.6}$$

Thirdly, noting that

$$\left\{\begin{aligned} & \sum_{K \in \mathcal{E}_h} \frac{-\alpha\, h_K^2\, (\underline{a} \bullet \underline{\nabla}\, \eta, \sigma\, e_h)_{0,K}}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & \le \varepsilon \sum_{K \in \mathcal{E}_h} \frac{\sigma\, |e_h|_{0,K}^2\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & + \frac{1}{4\,\varepsilon} \sum_{K \in \mathcal{E}_h} \frac{\alpha^2\, \sigma\, h_K^4\, |\underline{a} \bullet \underline{\nabla}\, \eta|_{0,K}^2}{(\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K)\, (\kappa + h_K\, [\underline{a}]_K)} \\ & \le \varepsilon \sum_{K \in \mathcal{E}_h} \frac{\sigma\, |e_h|_{0,K}^2\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} + \frac{1}{4\,\varepsilon}\, \alpha\, C_3 \sum_{K \in \mathcal{E}_h} h_K\, [\underline{a}_K]\, |\underline{\nabla}\, \eta|_{0,K}^2 \end{aligned}\right. \tag{3.7}$$

Here we have used the following inquality by the equivalence of norms on IR^2

$$|\underline{a} \bullet \underline{\nabla}\, \eta|_{0,K}^2 \le C_3\, [\underline{a}]_K^2\, |\underline{\nabla}\, \eta|_{0,K}^2$$

and similarly

$$\left\{\begin{aligned} & \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, (\kappa\, \Delta\, \eta, \sigma\, e_h)_{0,K}}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & \le \varepsilon \sum_{K \in \mathcal{E}_h} \frac{\sigma\, |e_h|_{0,K}^2\, (\kappa + h_K\, [\underline{a}]_K)}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & + \frac{1}{4\,\varepsilon}\, \alpha \sum_{K \in \mathcal{E}_h} \kappa\, h_K^2\, |\Delta\, \eta|_{0,K}^2 \end{aligned}\right. \tag{3.8}$$

and in addition it can be easily seen that

$$\left\{\begin{aligned} & \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, (\underline{a} \bullet \underline{\nabla}\, \eta - \kappa\, \Delta\, \eta, \underline{a} \bullet \underline{\nabla}\, e_h)_{0,K}}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & \le \varepsilon \sum_{K \in \mathcal{E}_h} \frac{\alpha\, h_K^2\, |\underline{a} \bullet \underline{\nabla}\, e_h|_{0,K}^2}{\alpha\, \sigma\, h_K^2 + \kappa + h_K\, [\underline{a}]_K} \\ & + \frac{1}{2\,\varepsilon}\, \alpha \max(C_3, 1) \sum_{K \in \mathcal{E}_h} h_K\, [\underline{a}]_K\, |\underline{\nabla}\, \eta|_{0,K}^2 + \kappa\, h_K^2\, |\Delta\, \eta|_{0,K}^2 \end{aligned}\right. \tag{3.9}$$

thus by combining (3.7)∼(3.9),

$$\left\{\begin{array}{l} E_3(\eta, e_h) \\ \leq 2\,\varepsilon \left(\sum_{K\in\mathcal{E}_h} \dfrac{\sigma\,|e_h|^2_{0,K}\,(\kappa + h_K\,[\underline{a}]_K)}{\alpha\,\sigma\,h_k^2 + \kappa + h_K\,[\underline{a}]_K} + \sum_{K\in\mathcal{E}_h} \dfrac{\alpha\,h_K^2\,|\underline{a}\bullet\underline{\nabla}\,e_h|^2_{0,K}}{\alpha\,\sigma\,h_k^2 + \kappa + h_K\,[\underline{a}]_K} \right) \\ +\dfrac{1}{\varepsilon}\,\alpha\,\max(1, C_3) \sum_{K\in\mathcal{E}_h} h_K\,[\underline{a}]_K\,|\underline{\nabla}\,\eta|^2_{0,K} + \kappa\,h_K^2\,|\Delta\,\eta|^2_{0,K} \end{array}\right. \tag{3.10}$$

Finally, summarizing (3.5) and (3.6) and (3.10), and choosing

$$0 < \varepsilon < C/\max(4, 3 + \alpha^{-1}) \tag{3.11}$$

we have a constant $C_4 > 0$ such that

$$\left\{\begin{array}{l} C_4\,\kappa\,|||e_h|||_h^2 \\ \leq \kappa\,|\underline{\nabla}\,\eta|_0^2 + \sum_{K\in\mathcal{E}_h} h_K^{-2}\,|\eta|^2_{0,K}\,(\alpha\,\sigma\,h_K^2 + \kappa + h_K\,[\underline{a}]_K) \\ + \sum_{K\in\mathcal{E}_h} \alpha\,\sigma\,h_K^2\,h_K^{-2}\,|\eta|^2_{0,K} \\ + \sum_{K\in\mathcal{E}_h} h_K\,[\underline{a}]_K\,|\underline{\nabla}\,\eta|^2_{0,K} + \kappa\,h_K^2\,|\Delta\,\eta|^2_{0,K} \end{array}\right. \tag{3.12}$$

With (3.12) and (3.2), by using the triangle–inquality we have a general estimates for the difference between u_h and u. The following theorem includes the above results.

Theorem 3.1. *Let u_h and u solve (2.10) and (2.1), respectively. In addition, (3.2) is satisfied. Then*

$$C\,\kappa\,|||u - u_h|||_h^2 \leq \sum_{K\in\mathcal{E}_h} h_K^{2\,m}\,|u|^2_{m+1,K}\,(\alpha\,\sigma\,h_K^2 + \kappa + h_K\,[\underline{a}]_K) \tag{3.13}$$

where $C > 0$ is a constant, not depend on $h, \kappa, \sigma, [\underline{a}]_K$.

In conclusion, we remark that $\alpha \in (0, 4/C_I)$ can take its value in a definite range, where C_I has its some values given by [9], and a unified computaional formula is presented by [6]. In [5], α precisely takes its value as any one belonging to $(0, 1]$.

4. Discussions

We close the paper by briefly considering some usual problems corresponding to (1.1). Accordingly, some sharp estimates for $\kappa\,|||u - u_h|||_h$ are obtained. And some comments are also made.

Let us firstly consider the advection–diffusion equation with $\sigma = 0$. In this case,

$$C\,\kappa\,|||u - u_h|||_h^2 \leq \sum_{K\in\mathcal{E}_h} h_K^{2\,m}\,|u|^2_{m+1,K}\,(\kappa + h_K\,[\underline{a}]_K) \tag{4.1}$$

We remark that a similar result to (4.1) was obtained by [5], but therein a prerequisite for $\underline{a}$ is that it must be a locally piecewise polynomial function.

Furthermore, we point out that if we introduce a mesh–Peclet number analogous to [8] [5],

$$Pe_K = \frac{h_K\,[\underline{a}]_K}{\kappa} \tag{4.2}$$

and applying an analogous argument to [5], it immediately follows that

$$\begin{aligned}&\kappa\,|u-u_h|_1^2+\sum_{K\in\mathcal{E}_h}\frac{\alpha\,h_K^2\,|\underline{a}\bullet\underline{\nabla}\,(u-u_h)|_{0,K}^2}{\kappa+h_K\,[\underline{a}]_K}\\ &\le C\sum_{K\in\mathcal{E}_h}h_K^{2\,m}\,|u|_{m+1,K}^2\,\{H(Pe_K-1)\,[\underline{a}]_K\,h_K+\kappa\,H(1-Pe_K)\}\end{aligned}\tag{4.3}$$

and

$$\left\{\begin{aligned}&\sum_{K\in\mathcal{E}_h}\frac{\alpha\,h_K^2\,|\underline{a}\bullet\underline{\nabla}\,(u-u_h)|_{0,K}^2}{\kappa+h_K\,[\underline{a}]_K}\\ &\ge C\left(\sum_{K\in\mathcal{E}_h}H(Pe_K-1)\,\frac{h_K\,|\underline{a}\bullet\underline{\nabla}\,(u-u_h)|_{0,K}^2}{[\underline{a}]_K}\right.\\ &\qquad\left.+\sum_{K\in\mathcal{E}_h}H(1-Pe_K)\,\frac{h_K^2\,|\underline{a}\bullet\underline{\nabla}\,(u-u_h)|_{0,K}^2}{\kappa}\right)\end{aligned}\right.\tag{4.4}$$

where $H(x-y)$ is the Heaviside function. So, from (4.3) and (4.4) we know that our result (4.1) is better than that of [8], and (4.1) no longer depends on the mesh–Peclet number. Moreover, in our paper we have no need to design those complicated stabilizing parameters such as [8]. We also see that the $[\underline{a}]_K$ in [5] is defined by

$$[\underline{a}]_K^2=\sum_{i=1}^{r}|\underline{a}\bullet\underline{\nabla}\,\varphi_i^K|_{0,K}^2\tag{4.5}$$

where $\varphi_i^K,(1\le i\le r)$ are a local bubble basis function group being assumed to be linear–independent. Such definition has its merits that it may be adapted for problems with a wildly oscillatory convection field becasue it has an average meaning in L^2–norm.

Next, we consider the diffusion problem with $\underline{a}=0$ and $\sigma\ne0$. In this case, $\tau_{K,\alpha}$ of this paper is the same as [5], (for P_1 element and $\alpha=1$, it can be also found in [7]), that is

$$\tau_{K,\alpha}=\frac{h_K^2}{\alpha\,\sigma\,h_K^2+\kappa}\tag{4.6}$$

we should say that in the diffusion case with a zeroth order term $\sigma\,u$, our method here is somewhat weaker than that of [5], for therein T is symmetry, i.e.,

$$T(u,v)=-\alpha\sum_{K\in\mathcal{E}_h}\tau_{K,\alpha}\,(\sigma\,u-\kappa\,\Delta\,u,\sigma\,v-\kappa\,\Delta\,v)_{0,K}\tag{4.7}$$

Of course, both error estimates, including those of [7] for P_1 element method with $\alpha=1$, are the same. For the sake of convenience for readers, we only give error estimates in the following with not any detail. The readers are referred to [5] [7] for details.

$$|u-u_h|_1^2+\sum_{K\in\mathcal{E}_h}\frac{\sigma\,|u-u_h|_{0,K}^2}{\alpha\,\sigma\,h_K^2+\kappa}\le C\sum_{K\in\mathcal{E}_h}h_K^{2\,m}\,|u|_{m+1,K}^2\tag{4.8}$$

where $C>0$ does not depend on h and σ and κ.

Finally, for $\kappa=0$ the limiting problem, the advection dominated equation (1.1) with $\kappa=0$ is

$$\left\{\begin{aligned}\underline{a}\bullet\underline{\nabla}\,u+\sigma\,u&=f&&\text{in }\Omega\\ u&=0&&\text{on }\partial\,\Omega_{in}\end{aligned}\right.\tag{4.9}$$

where $\partial\Omega_{in}$ is the $inflow$ boundary

$$\partial\Omega_{in} = \{x \in \partial\Omega \mid \underline{a}(x) \bullet \nu(x) > 0\} \tag{4.10}$$

and $\nu(x)$ is the outer unit normal to the boundary. So, the error between u_h and u is

$$\left\{ \begin{array}{l} \sum\limits_{K\in\mathcal{E}_h} \dfrac{\sigma\, h_K\, [\underline{a}]_K\, |u-u_h|_{0,K}^2 + \alpha\, h_K^2\, |\underline{a}\bullet\underline{\nabla}\,(u-u_h)|_{0,K}^2}{\alpha\,\sigma\, h_K^2 + h_K\,[\underline{a}]_K} \\ \leq C \sum\limits_{K\in\mathcal{E}_h} h_K^{2m}\, |u|_{m+1,K}^2\, (\sigma\, h_K^2 + h_K\, [\underline{a}]_K) \end{array} \right. \tag{4.11}$$

At the end of our paper, we point out that, in fact, an alternative definition of T and R can be used, i.e., T and R can be respectively setted as

$$T(u,v) = -\alpha \sum_{K\in\mathcal{E}_h} \tau_{K,\alpha}\,(\sigma\, u + \underline{a}\bullet\underline{\nabla}\, u - \kappa\,\Delta\, u, \sigma\, v - \underline{a}\bullet\underline{\nabla}\, v\ \pm\ \beta\,\kappa\,\Delta\, v)_{0,K} \tag{4.12}$$

and

$$R(v) = -\alpha \sum_{K\in\mathcal{E}_h} \tau_{K,\alpha}\,(f, \sigma\, v - \underline{a}\bullet\underline{\nabla}\, v\ \pm\ \beta\,\kappa\,\Delta\, v)_{0,K} \tag{4.13}$$

instead of (2.8) and (2.9). All the results obtained in the previous sections are still true. Here $\beta \in [0,1]$ is an optional parameter.

References

[1] Brezzi F., Bristeau M.O, Franca L.P., Mallet M., Rosé G., A relationship between stabilized finite element methods and the Gal erkin method with bubble functions, *Comput. Methods Appl. Mech. Engrg.,* **96** (1992), 117–129.

[2] Brezzi F., Russo A., Choosing bubbles for advection-diffusion problems, *Math. Models Methods Appl. Sci.,* **4** (1994), 571–587.

[3] Ciarlet P.G., The finite element method for elliptic problems, North-Holland, Amsterdam, (1978).

[4] Duan Huo-yuan, New Mixed and Hybrid Finite Element Methods, Thesis, Institute of Computational Mathematics and Scientific /Engineering Computing, Chinese Academy of Sciences, Beijing, May, 1997.

[5] Duan Huo-yuan, Bubble stabilization finite element methods for advective–diffusive models (in Chinese), *Commu. Appl. Math. Comp.,* **12** (1998), 1-10.

[6] Duan Huo-yuan, A technique for inverse estimation emanating from stabilized finite element methods (in Chinese), *Numer. Math. Sinica,* **20** (1998), 403-408.

[7] Franca L.P., Farhat C., Bubble function prompt unusual stabilized finite element methods, *Comput. Methods Appl. Mech. Engrg.,* **123** (1995), 299–308.

[8] Franca L.P., Frey S.L., Hughes T.J.R., Stabilized finite element methods: I. application to the advective-diffusive model, *Comput. Methods Appl. Mech. Engrg.,* **95** (1992), 253–276.

[9] Harari I., Hughes T.J.R, What are C and h?: inqualities for the analysis and design of finite element mthods, *Comput. Methods Appl. Mech. Engrg.,* **97** (1992), 157–192.

[10] Russo A., Bubble stabilization of finite element methods for the linearized incompressible Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.,* **132** (1996), 335-343.

[11] Franca L.P., Farhat C., Lesoinne M., Russo A., Unusual stabilized finite element methods and residual free bubbles, *Int. J. Numer. Methods Fluids,* **27** (1998), 159–168.