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A NEW MULTI-SYMPLECTIC SCHEME FOR NONLINEAR "GOOD" BOUSSINESQ EQUATION *1)

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Abstract

The Hamiltonian formulations of the linear "good" Boussinesq (L.G.B.) equation and the multi-symplectic formulation of the nonlinear "good" Boussinesq (N.G.B.) equation are considered. For the multi-symplectic formulation, a new fifteen-point difference scheme which is equivalent to the multi-symplectic Preissmann integrator is derived. We also present numerical experiments, which show that the symplectic and multi-symplectic schemes have excellent long-time numerical behavior.

Key words: Nonlinear "good" Boussinesq equation, Multi-symplectic scheme, Preissmann integrator, Conservation law.

1. Introduction

In recent years a remarkable development has taken place in the study of nonlinear evolutionary partial differential equations. An example is the nonlinear "good" Boussinesq (N.G.B.) equation

$$u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx} \tag{1}$$

which describes shallow water waves propagating in both directions. The analytic expression of such solutions is

$$u(x,t) = -A sech^2[(P/2)(\xi - \xi_0)], \qquad \xi = x - ct; \tag{2}$$

where ξ_0 and $P > 0$ are free real parameters and the amplitude A and velocity c of the wave are related to P through the formulas

$$A = 3P^2/2, \qquad c = \pm\sqrt{1 - P^2} \tag{3}$$

Note that ξ_0 determines the initial position of the wave, and that, due to the square root in (3), the parameter P can only take values in $0 < P \leq 1$. Thus, the solitary waves (2) only exist for a finite range of velocities $-1 < c < 1$. Of course, a positive (respectively negative) velocity corresponds to a wave moving to the right (respectively to the left). From the available literature we find that for the Korteweg-de Vries(KdV) or cubic schrödinger (CS) equations, the literature is very large, while the study of the N.G.B. equation is only beginning [1, 8, 9, 10].

Hamiltonian systems are canonical systems on phase space endowed with symplectic structures. The dynamical evolutions, i.e., the phase flow of the Hamiltonian systems are symplectic transformations that are area-preserving. The importance of the Hamiltonian systems and their

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special property require the numerical algorithms for them should preserve as much as possible the relevant symplectic properties of the original systems.

Feng Kang[15]–[17] proposed in 1984 a new approach to computing Hamiltonian systems from the view point of symplectic geometry. He systematically described the general method for constructing symplectic schemes with any order accuracy via generating functions. A generalization of the above theory and methods for canonical Hamiltonian equations in infinite dimension can be found in paper [18].

The L.G.B. equation (10) can be written as three infinite dimension Hamiltonian systems. Therefore, it is natural to require a discretization or a semi-discretization to reflect this property. The basic idea is to find a finite dimensional spatial truncation of (10) so that the resulting semi-discretization equation can be cast into a finite dimensional Hamiltonian system. Next, we can integrate the finite dimensional Hamiltonian system in time using a symplectic discretization [11].

However, there are limitations in this approach to developing a symplectic method for PDEs. The disadvantage of this approach is that it is global. To overcome this limitation, Bridges and Reich introduced the concept of multi-symplectic integrators based on a multi-symplectic structure of some conservative PDEs [3, 4]. The theoretical results indicated [4] that the nice features of the multi-symplctic structure are that it is a strictly local concept and that it can be formulated as a conservation law involving differential two forms. Thus the multi-symplectic integrators have excellent local invariant conserving properties. The N.G.B. equation has multi-symplectic structures, therefore we can apply this approach to obtain multi-symplectic integrators.

The purpose of this paper is to present symplectic integrators based on the Hamiltonian formulations of (10) and multi-symplectic integrator based on the multi-symplectic formulation of (1). The outline of this paper is as follows. In section two, we derive the multi-symplectic formulation of the N.G.B. equation and obtain a new fifteen-point multi-symplectic scheme. In section three, we give out three Hamiltonian formulations of the L.G.B. equation (10) with periodic boundary condition and use the hyperbolic function $\tanh(x)$ to construct symplectic schemes of arbitrary order for them. Numerical experiments are presented in section four.

2. Multi-symplectic Formulation of N.G.B. Equation and Multi-symplectic Integrator

We first present the concept of multi-symplectic integrators introduced by Bridges and Reich in [3, 4]. A large class of PDEs (for simplicity,we only consider one space dimension) can be reformulated as a system of the form

$$M\mathbf{z}_t + K\mathbf{z}_x = \bigtriangledown_{\mathbf{z}} S(\mathbf{z}), \mathbf{z} \in \mathbf{R}^n, (x,t) \in \mathbf{R}^2, \tag{4}$$

where M and K are skew-symmetric matrices on $\mathbf{R}^n, n \geq 3$ and $S : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. We call the above system a multi-symplectic Hamiltonian system on a multi-symplectic structure, since it has a multi-symplectic conversation law

$$\frac{\partial}{\partial t}\omega + \frac{\partial}{\partial x}\kappa = 0 \tag{5}$$

where ω and κ are the pre-symplectic forms

$$\omega = \frac{1}{2} dz \wedge M dz \quad \text{and} \quad \kappa = \frac{1}{2} dz \wedge K dz$$

The most significant aspect of the multi-symplectic formulation (4) is that its multi-symplecticity is completely local, which characterizes the system more deeply.

Multi-symplecticity is a geometric property of the PDEs, and we naturally require a discretization to reflect this property. Based on this idea, Bridges and Reich introduced the concept of multi-symplectic integrators, i.e., numerical methods which preserve a discrete version of multi-symplectic conservation law in [3]. It has been shown that popular methods such as the center Preissmann scheme [5] and the leap-frog method are multi-symplectic and that such schemes having remarkable local energy and momentum conserving properties [3].

Now consider the following generalized nonlinear "good" Boussinesq equation

$$u_{tt} = -u_{xxxx} + u_{xx} + (G'(u))_{xx} \tag{6}$$

where $G(u) : \mathbf{R} \to \mathbf{R}$ is some nonlinear smooth function. Especially, choosing $G(u) = \frac{1}{3}u^3$, then equation (6) is just equation (1).

Introducing the canonical momenta $u_t = w_{xx}, u_x = p$, we can obtain the multi-symplectic PDEs

$$\begin{cases} -w_t - p_x = -u - G'(u) \\ u_t = w_{xx} \\ u_x = p \end{cases}$$

with state variable $z = (u, w, p)^T$ and the Hamiltonian

$$S = \frac{1}{2}(-u^2 - w_x^2 + p^2) - G(u)$$

In this case,

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The corresponding multi-symplectic conservation law is

$$\frac{\partial}{\partial t}(du \wedge dw) + \frac{\partial}{\partial x}(du \wedge dp) = 0 \tag{7}$$

Proposition. Upon introducing canonical moments $\frac{1}{\triangle t}(u_{i+\frac{1}{2}}^{j+1} - u_{i+\frac{1}{2}}^{j}) = \frac{1}{\triangle x^2}(w_{i+\frac{3}{2}}^{j+\frac{1}{2}} - 2w_{i+\frac{1}{2}}^{j+\frac{1}{2}} + w_{i-\frac{3}{2}}^{j+\frac{1}{2}}), p_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{\triangle x}(u_{i+1}^{j+\frac{1}{2}} - u_{i}^{j+\frac{1}{2}})$, the following two discrete versions are equivalent

$$\begin{aligned} \frac{\partial_t^2 u_{i+1}^j + 2\partial_t^2 u_i^j + \partial_t^2 u_{i-1}^j}{4 \triangle t^2} + \frac{\partial_x^4 u_i^{j+1} + 2\partial_x^4 u_i^j + \partial_x^4 u_i^{j-1}}{4 \triangle x^4} &= \frac{1}{4}(\overline{u}_i^j + \overline{u}_{i-1}^{j-1} + \overline{u}_i^{j-1} + \overline{u}_{i-1}^j)_{x\bar{x}} \\ &+ \frac{1}{4}([G'(\overline{u}_i^j)] + [G'(\overline{u}_{i-1}^{j-1})] + [G'(\overline{u}_i^{j-1})] + [G'(\overline{u}_{i-1}^j)])_{x\bar{x}}, \end{aligned} \tag{8}$$

$$\frac{1}{\triangle t}\begin{pmatrix} -w_{i+\frac{1}{2}}^{j+1} + w_{i+\frac{1}{2}}^j \\ u_{i+\frac{1}{2}}^{j+1} - u_{i+\frac{1}{2}}^j \\ 0 \end{pmatrix} + \frac{1}{\triangle x}\begin{pmatrix} -p_{i+1}^{j+\frac{1}{2}} + p_i^{j+\frac{1}{2}} \\ 0 \\ u_{i+1}^{j+\frac{1}{2}} - u_i^{j+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} -u_{i+\frac{1}{2}}^{j+\frac{1}{2}} - G'(u_{i+\frac{1}{2}}^{j+\frac{1}{2}}) \\ (w_{i+\frac{1}{2}}^{j+\frac{1}{2}})_{x\bar{x}} \\ p_{i+\frac{1}{2}}^{j+\frac{1}{2}} \end{pmatrix}, \tag{9}$$

where $u_i^j \approx u(i\triangle x, j\triangle t)$, $\triangle x$ and $\triangle t$ are the space step length and time step length respectively. $(u_i^j)_{x\bar{x}} \equiv \partial_x^2 u_i^j = u_{i+1}^j - 2u_i^j + u_{i-1}^j$, $\partial_t^2 u_i^j = u_i^{j+1} - 2u_i^j + u_i^{j-1}, u_{i+\frac{1}{2}}^{j+1} = \frac{1}{2}(u_i^{j+1} + u_{i+1}^{j+1})$, $\partial_x^4 u_i^j = u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j$, $\bar{u}_i^j = \frac{1}{4}(u_i^j + u_i^{j+1} + u_{i+1}^{j+1} + u_{i+1}^j), u_{i+\frac{1}{2}}^{j+\frac{1}{2}} = \frac{1}{2}(u_{i+\frac{1}{2}}^{j+1} + u_{i+\frac{1}{2}}^j), u_{i+1}^{j+\frac{1}{2}} = \frac{1}{2}(u_{i+1}^j + u_{i+1}^{j+1})$, etc.

Remark 2.1. The scheme (9) is the central Preissmann scheme [3] with the discrete multi-symplectic conservation law [6]

$$\frac{du_{i+\frac{1}{2}}^{j+1} \wedge dw_{i+\frac{1}{2}}^{j+1} - du_{i+\frac{1}{2}}^j \wedge dw_{i+\frac{1}{2}}^j}{\triangle t} + \frac{du_{i+1}^{j+\frac{1}{2}} \wedge dp_{i+1}^{j+\frac{1}{2}} - du_i^{j+\frac{1}{2}} \wedge dp_i^{j+\frac{1}{2}}}{\triangle x} = 0$$

Remark 2.2. The scheme (8) is a new fifteen-point difference scheme for the N.G.B. equation, which we have not encountered in the literature. This new scheme is a multi-symplectic integrator, since it is equivalent to the central Preissmann scheme (9).

3. Hamiltonian Formulation of L.G.B. Equation and Symplectic Integrators

Before this section, we introduce some notations that will be used in the following paper. $\bigtriangledown^n(2m)$ and $\triangle^n(2m)$ are the $2m^{th}$ order central difference operator for $\frac{\partial^n}{\partial x^n}$ and $\frac{\partial^{2n}}{\partial x^{2n}}$ respectively, and we note $\bigtriangledown(2m) = \bigtriangledown^1(2m)$, $\triangle(2m) = \triangle^1(2m)$, $\bigtriangledown^{2n}(2m) = \triangle^n(2m)$, $u_x^{(n)} = \frac{\partial^n u}{\partial x^n}$, $M(n,2)$ and $M(n,4)$ are the corresponding matrices of the second and fourth order central difference operators for $\frac{\partial^n}{\partial x^n}$, I is the identity operator.

3.1. Hamiltonian Formulation of the L.G.B. Equation

Let's consider the L.G.B. equation

$$u_{tt} = u_{xx} - u_{xxxx}, \qquad 0 \le x \le 2\pi,\, t \ge 0 \tag{10}$$

with an initial condition

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \qquad 0 \le x \le 2\pi, \tag{11}$$

and periodic boundary conditions

$$u(0,t) = u(2\pi,t), \qquad t \ge 0 \tag{12}$$

We begin with rewriting the equation (10) in three Hamiltonian forms.

Hamiltonian form 1.

The first Hamiltonian form for the L.G.B. equation is

$$\frac{dz}{dt} = J^{-1}H_z \tag{13}$$

where the Hamiltonian is $H = -\frac{1}{2}\int(v^2 + u_x^2 + u_{xx}^2)dx$ and

$$z = \begin{bmatrix} u \\ v \end{bmatrix}, J^{-1} = J' = -J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, H_z = \begin{bmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{bmatrix} = \begin{bmatrix} -u_{xxxx} + u_{xx} \\ -v \end{bmatrix}.$$

(13) can be rewritten as

$$\frac{dz}{dt} = J^{-1}Az, \tag{14}$$

where

$$J^{-1}A = \begin{bmatrix} 0 & 1 \\ \triangle - \triangle^2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \triangle - \triangle^2 & 0 \\ 0 & -1 \end{bmatrix},$$

and $\triangle^n$ is the central difference operator for $\frac{\partial^{2n}}{\partial x^{2n}}$.

Let $\triangle(2m)$ be the $2m-$th order central difference operator for $\frac{\partial^2}{\partial x^2}$, we have

$$\triangle(2m) = \bigtriangledown_+ \bigtriangledown_- \sum_{j=0}^{m-1} (-1)^j \beta_j \Big(\frac{\triangle x^2 \bigtriangledown_+ \bigtriangledown_-}{4}\Big)^j \tag{15}$$

where $\beta_j = [(j!)^2 2^{2j}]/[(2j+1)!(j+1)]$, $\bigtriangledown_+$ and $\bigtriangledown_-$ are forward and backward difference operators, respectively.

Denoting $U = [u_1, u_2, \cdots, u_N], V = [v_1, v_2, \cdots, v_N]$, system (14) will be

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & I \\ M(2,2m) - M(4,2m) & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} V \\ M(2,2m) - M(4,2m)U \end{bmatrix} \tag{16}$$

where I is $N \times N$ identity matrix, $M(2,2m)$ and $M(4,2m)$ are $N \times N$ matrix corresponding to $\triangle(2m)$ and $\triangle^2(2m)$, respectively. System (16) approximates (14) of accuracy $o(\triangle x^{2m})$ in the space discretization.

In practice, the second and fourth order central difference approximations are usually applied. Let $\triangle(2)$ and $\triangle(4)$ be the second and fourth order central difference operators for $\frac{\partial^2}{\partial x^2}$, respectively. In these two cases, we have

$$\triangle(2)u_i^j = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\triangle x^2}, \tag{17}$$

$$\triangle(4)u_i^j = \frac{-u_{i+2}^j + 16u_{i+1}^j - 30u_i^j + 16u_{i-1}^j - u_{i-2}^j}{12\triangle x^2}. \tag{18}$$

Their corresponding matrices are

$$M(2,2): \quad \frac{1}{\triangle x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & \cdots & -2 & 1 \\ 1 & 0 & \cdots & \cdots & 1 & -2 \end{pmatrix}. \tag{19}$$

$$M(2,4): \quad \frac{1}{12\triangle x^2}\begin{pmatrix} -30 & 16 & -1 & 0 & \cdots & -1 & 16 \\ 16 & -30 & 16 & -1 & \cdots & 0 & -1 \\ -1 & 16 & -30 & 16 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & 16 & -30 & 16 \\ 16 & -1 & 0 & \cdots & -1 & 16 & -30 \end{pmatrix}. \tag{20}$$

Similarly, Let $\triangle^2(2)$ and $\triangle^2(4)$ be the second and fourth order central difference operators for $\frac{\partial^4}{\partial x^4}$ respectively, we have

$$\triangle^2(2)u_i^j = \frac{u_{i+2}^j - 4u_{i+1}^j + 6u_i^j - 4u_{i-1}^j + u_{i-2}^j}{\triangle x^4}, \tag{21}$$

$$\triangle^2(4)u_i^j = \frac{-u_{i+3}^j + 12u_{i+2}^j - 39u_{i+1}^j + 56u_i^j - 39u_{i-1}^j + 12u_{i-2}^j - u_{i-3}^j}{6\triangle x^4}. \tag{22}$$

Their corresponding matrices are

$$M(4,2): \quad \frac{1}{\triangle x^4}\begin{pmatrix} 6 & -4 & 1 & 0 & \cdots & 1 & -4 \\ -4 & 6 & -4 & 1 & \cdots & 0 & 1 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & -4 & 6 & -4 \\ -4 & 1 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix}, \tag{23}$$

$$M(4,4): \quad \frac{-1}{6\triangle x^4}\begin{pmatrix} -56 & 39 & -12 & 1 & 0 & 0 & \cdots & 1 & -12 & 39 \\ 39 & -56 & 39 & -12 & 1 & 0 & \cdots & 0 & 1 & -12 \\ -12 & 39 & -56 & 39 & -12 & 1 & \cdots & 0 & 0 & 1 \\ 1 & -12 & 39 & -56 & 39 & -12 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -12 & 1 & 0 & 0 & \cdots & 1 & -12 & 39 & -56 & 39 \\ 39 & -12 & 1 & 0 & \cdots & 0 & 1 & -12 & 39 & -56 \end{pmatrix}. \tag{24}$$

Hamiltonian form 2.

The second Hamiltonian form for the L.G.B. equation is

$$\frac{dz}{dt} = J^{-1}H_z \tag{25}$$

where the Hamiltonian is $H = \frac{1}{2}\int(u^2 + v_x^2 + u_x^2)dx$ and

$$z = \begin{bmatrix} u \\ v \end{bmatrix}, J^{-1} = J' = -J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, H_z = \begin{bmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{bmatrix} = \begin{bmatrix} -u_{xx} + u \\ -v_{xx} \end{bmatrix}.$$

(25) can be rewritten as

$$\frac{dz}{dt} = J^{-1}Az, \tag{26}$$

where

$$J^{-1}A = \begin{bmatrix} 0 & \triangle \\ 1 - \triangle & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 - \triangle & 0 \\ 0 & -\triangle \end{bmatrix},$$

Denoting $U = [u_1, u_2, \cdots, u_N], V = [v_1, v_2, \cdots, v_N]$, system (26) will be

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & M(2,2m) \\ I - M(2,2m) & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} M(2,2m)V \\ U - M(2,2m)U \end{bmatrix} \tag{27}$$

Hamiltonian form 3.

The third Hamiltonian form for the L.G.B. equation is

$$\frac{dz}{dt} = D\frac{\delta H}{\delta z} \tag{28}$$

where the Hamiltonian is $H = \frac{1}{2}\int(u^2 + v^2 + u_x^2)dx$ and

$$z = \begin{bmatrix} u \\ v \end{bmatrix}, D = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix}, \frac{\delta H}{\delta z} = \begin{bmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{bmatrix} = \begin{bmatrix} -u_{xx} + u \\ v \end{bmatrix}.$$

Obviously the operator D is skew-adjoint and system (28) can be rewritten in the form similar to (14), but in this case

$$\frac{dz}{dt} = K^{-1}Az, \tag{29}$$

where

$$K = \begin{bmatrix} 0 & \nabla \\ \nabla & 0 \end{bmatrix},$$

which is a nonsingular skew-symmetric matrix, and

$$K^{-1}A = \begin{bmatrix} 0 & \nabla \\ \nabla - \nabla^3 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \nabla^2 - \nabla^4 & 0 \\ 0 & \nabla^2 \end{bmatrix},$$

where ∇, ∇^2, ∇^3 and ∇^4 are the central difference operator for $\frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^3}{\partial x^3}$ and $\frac{\partial^4}{\partial x^4}$, respectively.

Denoting $U = [u_1, u_2, \cdots, u_N], V = [v_1, v_2, \cdots, v_N]$, system (29) will be

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 & M(1,2m) \\ M(1,2m) - M(3,2m) & 0 \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} M(1,2m)V \\ M(1,2m)U - M(3,2m)U \end{bmatrix} \tag{30}$$

where I is $N \times N$ identity matrix, $M(1,2m)$ and $M(3,2m)$ are $N \times N$ matrix corresponding to $\nabla(2m)$ and $\nabla^3(2m)$, respectively. System (30) approximates (29) of accuracy $o(\triangle x^{2m})$ in the space discretization.

Let $\nabla(2)$ and $\nabla(4)$ be the second and fourth order central difference operators for $\frac{\partial}{\partial x}$ respectively, we have

$$\nabla(2)u_i^j = \frac{u_{i+1}^j - u_{i-1}^j}{2\triangle x}, \tag{31}$$

$$\nabla(4)u_i^j = \frac{-u_{i+2}^j + 8u_{i+1}^j - 8u_{i-1}^j + u_{i-2}^j}{12\triangle x}. \tag{32}$$

Their corresponding matrices are

$$M(1,2): \quad \frac{1}{2\triangle x}\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & -1 \\ -1 & 0 & 1 & \cdots & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & -1 & 0 \end{pmatrix}, \tag{33}$$

$$M(1,4): \quad \frac{1}{12\triangle x}\begin{pmatrix} 0 & 8 & -1 & 0 & \cdots & 1 & -8 \\ -8 & 0 & 8 & -1 & \cdots & 0 & 1 \\ 1 & -8 & 0 & 8 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & -8 & 0 & 8 \\ 8 & -1 & 0 & \cdots & 1 & -8 & 0 \end{pmatrix}. \tag{34}$$

Similarly, Let $\bigtriangledown^3(2)$ and $\bigtriangledown^3(4)$ be the second and fourth order central difference operators for $\frac{\partial^3}{\partial x^3}$ respectively, we have

$$\bigtriangledown^3(2)u_i^j = \frac{u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j}{2\triangle x^3}, \tag{35}$$

$$\bigtriangledown^3(4)u_i^j = \frac{-u_{i+3}^j + 8u_{i+2}^j - 13u_{i+1}^j + 13u_{i-1}^j - 8u_{i-2}^j + u_{i-3}^j}{8\triangle x^3}. \tag{36}$$

Their corresponding matrices are

$$M(3,2): \quad \frac{1}{2\triangle x^3}\begin{pmatrix} 0 & -2 & 1 & 0 & \cdots & -1 & 2 \\ 2 & 0 & -2 & 1 & \cdots & 0 & -1 \\ -1 & 2 & 0 & -2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 2 & 0 & -2 \\ -2 & 1 & 0 & \cdots & -1 & 2 & 0 \end{pmatrix}. \tag{37}$$

$$M(3,4): \quad \frac{1}{8\triangle x^3}\begin{pmatrix} 0 & -13 & 8 & -1 & 0 & 0 & \cdots & 1 & -8 & 13 \\ 13 & 0 & -13 & 8 & -1 & 0 & \cdots & 0 & 1 & -8 \\ -8 & 13 & 0 & -13 & 8 & -1 & \cdots & 0 & 0 & 1 \\ 1 & -8 & 13 & 0 & -13 & 8 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 8 & -1 & 0 & 0 & \cdots & 1 & -8 & 13 & 0 & -13 \\ -13 & 8 & -1 & 0 & \cdots & 0 & 1 & -8 & 13 & 0 \end{pmatrix}. \tag{38}$$

We can apply generating function methods to construct symplectic integrators of (14), (26) and (29). As paper [11], we can construct symplectic schemes from Hyperbolic Functions $\tanh(x)$, $\sinh(x)$, and $\cosh(x)$. For example, we only present the symplectic schemes from Hyperbolic Function $\tanh(x)$.

3.2. Symplectic Schemes from Hyperbolic Function $\tanh(x)$

Now, we consider symplectic schemes generated from $\tanh(x)$ for the linear Hamiltonian system

$$\frac{dz}{dt} = J^{-1}A(2m)z, \tag{39}$$

where $A(2m)$ is a symmetric matrix which has $2m-$th order approximation to matrix A. Then the exact solution of (39) at time $t+\triangle t$ and t have the relation

$$z(t+\triangle t) = e^{\triangle t J^{-1}A(2m)}z(t).$$

since

$$z(t+\triangle t)-z(t)=(e^{\triangle tJ^{-1}A(2m)}-1)z(t),\quad z(t+\triangle t)+z(t)=(e^{\triangle tJ^{-1}A(2m)}+1)z(t).$$

we then have

$$\begin{aligned} z(t+\triangle t)-z(t) &= \frac{e^{\triangle tJ^{-1}A(2m)}-1}{e^{\triangle tJ^{-1}A(2m)}+1}(z(t+\triangle t)+z(t)) \\ &= \tanh\Big(\frac{\triangle t}{2}J^{-1}A(2m)\Big)(z(t+\triangle t)+z(t)), \end{aligned} \tag{40}$$

where

$$\tanh(x)=x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7+\cdots=\sum_{k=1}^{\infty}a_{2k-1}x^{2k-1}, \tag{41}$$

$$a_{2k-1}=2^{2k}(2^{2k}-1)\frac{B_{2k}}{(2k)!},\quad B_{2k}: \text{Bernoulli numbers.}$$

We know scheme (40) is of arbitrary order, while the follow scheme

$$z_{n+1}-z_n=\tanh\Big(\frac{\triangle t}{2}J^{-1}A(2m)\Big)(z_{n+1}+z_n) \tag{42}$$

given by the following $2s-$th order truncation of $\tanh(x)$

$$\tanh\Big(2s,\frac{\triangle t}{2}J^{-1}A(2m)\Big)=\sum_{k=1}^{s}a_{2k-1}\Big(\frac{\triangle t}{2}J^{-1}A(2m)\Big)^{2k-1},$$

is obviously of accuracy $o(\triangle t^{2s}+\triangle x^{2m})$. We now prove (42) is also symplectic.

We have the following two lemma from [15].

Lemma 1. *If $f(x)$ is an odd polynomial and L is an infinitesimal symplectic matrix, i.e., $L'J+JL=0$, then $f(L)$ is also an infinitesimal symplectic matrix.*

Lemma 2. *If Φ is an infinitesimal symplectic matrix, $|I+\Phi|\neq 0$, then $F=(I+\Phi)^{-1}(I-\Phi)$ is also a symplectic matrix.*

Since matrix $L=J^{-1}A(2m)$, where $J^{-1}=J'=-J=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ or $J=K$, is an infinitesimal symplectic matrix, we know from Lemma 1, $\Phi=\tanh(2s,\frac{\triangle t}{2}J^{-1}A(2m))$ is an infinitesimal symplectic matrix. Then from Lemma 2, the scheme (42) is obviously symplectic. Thus we get the following theorem.

Theorem. *Scheme (42) is a symplectic scheme with the accuracy of $o(\triangle t^{2s}+\triangle x^{2m})$.*

We give out two schemes of order $o(\triangle t^2+\triangle x^{2m})$ and $o(\triangle t^4+\triangle x^{2m})$

$$z_{n+1}-z_n=\frac{\triangle t}{2}J^{-1}A(2m)(z_{n+1}+z_n) \tag{43}$$

$$z_{n+1}-z_n=\Big(\frac{\triangle t}{2}J^{-1}A(2m)-(\frac{\triangle t}{2}J^{-1}A(2m))^3\Big)(z_{n+1}+z_n) \tag{44}$$

We note scheme (43) is just the centered Euler scheme, especially, when m=2, which is

$$\frac{u^{n+1}-2u^n+u^{n-1}}{\triangle t^2}=-\frac{\partial_x^4u^{n+1}+2\partial_x^4u^n+\partial_x^4u^{n-1}}{4\triangle x^4}+\frac{\partial_x^2u^{n+1}+2\partial_x^2u^n+\partial_x^2u^{n-1}}{4\triangle x^2} \tag{45}$$

4. Numerical Experiments

In this section, we first present the numerical simulation results of the L.G.B. equation (10) using Euler symplectic integrator (43) ($m=2$) (i.e. (45)) . Then the non-symplectic scheme (5.1) of [1] with $\theta=1/3$, i.e.

$$\begin{aligned} \frac{u^{n+1}-2u^n+u^{n-1}}{\triangle t^2} &= -\frac{\partial_x^4u^{n+1}+\partial_x^4u^n+\partial_x^4u^{n-1}}{3\triangle x^4}+\frac{\partial_x^2u^{n+1}+\partial_x^2u^n+\partial_x^2u^{n-1}}{3\triangle x^2} \\ &\quad +\frac{\partial_x^2(u^{n+1})^2+\partial_x^2(u^n)^2+\partial_x^2(u^{n-1})^2}{3\triangle x^2} \end{aligned} \tag{46}$$

and the multi-symplectic scheme (8) analyzed above have been tested in the long-time integration of solitary waves for the N.G.B. equation (1). Equation (2) shows that this kind of solution decays exponentially as $|x| \to \infty$ and therefore, for numerical purposes we have employed the scheme on an interval (X_L, X_R), where the artificial boundaries X_L and X_R are located far enough for the theoretical solution to satisfy the periodic boundary conditions, except for a negligible remainder. The numerical experiments were implemented with Matlab 5.

4.1. Numerical Experiments for Symplectic (45)

This part, we use Euler Symplectic scheme (45) to solve (10) with initial value $f(x) = \sin(x)$ and compare the numerical results with the exact solution $u(x,t) = \sin(x)\cos(\sqrt{2}t)$. We take a time step-length $\triangle t = \pi^2/1600$ and a space step-length $\triangle x = \pi/40$ (i.e., The number of collocation points used here is 80), for simplicity, the missing starting level U^1 was obtained by the theoretical solutions. The integrator is calculated until 10000 time steps, the results are plotted out in Figure 1 in which the ranges of x and t are $[\pi/40, 2\pi]$ and $[61.1052, 61.6850]$, respectively. The compare results are listed in Table 1 ($\pi/40 \leq x \leq 10\pi/40$, $t = 10000 \times \pi^2/1600 = 61.6850$). When $t = 61.6850, \pi/40 \leq x \leq 2\pi$ The compare results are also plotted out in Figure 2.

Table 1. Compare the numerical results of Euler Symplectic scheme (45) with the exact solution at $t = 61.6850$, $\pi/40 \leq x \leq 10\pi/40$ ($x_j = j \times \triangle x, j = 1, 2, \cdots, 80$).

x_j	Exa. Solu. $U(x_j)$	Num. Solu. U_j	Error=$U(x_j) - U_j$
$\pi/40$	5.8529e-002	5.8498e-002	3.0658e-005
$2\pi/40$	1.7415e-001	1.7405e-001	9.1220e-005
$3\pi/40$	2.8547e-001	2.8532e-001	1.4954e-004
$4\pi/40$	3.8977e-001	3.8957e-001	2.0417e-004
$5\pi/40$	4.8447e-001	4.8422e-001	2.5378e-004
$6\pi/40$	5.6725e-001	5.6695e-001	2.9713e-004
$7\pi/40$	6.3605e-001	6.3572e-001	3.3317e-004
$8\pi/40$	6.8920e-001	6.8883e-001	3.6101e-004
$9\pi/40$	7.2537e-001	7.2499e-001	3.7996e-004
$10\pi/40$	7.4368e-001	7.4329e-001	3.8955e-004

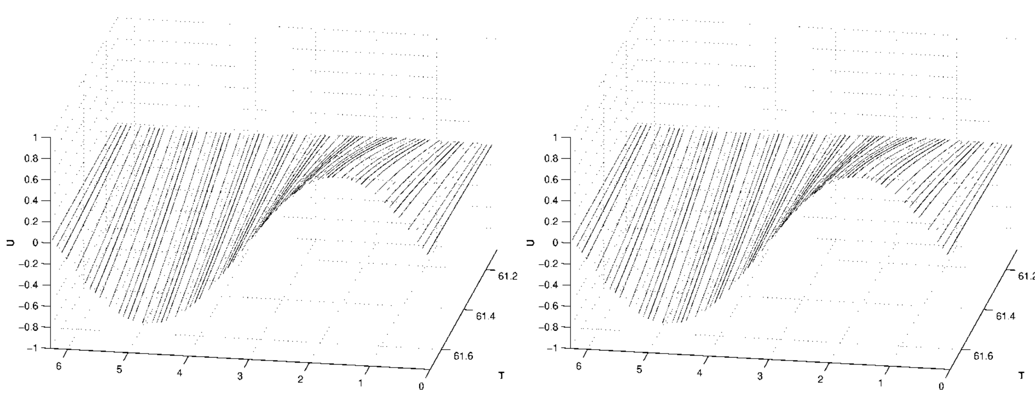

Figure 1: Compare the numerical results of Euler Symplectic scheme (45) with the exact solution at $61.1052 \leq t \leq 61.6850$, $\pi/40 \leq x \leq 2\pi$. The left plot: obtained with (45). The right plot: with exact solution.

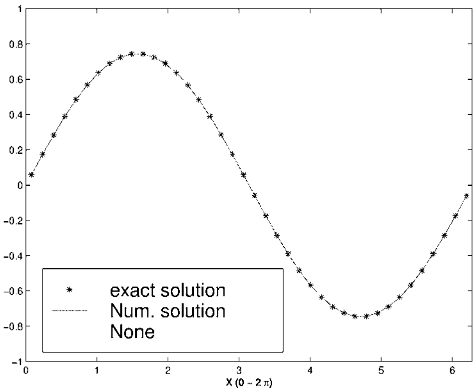

Figure 2: Compare the numerical results of Euler Symplectic scheme (45) with the exact solution at $t = 61.6850$, $\pi/40 \leq x \leq 2\pi$.

4.2. Numerical Experiments for Multi-symplectic Integrator

The single soliton solution First, we consider the propagation of single soliton. To do so, we use the soliton solution (2) with an amplitude $A = 0.5$ and an initial phase $\xi_0 = 0$, the missing starting level U^1 was obtained by the theoretical solution, boundaries were placed at $X_L = -60, X_R = 60$, with a time step $\triangle t = 0.125$ and a space step $\triangle x = 0.5$. After 10000 time steps, we find that the motion of the soliton is well simulated by integrator (8), and there has no oscillation phenomena appearance. While the non-symplectic scheme (46) appears instability. The left plot of Figure 3 shows the numerical results of multi-symplectic scheme (8) with the time from $t = 1125$ to $t = 1250$ on a spatial interval $[-40, 40]$. The right plot of Figure 3 shows the compare results of multi-symplectic scheme (8) with non-symplectic scheme (46) at $t = 1200$.

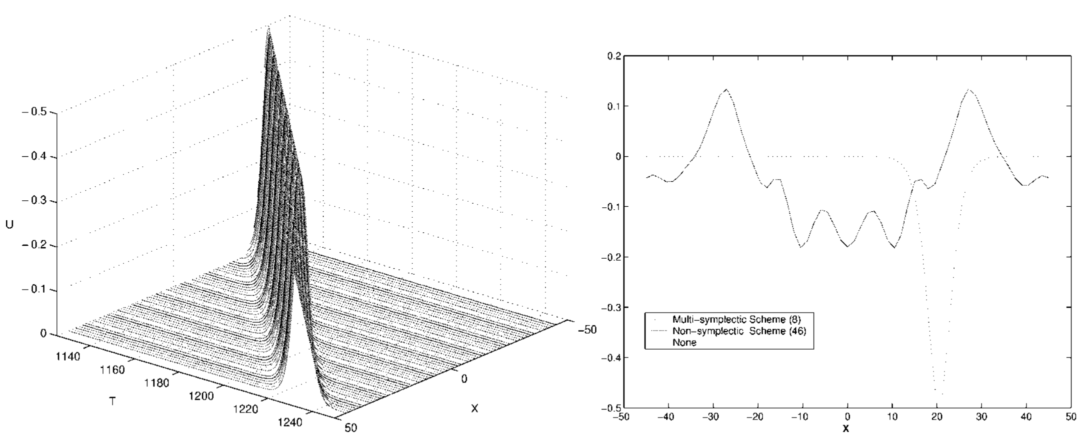

Figure 3: The left plot: The simulation results of the propagation of one soliton obtained with (8). The right plot: The compare results of the propagation of one soliton obtained with (8) and (46)at $t = 1200$.

The two-soliton solution From now on, we consider the case of two solitons of the same amplitudes $A = 0.369$ placed reasonably apart to move towards each other with the integrator (8). At this case, the initial positions are $\xi_0 = 0$ and $\xi_0 = -50$ respectively. Again, the missing starting level U^1 was taken from the theoretical solution, but now the boundaries were placed at $X_L = -100, X_R = 100$, with a time step length $\triangle t = 0.125$ and a space step length $\triangle x = 0.5$. After 10000 time steps, we observe that the integrator (8) simulates the interaction of the solitons well, after the interaction, the waves seem to emerge from the collision with their original shape and speed as if the collision had not taken place. The numerical results with the time from $t = 1126.5$ to $t = 1250$ on a spatial interval $[-100, 100]$ are also plotted in Figure 4.

Finally, the collision process of the two solitons with (8) are plotted out in Figure 5.

The numerical results show that the symplectic and multi-symplectic scheme have excellent long-time numerical behavior, which are coincident with theoretical results.

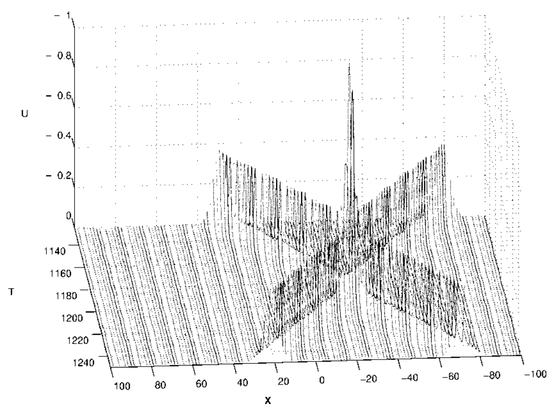

Figure 4: The simulation results of the interaction of two solitons with (8).

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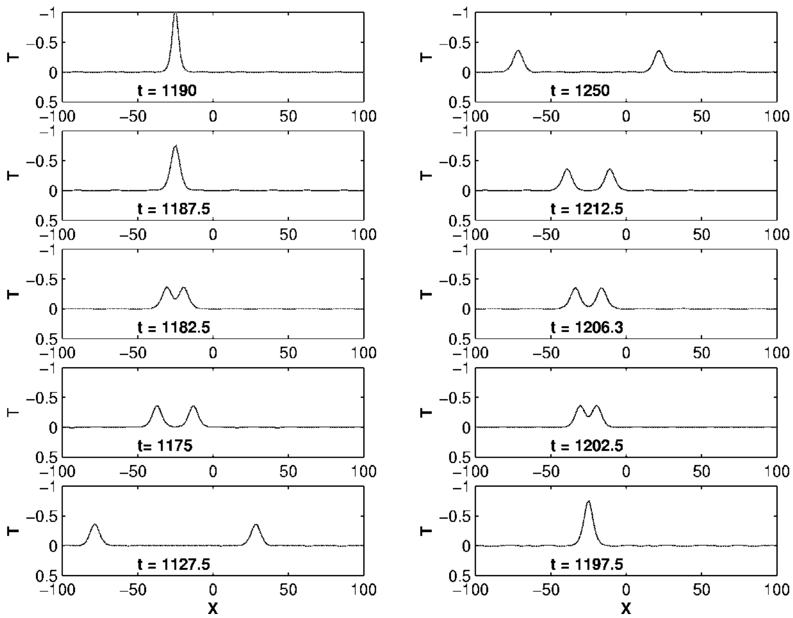

Figure 5: The collision process of the two solitons with (8) at various t.

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