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hp–VERSION INTERIOR PENALTY DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS ON ANISOTROPIC MESHES

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Abstract. We consider the hp-version interior penalty discontinuous Galerkin finite element method (hp-DGFEM) for linear second-order elliptic reaction-diffusion-advection equations with mixed Dirichlet and Neumann boundary conditions. Our main concern is the extension of the error analysis of the hp-DGFEM to the case when anisotropic (shape-irregular) elements and anisotropic polynomial degrees are used. For this purpose, extensions of well known approximation theory results are derived. In particular, new error bounds for the approximation error of the L^2- and H^1-projection operators are presented, as well as generalizations of existing inverse inequalities to the anisotropic setting. Equipped with these theoretical developments, we derive general error bounds for the hp-DGFEM on anisotropic meshes, and anisotropic polynomial degrees. Moreover, an improved choice for the (user-defined) discontinuity-penalisation parameter of the method is proposed, which takes into account the anisotropy of the mesh. These results collapse to previously known ones when applied to problems on shape-regular elements. The theoretical findings are justified by numerical experiments, indicating that the use of anisotropic elements, together with our newly suggested choice of the discontinuity-penalisation parameter, improves the stability, the accuracy and the efficiency of the method.

Key Words. discontinuous Galerkin, finite element methods, anisotropic meshes, equations with non-negative characteristic form.

1. Introduction

In recent years, there has been an increasing interest in a class of non-conforming finite element approximations of elliptic boundary-value problems, usually referred to as *discontinuous Galerkin finite element methods*. Justifications for the renewed interest in these methods, which date back to the 1970s and the early 1980s [23, 29, 2], can be found in the attractive properties they exhibit, such as increased flexibility in mesh design (irregular grids are admissible), the freedom of choosing the elemental polynomial degrees without the need to enforce any conformity requirements, good local conservation properties of the state variable, and good stability properties near boundary/interior layers or even discontinuities [6]. The first two reasons mentioned above make discontinuous Galerkin methods very suitable contenders for hp-adaptivity, whereas the last two render these methods attractive when convection is the dominant feature of the problem. New error analyses for various DGFEMs have been presented in the literature during the last

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decade: see [7] for the developments until 1999 and [3] for the contemporary unified approach; see also [5].

In this work we analyze the use of anisotropic finite elements for the numerical approximation of second-order equations with non-negative characteristic form [24]. On isotropic meshes, discontinuous Galerkin finite element methods for such equations were considered in [18]. In many practical examples of boundary value problems for partial differential equations with non-negative characteristic form, diffusion can be small, degenerate, or even identically equal to zero in subregions of the computational domain. Hence, computationally demanding features may appear in their analytical solutions; these include boundary/interior layers or discontinuities in subregions where the problem becomes of first-order hyperbolic type. When structures such as layers or discontinuities are present in the solution, the use of anisotropic elements aims to provide the necessary resolution in the directions along these structures in order to reduce the number of degrees of freedom required to accurately capture them. Therefore, the combination of discontinuous Galerkin finite element methods, that produce stable approximations in the unresolved regimes, and of the use of anisotropic elements and elemental bases with anisotropic polynomial degree, that aim to provide the desired resolution only in the space directions required, is an appealing technique for the numerical solution of these problems. This work extends the arguments presented in [18] to the anisotropic setting; much of our discussion is inspired by that paper.

Anisotropic bounds for various types of FEMs have been presented in the literature, addressing mainly the question of designing structured meshes for the robust approximation of solutions to singularly perturbed boundary-value problems that admit boundary or interior layers (see, e.g., [1, 21, 26] and the references therein). Analogous results for certain DGFEMs can be found in [30, 20]. Our approach focuses on the development of general approximation-theory-tools for anisotropic elements and their subsequent application to the error analysis of the DGFEM. Potentially, the anisotropic approximation theory developed here can also be used in other applications.

The paper is structured as follows. We begin by introducing the model problem (Section 2) and the functional analytic framework used in this work (Section 3). Along with standard Sobolev spaces, we shall make use of augmented Sobolev spaces (see [12, 13] for details), as they appear to be suitable for proving hp-optimal error bounds for interior penalty versions of discontinuous Galerkin finite element methods. After introducing the appropriate weak formulation (from which the method will emerge) in Section 4 and the admissible finite element spaces in Section 5, the hp-version interior penalty discontinuous Galerkin finite element method is introduced in Section 6. Next, we present new anisotropic approximation theory results, including bounds on the projection errors of the L^2- and H^1-projection operators in various norms (Section 7). The latter will be used in the derivation of anisotropic a-priori error bounds for the hp-DGFEM in the energy norm (Section 8). We shall conclude with some numerical experiments indicating that the use of anisotropic elements improves the efficiency of the method, and that the analysis presented herein, yielding an new choice of the discontinuity-penalisation parameter, improves the stability, the accuracy and the efficiency of the method.

2. Model Problem

Let Ω be a bounded open (curvilinear) polygonal domain in $\mathbb{R}^2$, and let Γ_∂ signify the union of its one-dimensional open edges, which are assumed to be sufficiently smooth (in a sense defined rigorously later). We consider the convection-diffusion-reaction equation

$$\mathcal{L}u \equiv -\nabla \cdot (a\nabla u) + b \cdot \nabla u + cu = f \quad \text{in } \Omega, \tag{1}$$

where $f \in L^2(\Omega)$, $c \in L^\infty(\Omega)$, $b = (b_1, b_2)^T$ is a vector function whose entries $b_i, i = 1, 2$, are Lipschitz continuous real-valued functions on Ω, and $a = \{a_{ij}\}_{i,j=1}^2$ is a *symmetric* matrix whose entries a_{ij} are bounded, piecewise continuous real-valued functions defined on $\bar{\Omega}$, with

$$\zeta^T a(x)\zeta \geq 0 \quad \forall \zeta \in \mathbb{R}^2, \qquad \text{a.e. } x \in \bar{\Omega}. \tag{2}$$

Under this hypothesis, (1) is termed a *partial differential equation with nonnegative characteristic form.* By $\mu(x) = (\mu_1, \mu_2)^T$ we denote the unit outward normal vector to Γ_∂ at $x \in \Gamma_\partial$. On introducing the so called *Fichera function* $b \cdot \mu$ (cf. [8]), we define

$$\Gamma_0 = \left\{x \in \Gamma_\partial : \mu^T(x)a(x)\mu(x) > 0\right\},$$

$$\Gamma_- = \{x \in \Gamma_\partial \backslash \Gamma_0 : b(x) \cdot \mu(x) < 0\}, \quad \Gamma_+ = \{x \in \Gamma_\partial \backslash \Gamma_0 : b(x) \cdot \mu(x) \geq 0\}.$$

The sets Γ_- and Γ_+ are referred to as *inflow* and *outflow* boundary, respectively. We can also see that $\Gamma_\partial = \Gamma_0 \cup \Gamma_- \cup \Gamma_+$. If Γ_0 has positive one-dimensional Hausdorff measure, we also decompose Γ_0 into two parts $\Gamma_{\mathrm{D}}, \Gamma_{\mathrm{N}}$ and we impose Dirichlet and Neumann boundary conditions, respectively, via

$$\begin{aligned} u &= g_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}} \cup \Gamma_-, \\ (a\nabla u) \cdot \mu &= g_{\mathrm{N}} \text{ on } \Gamma_{\mathrm{N}}, \end{aligned} \tag{3}$$

where we adopt the (physically reasonable) hypothesis that $b \cdot \mu \geq 0$ on Γ_{N}, whenever the latter is nonempty.

Existence and uniqueness of solutions (in the weak sense) for the corresponding homogeneous problem were considered by Fichera [8, 9], Oleĭnik & Radkevič [24] and Houston & Süli [19], under the assumption that there exists a positive constant γ_0 such that

$$c(x) - \frac{1}{2}\nabla \cdot b(x) \geq \gamma_0 \;\; \text{for almost every } x \in \Omega. \tag{4}$$

3. Function Spaces

We shall denote by $H^s(\Omega)$ the standard Hilbertian Sobolev space of index $s \geq 0$ of real-valued functions defined on Ω.

Let $\mathcal{T}$ be a subdivision of the polygonal domain Ω into disjoint open (curvilinear) quadrilateral elements κ constructed via mappings $Q_\kappa \circ F_\kappa$, where $F_\kappa : \hat{\kappa} := (-1, 1)^2 \to \tilde{\kappa}$ is an affine mapping of the form

$$F_\kappa(\vec{x}) := A_\kappa \vec{x} + \vec{b}_\kappa, \tag{5}$$

with $A_\kappa := \frac{1}{2}\operatorname{diag}(h_1^\kappa, h_2^\kappa)$, where h_1^κ and h_2^κ are the lengths of the edges of $\tilde{\kappa}$ parallel to the $\tilde{x}_1$- and $\tilde{x}_2$-axes, respectively, $\vec{b}_\kappa$ is a two-component real-valued vector, and $Q_\kappa : \tilde{\kappa} \to \kappa$ is a C^1-diffeomorphism (cf. Figure 1).

Heuristically, we can say that the affine mapping F_κ defines the size of the element κ and the diffeomorphism Q_κ defines the "shape". For this reason, we shall be working with diffeomorphisms that are close to the identity in the following sense: the Jacobian J_{Q_κ} of Q_κ satisfies

$$C_1^{-1} \leq \det J_{Q_\kappa} \leq C_1, \quad \|(J_{Q_\kappa})_{ij}\|_{L^\infty(\kappa)} \leq C_2, \; i, j = 1, 2 \;\; \text{for all} \;\; \kappa \in \mathcal{T},$$

uniformly throughout the mesh for some positive constants C_1, C_2.

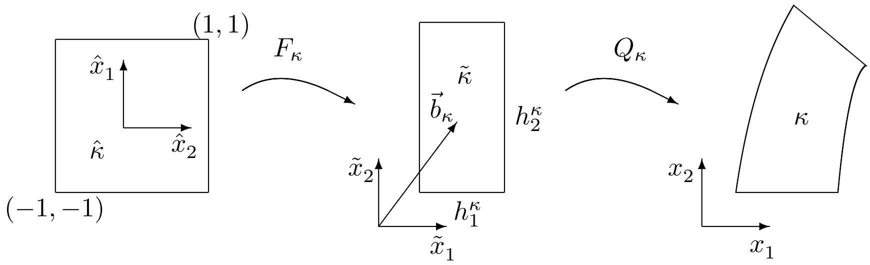


FIGURE 1. Construction of elements via composition of affine maps and diffeomorphisms.

The above maps are assumed to be constructed so as to ensure that the union of the closures of the disjoint open elements $\kappa \in \mathcal{T}$ forms a covering of the closure of Ω, i.e., $\bar{\Omega} = \cup_{\kappa\in\mathcal{T}}\bar{\kappa}$.

We shall also make use of *augmented Sobolev spaces*, introduced in [13].

Definition 3.1. *Let s be a nonnegative integer and κ an open (curvilinear) quadrilateral domain constructed as the image of an axiparallel rectangle $\tilde{\kappa}$ through a C^{s+1}-diffeomorphism Q_κ, as indicated above. We define the* augmented Sobolev space of order *s on κ by*

$$\tilde{H}^s(\kappa) := \{(u\circ Q_\kappa) \in H^s(\tilde{\kappa}) : \textit{for } (\alpha,\beta)\in I_A,\ \partial_1^\alpha\partial_2^\beta(u\circ Q_\kappa)\in L^2(\tilde{\kappa})\}, \tag{6}$$

where

$$I_A := \{(\alpha,\beta)\in\mathbb{N}_0^2 : \alpha+\beta = s+1,\ \textit{excluding } (s+1,0) \textit{ and } (0,s+1)\},$$

with associated norm $\|\cdot\|_{\tilde{H}^s(\kappa)}$ and seminorms $|\cdot|_{\tilde{H}^s(\kappa),i}$:

$$\|u\|_{\tilde{H}^s(\kappa)} := \left(\|u\circ Q_\kappa\|^2_{H^s(\tilde{\kappa})} + \sum_{(\alpha,\beta)\in I_A}\|\partial_1^\alpha\partial_2^\beta(u\circ Q_\kappa)\|^2_{\tilde{\kappa}}\right)^{\frac{1}{2}},$$

$$|u|_{\tilde{H}^s(\kappa),i} := \left(\|\partial_i^s(u\circ Q_\kappa)\|^2_{\tilde{\kappa}} + h_\kappa^2\|\partial_i^s\partial_j(u\circ Q_\kappa)\|^2_{\tilde{\kappa}} + h_\kappa^2\|\partial_i^{s-1}\partial_j^2(u\circ Q_\kappa)\|^2_{\tilde{\kappa}}\right)^{\frac{1}{2}},$$

for $i,j = 1,2$, with $i\neq j$, and $h_\kappa :=$ diam(κ).

Augmented Sobolev spaces were introduced in [13] to provide sufficient regularity assumptions for the first fully *hp*-optimal error analysis of the *hp*-DGFEM for the boundary-value problem (1), (3) in the special case when a is strictly positive definite and $b \equiv \vec{0}$.

Since the *hp*-DGFEM is a non-conforming method, it is necessary to introduce the notion of a *broken* Sobolev space.

Definition 3.2. *We define the* broken Sobolev space *of composite order $\boldsymbol{s}$ on an open set Ω, subject to a subdivision $\mathcal{T}$ of Ω, as*

$$H^{\boldsymbol{s}}(\Omega,\mathcal{T}) = \{u\in L^2(\Omega) : u|_\kappa \in \mathcal{H}^{s_\kappa}(\kappa)\ \forall\kappa\in\mathcal{T}\},$$

with $\mathcal{H}^{s_\kappa}(\kappa)\in\{H^{s_\kappa}(\kappa),\tilde{H}^{s_\kappa}(\kappa)\}$, s_κ being the local *Sobolev index on the element κ, and $\boldsymbol{s} := (s_\kappa : \kappa\in\mathcal{T})$; when $s_\kappa = s$ for all $\kappa\in\mathcal{T}$, we shall write $H^s(\Omega,\mathcal{T})$.*

4. Weak Formulation

Let us first introduce some notation. Let $\mathcal{T}$ be a subdivision of Ω into elements κ, as described in the previous section. By $\mathcal{E}$ we denote the set of all open one-dimensional element faces associated with the subdivision $\mathcal{T}$, and we define $\Gamma := \cup_{e\in\mathcal{E}} e$. We also assume that $\mathcal{E}$ is decomposed into two subsets, namely $\mathcal{E}_{\rm int}$ and $\mathcal{E}_\partial$, which contain the set of all elements of $\mathcal{E}$ that are not subsets of $\partial\Omega$ and the set of all elements of $\mathcal{E}$ that are subsets of $\partial\Omega$, respectively. $\mathcal{E}_\partial$ is further decomposed into $\mathcal{E}_{\rm D}$ and $\mathcal{E}_{\rm N}$, such that $\Gamma_{\rm D} := \cup_{e\in\mathcal{E}_{\rm D}} e$, $\Gamma_{\rm N} := \cup_{e\in\mathcal{E}_{\rm N}} e$ and $\Gamma_{\rm int} := \cup_{e\in\mathcal{E}_{\rm int}} e$, all with the obvious meanings. Thus, introducing an element numbering, and given an interface $e \in \mathcal{E}_{\rm int}$, there exist indices i and j such that $i > j$ and the elements $\kappa := \kappa_i$ and $\kappa' := \kappa_j$ share the edge e. Then we define the *jump* of a function $u \in H^1(\Omega, \mathcal{T})$ across e and the *mean value* of u on e by

$$[u]_e := u|_{\partial\kappa\cap e} - u|_{\partial\kappa'\cap e} \quad \text{and} \quad \langle u\rangle_e := \frac{1}{2}(u|_{\partial\kappa\cap e} + u|_{\partial\kappa'\cap e}),$$

respectively, where $\partial\kappa$ denotes the union of all open edges of the element κ. With each edge we associate the unit normal vector ν pointing from element κ_i to κ_j when $i > j$; we choose ν to be the unit outward normal μ when $e \in \mathcal{E}_\partial$.

Also, we divide the union of all open edges $\partial\kappa$ of every element κ into two subsets

$$\partial_-\kappa := \{x \in \partial\kappa : b(x)\cdot\mu(x) < 0\}, \quad \partial_+\kappa := \{x \in \partial\kappa : b(x)\cdot\mu(x) > 0\},$$

where $\mu(\cdot)$ denotes the unit outward normal vector function associated with the element κ; we call these the *inflow* and *outflow* parts of $\partial\kappa$ respectively. Then, for every element $\kappa \in \mathcal{T}$, we denote by u_κ^+ the trace of u on $\partial\kappa$ taken from within the element κ (interior trace). We also define the exterior trace u_κ^- of $u \in H^1(\Omega,\mathcal{T})$ for almost all $x \in \partial_-\kappa\backslash\Gamma$ to be the interior trace $u_{\kappa'}^+$ of u on the element(s) κ' that share the edges contained in $\partial_-\kappa\backslash\Gamma$ of the boundary of element κ. Then, the *jump* of u across $\partial_-\kappa\backslash\Gamma$ is defined by

$$\lfloor u\rfloor_\kappa := u_\kappa^+ - u_\kappa^-.$$

We note that this definition of jump is not the same as the one define above; here the sign of the jump depends on the direction of the flow, whereas $[\cdot]$ depends only on the element-numbering. We note that the subscripts in these definitions will be suppressed when no confusion is likely to occur.

The broken weak formulation of the problem (1), (3), from which the interior penalty DGFEM will emerge, reads

$$\text{find } u \in A \text{ such that } B(u,v) = l(v) \quad \forall v \in H^2(\Omega,\mathcal{T}), \tag{7}$$

where

$$A := \{u \in H^2(\Omega,\mathcal{T}) : u,\ (a\nabla u)\cdot\nu \text{ are continuous across all } e \in \mathcal{E}_{\rm int}\},$$

$$\begin{aligned} B(u,v) \quad := \quad & \sum_{\kappa\in\mathcal{T}} \int_\kappa a\nabla u\cdot\nabla v\,\mathrm{d}x + \sum_{\kappa\in\mathcal{T}}\int_\kappa (b\cdot\nabla u + cu)v\,\mathrm{d}x \\ & - \sum_{\kappa\in\mathcal{T}} \int_{\partial_-\kappa\cap(\Gamma_-\cup\Gamma_{\rm D})} (b\cdot\mu)u^+v^+\mathrm{d}s - \sum_{\kappa\in\mathcal{T}}\int_{\partial_-\kappa\backslash\Gamma_\partial} (b\cdot\mu)\lfloor u\rfloor v^+\mathrm{d}s \\ & + \int_{\Gamma_{\rm D}} \{\theta((a\nabla v)\cdot\mu)u - ((a\nabla u)\cdot\mu)v\}\mathrm{d}s + \int_{\Gamma_{\rm D}} \sigma uv\,\mathrm{d}s \\ & + \int_{\Gamma_{\rm int}} \{\theta\langle(a\nabla v)\cdot\nu\rangle[u] - \langle(a\nabla u)\cdot\nu\rangle[v]\}\mathrm{d}s + \int_{\Gamma_{\rm int}} \sigma[u][v]\,\mathrm{d}s, \end{aligned}$$

and

$$
\begin{aligned}
l(v) :\quad = \quad & \sum_{\kappa\in\mathcal{T}} \int_\kappa f v \,\mathrm{d}x - \sum_{\kappa\in\mathcal{T}} \int_{\partial_-\kappa\cap(\Gamma_-\cup\Gamma_\mathrm{D})} (b\cdot\mu) g_\mathrm{D} v^+ \mathrm{d}s \\
& + \int_{\Gamma_\mathrm{D}} \theta((a\nabla v)\cdot\mu) g_\mathrm{D}\,\mathrm{d}s + \int_{\Gamma_\mathrm{N}} g_\mathrm{N} v\,\mathrm{d}s + \int_{\Gamma_\mathrm{D}} \sigma g_\mathrm{D} v\,\mathrm{d}s, \qquad (8)
\end{aligned}
$$

for $\theta \in \{-1, 1\}$, with the function σ to be defined in the error analysis below.

5. Finite Element Spaces

We shall restrict ourselves to meshes that are unions of diffeomorphic images of rectangles and to tensor-product polynomial spaces. In the case of shape-regular triangular and rectangular elements, the error analysis has already been carried out in [18]. For considerations on anisotropic triangles for the h-version FEM we refer to [1, 10] and the references therein.

Let $\hat{I} \equiv (-1, 1)$ and $\hat{\kappa} \equiv \hat{I} \times \hat{I} = (-1, 1)^2$. On the interval $\hat{I}$ we denote the space of polynomials of degree p or less by $\mathcal{P}_p(\hat{I})$. Then, for $\vec{p} := (p_1, p_2)$, the *anisotropic tensor-product polynomial space* $\mathcal{Q}_{\vec{p}}$ is defined by $\mathcal{Q}_{\vec{p}}(\hat{\kappa}) := \mathcal{P}_{p_1}(\hat{I}) \otimes \mathcal{P}_{p_2}(\hat{I})$, where $\otimes$ denotes the standard functional tensor product.

Let $\mathcal{T}$ be a subdivision of the computational domain Ω into elements $\kappa \in \mathcal{T}$ and let $\mathbf{F} = \{F_\kappa : \kappa \in \mathcal{T}\}$, $\mathbf{Q} = \{Q_\kappa : \kappa \in \mathcal{T}\}$, where F_κ, Q_κ are the maps defined in Section 3.

Definition 5.1. *Let $\vec{\mathbf{p}} := (\vec{p}_\kappa : \kappa \in \mathcal{T})$ be the $2 \times |\mathcal{T}|$-matrix containing the polynomial degree vectors $\vec{p}_\kappa$ of the elements in a given subdivision $\mathcal{T}$. We define the* finite element space *with respect to Ω, $\mathcal{T}$, $\mathbf{F}$ and $\vec{\mathbf{p}}$ by*

$$S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q}) := \{u \in L^2(\Omega) : u|_\kappa \circ Q_\kappa \circ F_\kappa \in \mathcal{Q}_{\vec{p}_\kappa}(\hat{\kappa})\}.$$

6. Discontinuous Galerkin Finite Element Method

Using the weak formulation stated in Section 4 and the finite element spaces from Section 5, the *discontinuous Galerkin finite element method* for the problem (1), (3) is defined as follows:

(9) find $u_\mathrm{DG} \in S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q})$ such that $B(u_\mathrm{DG}, v) = l(v) \quad \forall v \in S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q})$,

with the function σ contained in $B(\cdot, \cdot)$ and in $l(\cdot)$ to be defined in the error analysis. We shall refer to the DGFEM with $\theta = -1$ as the *symmetric interior penalty discontinuous Galerkin finite element method*, whereas for $\theta = 1$ the DGFEM will be referred to as the *non-symmetric interior penalty discontinuous Galerkin finite element method.* This terminology stems from the fact that when $b \equiv \vec{0}$, the bilinear form $B(\cdot, \cdot)$ is symmetric if, and only if, $\theta = -1$.

We make some assumptions on the regularity of the solution and on the functions in the finite element space $S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q})$. We assume that $p_i^\kappa \geq 1$, $i = 1, 2$, $\kappa \in \mathcal{T}$, whenever diffusion is present, in order to ensure that the matrix of the system of linear algebraic equations that arises from (9) is nonsingular. When the analytical solution $u \in A$, the following *Galerkin orthogonality property* holds: $B(u - u_\mathrm{DG}, v) = 0$ for every $v \in S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q})$. If the continuity assumptions involved in the definition of A are violated, as is the case, for example, in an elliptic transmission problem, the DGFEM has to be modified accordingly.

7. Projection Operators and Inverse Inequalities

7.1. L^2-Orthogonal Projection Operator. Let $\hat{u} \in L^2(\hat{I})$, with $\hat{I} \equiv (-1, 1)$. We define the L^2-orthogonal projection $\hat{\pi}_p$ on $\hat{I}$ in a standard fashion by means of truncated Legendre series (see, e.g., [27]). Also, we denote by $\Phi_1(p,s)$ and $\Phi_2(p,s)$ the quantities

$$\Phi_1(p,s) := \left(\frac{\Gamma(p-s+1)}{\Gamma(p+s+1)}\right)^{\frac{1}{2}} \quad \text{and} \quad \Phi_2(p,s) := \frac{\Phi_1(p,s)}{\sqrt{p(p+1)}}, \tag{10}$$

respectively, with p,s real numbers such that $0 \le s \le p$ and $\Gamma(\cdot)$ being the Gamma function; we also adopt the standard convention $\Gamma(1) = 0! = 1$. We remark on the asymptotic behaviour of $\Phi_1(p,s)$: making use of Stirling's formula we obtain $\Phi_1(p,s) \le C(s)p^{-s}$, for $p \ge 1$, with $0 \le s \le p$ and $C(s)$ denoting a generic constant depending on s. We recall the following approximation result (Theorem 3.11 in [27]).

Lemma 7.1. *Let $\hat{u} \in H^{k+1}(\hat{I})$, $k \ge 0$; then, for every integer s such that $0 \le s \le \min\{p+1, k+1\}$, the following estimate holds:*

$$\|\hat{u} - \hat{\pi}_p\hat{u}\|_{\hat{I}} \le \Phi_1(p+1,s)\|\hat{u}^{(s)}\|_{\hat{I}}. \tag{11}$$

Next, we recall a result concerning the estimation of the approximation error of the L^2-projection on the boundary (Lemma 3.5 in [18]).

Lemma 7.2. *Let $\hat{u} \in H^{k+1}(\hat{I})$ for some integer $k \ge 0$; then, for $0 \le t \le \min\{p,k\}$, $p \ge 0$, we have*

$$|(\hat{u} - \hat{\pi}_p\hat{u})(\pm 1)| \le \frac{\Phi_1(p,t)}{\sqrt{2p+1}}\|\hat{u}^{(t+1)}\|_{\hat{I}}. \tag{12}$$

Also, we shall make use of the following "commutation error" bound. For a proof we refer to [12]. A result of this form in several space dimensions (but with unspecified constants) can be found in [4] (Lemma 2.3).

Lemma 7.3. *Let $\hat{u} \in H^{k+1}(\hat{I})$, $k \ge 0$, and let $\hat{\pi}_p\hat{u} \in \mathcal{P}_p(\hat{I})$ be its L^2-projection with $p \ge 0$; then*

$$\|\hat{\pi}_p\hat{u}' - (\hat{\pi}_p\hat{u})'\|_{\hat{I}} \quad \le \quad C_p^{L^2}\Phi_1(p,s)\|\hat{u}^{(s+1)}\|_{\hat{I}}, \tag{13}$$

for any $0 \le s \le \min\{p,k\}$, with $C_0^{L^2} = 1$, and $C_p^{L^2} = (2p+2)^{\frac{1}{2}}$, for $p \ge 1$. Moreover,

$$\|\hat{u}' - (\hat{\pi}_p\hat{u})'\|_{\hat{I}} \quad \le \quad (1 + C_p^{L^2})\Phi_1(p,s)\|\hat{u}^{(s+1)}\|_{\hat{I}}, \tag{14}$$

for any $0 \le s \le \min\{p,k\}$.

We extend the notion of the L^2-orthogonal projection operator to two dimensions. Let $\hat{\kappa} \equiv (-1,1)^2$. Then the L^2-orthogonal projection with composite polynomial degree vector $\vec{p} = (p_1,p_2)$, is defined by

$$\hat{\Pi}_{\vec{p}} = \hat{\pi}^1_{p_1}\hat{\pi}^2_{p_2} := (\hat{\pi}^1_{p_1} \otimes I) \circ (I \otimes \hat{\pi}^2_{p_2}),$$

where the superscripts $1,2$ indicate the directions in which the one-dimensional projections are applied.

We derive error estimates for functions defined on unions of diffeomorphic images of axiparallel rectangular domains. The domains of our interest will be elements of a given subdivision $\mathcal{T}$ of the computational domain Ω, admitting the properties stated in Section 3. We shall *not* require any shape-regularity hypotheses of the form

$$C^{-1} \le R_{\tilde{\kappa}}/r_{\tilde{\kappa}} \le C, \tag{15}$$

for $\tilde{\kappa}$, where $R_{\tilde{\kappa}}, r_{\tilde{\kappa}}$ denote the radii of the circumcircle and inscribed circle of $\tilde{\kappa}$, respectively. Indeed, instead of working with the diameter, $h_\kappa := \mathrm{diam}(\kappa)$, of the element κ as a measure of the meshsize, we derive bounds on the approximation error in terms of the elemental directional magnitudes h_1^κ and h_2^κ.

Definition 7.4. *Let $\tilde{u} : \tilde{\kappa} \to \mathbb{R}$ and $u : \kappa \to \mathbb{R}$ and assume that there exist mappings $F_\kappa : \hat{\kappa} \to \tilde{\kappa}$, $Q_\kappa : \tilde{\kappa} \to \kappa$ as above. We define the L^2-projection operator $\tilde{\Pi}_{\vec{p}}$ on $\tilde{\kappa}$, with $\vec{p} = (p_1, p_2)$ being the composite polynomial degree vector, by the relation*

$$\tilde{\Pi}_{\vec{p}}\tilde{u} := (\hat{\Pi}_{\vec{p}}(\tilde{u} \circ F_\kappa)) \circ F_\kappa^{-1}, \quad \text{for } \tilde{u} \in L^2(\tilde{\kappa}),$$

where, as before, $\hat{\Pi}_{\vec{p}}$ denotes the L^2-orthogonal projection onto the reference element $\hat{\kappa}$. Moreover, we define the L^2-orthogonal projection operator $\Pi_{\vec{p}}$ on κ, with $\vec{p} = (p_1, p_2)$, by

$$\Pi_{\vec{p}}u := (\tilde{\Pi}_{\vec{p}}(u \circ Q_\kappa)) \circ Q_\kappa^{-1}, \quad \text{for } u \in L^2(\kappa).$$

We introduce some notation which we shall use in the approximation estimates below. Let $J_{Q_\kappa} = \big((J_{Q_\kappa})_{ij}\big)_{i,j=1,2}$ denote the Jacobian of Q_κ. We then define, for $i, j = 1, 2$:

$$C_\kappa := \| \det J_{Q_\kappa} \|_{L^\infty(\tilde{\kappa})}^{\frac{1}{2}}, \quad C'_\kappa := \|(\det J_{Q_\kappa})^{-1}\|_{L^\infty(\tilde{\kappa})}^{\frac{1}{2}}, \quad C_\kappa^{ij} := \|(J_{Q_\kappa})_{ij}\|_{L^\infty(\tilde{\kappa})},$$

$$J_{Q_\kappa}^i := \big((J_{Q_\kappa})_{ii}^2 + (J_{Q_\kappa})_{ij}^2\big)^{\frac{1}{2}} \text{ for } i \neq j, \; C_{\partial\kappa}^i := \|J_{Q_\kappa}^i\|_{L^\infty(\partial\tilde{\kappa}_i)}. \tag{16}$$

Lemma 7.5. *Let $u \in H^{k+1}(\kappa)$, for $k \geq 1$, and let Q_κ be a C^{k+1}-diffeomorphism; then, for $\tilde{u} := u \circ Q_\kappa$, $\vec{p} = (p_1, p_2)$ and $p_1, p_2 \geq 1$, we have*

$$(C_\kappa)^{-1}\|u - \Pi_{\vec{p}}u\|_\kappa \leq M_\kappa^0 := \sum_{i=1}^{2} \Phi_2(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i+1} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}}, \tag{17}$$

and

$$\|\partial_i(u - \Pi_{\vec{p}}u)\|_\kappa \leq C_{\kappa,i}^1 M_{\kappa,i}^1 + C_{\kappa,i}^2 M_{\kappa,j}^1, \tag{18}$$

with

$$M_{\kappa,i}^1 := 4p_i^{\frac{1}{2}}\Phi_1(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \Phi_1(p_j, s_j)\left(\frac{h_j}{2}\right)^{s_j} \|\tilde{\partial}_j^{s_j}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}, \tag{19}$$

where $i, j = 1, 2$, $i \neq j$, $0 \leq s_i \leq \min\{p_i, k\}$, for $i = 1, 2$, $\tilde{\partial}_i$ is the partial derivative in $\tilde{x}_i$-direction in the $\tilde{x}_1\tilde{x}_2$-plane, and

$$C_{\kappa,i}^1 := \begin{cases} 1, & \textit{if } Q_\kappa = \mathrm{id}, \\ \sqrt{2}C_\kappa^{jj}C'_\kappa, & \textit{otherwise} \end{cases}, \quad C_{\kappa,i}^2 := \begin{cases} 0, & \textit{if } Q_\kappa = \mathrm{id}, \\ \sqrt{2}C_\kappa^{ji}C'_\kappa, & \textit{otherwise} \end{cases}.$$

Proof. For (17), we have

$$\|u - \Pi_{\vec{p}}u\|_\kappa \leq C_\kappa\|\tilde{u} - \tilde{\Pi}_{\vec{p}}\tilde{u}\|_{\tilde{\kappa}} \leq C_\kappa\left(\frac{h_1}{2}\frac{h_2}{2}\right)^{\frac{1}{2}} \|\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u}\|_{\hat{\kappa}},$$

where $\hat{u} := \tilde{u} \circ F_\kappa$. Making use of the bound

$$\|\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u}\|_{\hat{\kappa}} \leq \sum_{i=1}^{2} \Phi_1(p_i + 1, t_i)\|\hat{\partial}_i^{t_i}\hat{u}\|_{\hat{\kappa}}, \tag{20}$$

with $0 \leq t_i \leq \min\{p_i + 1, k + 1\}$, whose proof can be found, e.g., in [18], noting that $\Phi_1(p_i + 1, t_i) \leq \Phi_2(p_i, s_i)$ for $t_i = s_i + 1$, and scaling the right-hand side back to $\tilde{\kappa}$, we obtain the result.

For (18), a change of variables yields

$$\begin{aligned} \|\partial_i(u - \Pi_{\vec{p}}u)\|_\kappa &\leq C_{\kappa,i}^1\|\tilde{\partial}_i(\tilde{u} - \tilde{\Pi}_{\vec{p}}\tilde{u})\|_{\tilde{\kappa}} + C_{\kappa,i}^2\|\tilde{\partial}_j(\tilde{u} - \tilde{\Pi}_{\vec{p}}\tilde{u})\|_{\tilde{\kappa}} \\ &\leq C_{\kappa,i}^1\left(\frac{h_j}{h_i}\right)^{\frac{1}{2}} \|\hat{\partial}_i(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\hat{\kappa}} + C_{\kappa,i}^2\left(\frac{h_i}{h_j}\right)^{\frac{1}{2}} \|\hat{\partial}_j(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\hat{\kappa}}, \end{aligned}$$

with $i, j = 1, 2$, $i \neq j$, where $\hat{\partial}_i$ has the obvious meaning. Without loss of generality, let $i = 1$; then

$$\|\hat{\partial}_1(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\hat{\kappa}} \leq \|\hat{\partial}_1\hat{u} - \hat{\Pi}_{\vec{p}}(\hat{\partial}_1\hat{u})\|_{\hat{\kappa}} + \|\hat{\Pi}_{\vec{p}}(\hat{\partial}_1\hat{u}) - \hat{\partial}_1\hat{\Pi}_{\vec{p}}\hat{u}\|_{\hat{\kappa}}.$$

Using (20), the first term on the right-hand side can be bounded as follows:

$$\|\hat{\partial}_1\hat{u} - \hat{\Pi}_{\vec{p}}(\hat{\partial}_1\hat{u})\|_{\hat{\kappa}} \leq \Phi_1(p_1 + 1, t_1)\|\hat{\partial}_1^{t_1+1}\hat{u}\|_{\hat{\kappa}} + \Phi_1(p_2 + 1, q_2)\|\hat{\partial}_2^{q_2}\hat{\partial}_1\hat{u}\|_{\hat{\kappa}}$$

with $0 \leq q_2 \leq \min\{p_2 + 1, k\}$, $0 \leq t_1 \leq \min\{p_1, k\}$. Also,

$$\|\hat{\Pi}_{\vec{p}}(\hat{\partial}_1\hat{u}) - \hat{\partial}_1\hat{\Pi}_{\vec{p}}\hat{u}\|_{\hat{\kappa}} = \|\hat{\pi}^2_{p_2}(\hat{\pi}^1_{p_1}(\hat{\partial}_1\hat{u}) - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u})\|_{\hat{\kappa}} \leq C^{L^2}_{p_1}\Phi_1(p_1, t_1)\|\hat{\partial}_1^{t_1+1}\hat{u}\|_{\hat{\kappa}};$$

in the equality, we used the commutativity of $\hat{\partial}_1$ with $\hat{\pi}^2_{p_2}$, and in the inequality we used the boundedness of $\hat{\pi}^2_{p_2}$, as well as (13) and $\Phi_1(p_1 + 1, t_1) \leq \Phi_1(p_1, t_1)$. Thus,

$$\|\hat{\partial}_i(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\hat{\kappa}} \leq (1 + C^{L^2}_{p_i})\Phi_1(p_i, t_i)\|\hat{\partial}_i^{t_i+1}\hat{u}\|_{\hat{\kappa}} + \Phi_1(p_j + 1, q_j)\|\hat{\partial}_j^{q_j}\hat{\partial}_i\hat{u}\|_{\hat{\kappa}}, \tag{21}$$

and the result follows, by scaling back to $\tilde{\kappa}$. □

Remark 7.6. *Here and in the subsequent discussion, we prefer to keep in the anisotropic bounds norms of derivatives of $\tilde{u}$ on $\tilde{\kappa}$ rather than norms of derivatives of u on κ, as we then obtain sharper bounds which indicate the different features of u in the actual directions of anisotropy.*

Lemma 7.7. *Let $u \in H^{k+1}(\kappa)$, with $k \geq 0$, and let Q_κ be a C^{k+1}-diffeomorphism; then, on defining $\partial\hat{\kappa}_1 := (-1, 1) \times \{\pm 1\}$, $\partial\hat{\kappa}_2 := \{\pm 1\} \times (-1, 1)$, $\partial\tilde{\kappa}_i := F_\kappa(\partial\hat{\kappa}_i)$ and $\partial\kappa_i := Q_\kappa(\partial\tilde{\kappa}_i)$, for $i = 1, 2$, we have*

$$\begin{aligned}(C^i_{\partial\kappa})^{-1}\|u - \Pi_{\vec{p}}u\|_{\partial\kappa_i} \leq M^0_{\partial\kappa,i} \quad := \quad & (2p_j)^{-\frac{1}{2}}\Phi_1(p_j, s_j)\left(\frac{h_j}{2}\right)^{s_j+\frac{1}{2}}\|\tilde{\partial}_j^{s_j+1}\tilde{u}\|_{\tilde{\kappa}} \\ & +\sqrt{2}p_i^{-\frac{1}{2}}\Phi_1(p_i, s_i)\left(\frac{h_i}{h_j}\right)^{\frac{1}{2}}\left(\frac{h_i}{2}\right)^{s_i+\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} \\ & +\sqrt{2}(p_i^{-\frac{1}{2}} + p_j^{-\frac{1}{2}})\Phi_1(p_i, s_i)\left(\frac{h_j}{2}\right)^{\frac{1}{2}}\left(\frac{h_i}{2}\right)^{s_i}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}},\end{aligned}$$

with $i, j = 1, 2$, $i \neq j$, $0 \leq s_i \leq \min\{p_i, k\}$, $p_i \geq 1$, for $i = 1, 2$, $\tilde{u} = u \circ Q_\kappa$ and $C^i_{\partial\kappa}$ as in (16).

Proof. For a complete proof we refer to [12] (Lemma 3.8, p. 49, and Lemma 3.17, p. 56). The proof there follows the ideas of the proof of Lemma 3.6 in [18] which is the corresponding result when $\vec{p} = (p, p)$. Here we only comment that the crucial ingredient of the proof is the use of the trace inequality

$$\|u\|_{\partial\hat{\kappa}_i} \leq \|u\|_{\hat{\kappa}} + 2\|u\|^{\frac{1}{2}}_{\hat{\kappa}}\|\hat{\partial}_j u\|^{\frac{1}{2}}_{\hat{\kappa}}, \tag{22}$$

which refines the trace inequality used therein; we then apply (22) with $u = \hat{u} - \hat{\Pi}_{\vec{p}}\hat{u}$. □

The above estimates hold also for $\vec{p} \in \{(0, 0), (0, 1), (1, 0)\}$, but these cases were not included in the bounds for simplicity of the exposition.

Next, we derive bounds on the H^1-norm of the approximation error on the boundary. We shall make use of the following inverse inequality [12].

Lemma 7.8. *Let $v \in \mathcal{Q}_{\vec{p}}(\hat{\kappa})$ with $\vec{p} = (p_1, p_2)$ and let $w_i \in C(\bar{\hat{\kappa}})$ be such that w_i is constant in the direction of x_j, for $i, j = 1, 2$, $i \neq j$; then,*

$$\|w_i v\|_{\partial\hat{\kappa}_i} \leq (p_j + 1)\|w_i v\|_{\hat{\kappa}}. \tag{23}$$

Lemma 7.9. *Let $u \in H^{k+1}(\kappa)$, with $k \geq 1$, and let Q_κ be a C^{k+1}-diffeomorphism; then the following error estimates hold:*

$$\|\partial_i(u - \Pi_{\vec{p}}u)\|_{\partial\kappa_i} \leq C^{1,i}_{\partial\kappa,i} M^1_{\partial\kappa,i} + C^{2,i}_{\partial\kappa,i} M^2_{\partial\kappa,i}, \tag{24}$$

$$\|\partial_j(u - \Pi_{\vec{p}}u)\|_{\partial\kappa_i} \leq C^{1,i}_{\partial\kappa,j} M^2_{\partial\kappa,i} + C^{2,i}_{\partial\kappa,j} M^1_{\partial\kappa,i}, \tag{25}$$

with

$$\begin{aligned} M^1_{\partial\kappa,i} &:= 4\sqrt{3}p_i\Phi_1(p_i,s_i)\Big(\frac{h_i}{2}\Big)^{s_i-\frac{1}{2}}\Bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} \\ &\quad + \Big(p_i^{-\frac{1}{2}} + p_j^{-\frac{1}{2}}\Big)\Big(\frac{h_j}{h_i}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\Bigg) \\ &\quad + \Big(\frac{15p_j}{28}\Big)^{\frac{1}{2}}\Phi_1(p_j,s_j)\Big(\frac{h_j}{2}\Big)^{s_j-\frac{1}{2}}\|\tilde{\partial}_j^{s_j}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}, \end{aligned} \tag{26}$$

for $i,j = 1,2,\ i \neq j,\ 0 \leq s_i \leq \min\{p_i,k\},\ p_i \geq 1,\ i = 1,2,$ and

$$\begin{aligned} M^2_{\partial\kappa,i} &:= (p_j+1)\Bigg(5p_j^{\frac{1}{2}}\Phi_1(p_j,s_j)\Big(\frac{h_j}{2}\Big)^{s_j-\frac{1}{2}}\|\tilde{\partial}_j^{s_j+1}\tilde{u}\|_{\tilde{\kappa}} \\ &\quad + \Phi_1(p_i,s_i)\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\Big(\frac{h_i}{2}\Big)^{s_i-\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\Bigg), \end{aligned} \tag{27}$$

for $1 \leq s_i \leq \min\{p_i,k\}$, and for $l = 1,2$,

$$C^{1,l}_{\partial\kappa,i} := \begin{cases} 1, & \text{if } Q_\kappa = \mathrm{id}, \\ \sqrt{2}C^l_{\partial\kappa}C^{jj}_\kappa(C'_\kappa)^2, & \text{otherwise} \end{cases}, \quad C^{2,l}_{\partial\kappa,i} := \begin{cases} 0, & \text{if } Q_\kappa = \mathrm{id}, \\ \sqrt{2}C^l_{\partial\kappa}C^{ji}_\kappa(C'_\kappa)^2, & \text{otherwise} \end{cases}.$$

Proof. Let $i = 1$; for $i = 2$ the proof is analogous. For the proof of (24), we first use the chain rule and then we apply a change of variables (cf. the proof of Lemma 7.5) to obtain

$$\begin{aligned} \|\partial_i(u - \Pi_{\vec{p}}u)\|_{\partial\kappa_i} &\leq C^{1,i}_{\partial\kappa,i}\|\tilde{\partial}_i(\tilde{u} - \tilde{\Pi}_{\vec{p}}\tilde{u})\|_{\partial\tilde{\kappa}_i} + C^{2,i}_{\partial\kappa,i}\|\tilde{\partial}_j(\tilde{u} - \tilde{\Pi}_{\vec{p}}\tilde{u})\|_{\partial\tilde{\kappa}_i} \\ &= C^{1,i}_{\partial\kappa,i}\Big(\frac{2}{h_i}\Big)^{\frac{1}{2}}\|\hat{\partial}_i(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\partial\hat{\kappa}_i} + \Big(\frac{2h_j^2}{h_i}\Big)^{\frac{1}{2}}C^{2,i}_{\partial\kappa,i}\|\hat{\partial}_j(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\partial\hat{\kappa}_i}. \end{aligned} \tag{28}$$

To bound $\|\hat{\partial}_i(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\partial\hat{\kappa}_i}$, we proceed as follows. Without loss of generality we assume that $i = 1$. We have, respectively,

$$\begin{aligned} \|\hat{\partial}_1(\hat{u} - \hat{\Pi}_{\vec{p}}\hat{u})\|_{\partial\hat{\kappa}_1} &= \|\hat{\partial}_1\hat{u} - \hat{\pi}^2_{p_2}(\hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u})\|_{\partial\hat{\kappa}_1} \\ &\leq \|\hat{\partial}_1\hat{u} - \hat{\pi}^2_{p_2}(\hat{\partial}_1\hat{u})\|_{\partial\hat{\kappa}_1} + \|\hat{\pi}^2_{p_2}(\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u})\|_{\partial\hat{\kappa}_1} \\ &\leq \|\hat{\partial}_1\hat{u} - \hat{\pi}^2_{p_2}(\hat{\partial}_1\hat{u})\|_{\partial\hat{\kappa}_1} + \|\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u}\|_{\partial\hat{\kappa}_1} \\ &\quad + \|(\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u}) - \hat{\pi}^2_{p_2}(\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u})\|_{\partial\hat{\kappa}_1}. \end{aligned} \tag{29}$$

The first and the third terms on the right-hand side of (29) can be bounded using (12) for $t = t_2$ and $t = 0$ respectively. For the second term on the right-hand side of (29), we use the trace inequality (22) and, observing that $\hat{\partial}_2\hat{\pi}^1_{p_1} = \hat{\pi}^1_{p_1}\hat{\partial}_2$, and using Cauchy's inequality in the form $2\beta\gamma \leq \beta^2/c + c\gamma^2$, with $c = ((p_i+1)^{\frac{1}{2}} - 1)^{-1}$, we deduce that

$$\|\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u}\|_{\partial\hat{\kappa}_1} \leq (c^{-1}+1)\|\hat{\partial}_1\hat{u} - \hat{\partial}_1\hat{\pi}^1_{p_1}\hat{u}\|_{\hat{\kappa}} + c\|\hat{\partial}_1(\hat{\partial}_2\hat{u}) - \hat{\partial}_1\hat{\pi}^1_{p_1}(\hat{\partial}_2\hat{u})\|_{\hat{\kappa}}.$$

Hence, we deduce that

$$\begin{aligned}\|\hat\partial_1(\hat u-\hat\Pi_{\vec p}\hat u)\|_{\partial\hat\kappa_1} &\le \frac{\Phi_1(p_2,t_2)}{\sqrt{2p_2+1}}\|\hat\partial_2^{t_2+1}\hat\partial_1\hat u\|_{\hat\kappa}+\frac{1}{\sqrt{2p_2+1}}\|\hat\partial_1(\hat\partial_2\hat u)-\hat\partial_1\hat\pi^1_{p_1}(\hat\partial_2\hat u)\|_{\hat\kappa}\\ &\quad +(c^{-1}+1)\|\hat\partial_1\hat u-\hat\partial_1\hat\pi^1_{p_1}\hat u\|_{\hat\kappa}+c\|\hat\partial_1(\hat\partial_2\hat u)-\hat\partial_1\hat\pi^1_{p_1}(\hat\partial_2\hat u)\|_{\hat\kappa};\end{aligned}$$

an application of (14) to the last three terms on the right-hand side of the last inequality yields

$$\begin{aligned}\|\hat\partial_i(\hat u-\hat\Pi_{\vec p}\hat u)\|_{\partial\hat\kappa_i} &\le \frac{\Phi_1(p_j,t_j)}{\sqrt{2p_j+1}}\|\hat\partial_j^{t_j+1}\hat\partial_i\hat u\|_{\hat\kappa}+(p_i+1)^{\frac12}(1+C^{L^2}_{p_i})\Phi_1(p_i,s_i)\|\hat\partial_i^{s_i+1}\hat u\|_{\hat\kappa}\\ &\quad +\Big(((p_i+1)^{\frac12}-1)^{-1}+(2p_j+1)^{-\frac12}\Big)(1+C^{L^2}_{p_i})\Phi_1(p_i,t_i)\|\hat\partial_i^{t_i+1}\hat\partial_j\hat u\|_{\hat\kappa},\end{aligned}\tag{30}$$

for $i=1$ and $j=2$, where $0\le t_i,q_i\le\min\{p_i,k-1\}$ and $0\le s_i\le\min\{p_i,k\}$ and $0\le r_i\le\min\{p_i+1,k\}$; the proof for $i=2$ and $j=1$ follows analogously.

To bound $\|\hat\partial_2(\hat u-\hat\Pi_{\vec p}\hat u)\|_{\partial\hat\kappa_1}$, we have:

$$\begin{aligned}\|\hat\partial_2(\hat u-\hat\Pi_{\vec p}\hat u)\|_{\partial\hat\kappa_1} &= \|\hat\partial_2\hat u-\hat\pi^1_{p_1}(\hat\partial_2\hat\pi^2_{p_2}\hat u)\|_{\partial\hat\kappa_1}\\ &\le \|\hat\partial_2\hat u-\hat\pi^2_{p_2}(\hat\partial_2\hat u)\|_{\partial\hat\kappa_1}+\|\hat\pi^2_{p_2}\hat\partial_2\hat u-\hat\pi^1_{p_1}(\hat\pi^2_{p_2}\hat\partial_2\hat u)\|_{\partial\hat\kappa_1}\\ &\quad +\|\hat\pi^1_{p_1}(\hat\pi^2_{p_2}\hat\partial_2\hat u-\hat\partial_2\hat\pi^2_{p_2}\hat u)\|_{\partial\hat\kappa_1}\\ &\le \|\hat\partial_2\hat u-\hat\pi^2_{p_2}(\hat\partial_2\hat u)\|_{\partial\hat\kappa_1}+(p_2+1)\|\hat\pi^2_{p_2}\hat\partial_2\hat u-\hat\pi^1_{p_1}(\hat\pi^2_{p_2}\hat\partial_2\hat u)\|_{\hat\kappa}\\ &\quad +(p_2+1)\|\hat\pi^1_{p_1}(\hat\pi^2_{p_2}\hat\partial_2\hat u-\hat\partial_2\hat\pi^2_{p_2}\hat u)\|_{\hat\kappa},\end{aligned}\tag{31}$$

with the last inequality emerging from applying (23), after noting that the quantities inside the norms of the last two terms on the right-hand side are polynomials in the $\hat x_2$-direction. The first term on the right-hand side of (31) can be bounded directly using (12). For the second term we have

$$\|\hat\pi^2_{p_2}\hat\partial_2\hat u-\hat\pi^1_{p_1}(\hat\pi^2_{p_2}\hat\partial_2\hat u)\|_{\hat\kappa}\le\Phi_1(p_i+1,r_i)\|\hat\pi^2_{p_2}\hat\partial_1^{r_i}\hat\partial_2\hat u\|_{\hat\kappa}\le\Phi_1(p_i+1,r_i)\|\hat\partial_1^{r_i}\hat\partial_2\hat u\|_{\hat\kappa}.$$

The last term on the right-hand side of (31), is bounded by first using the boundedness of the L^2-orthogonal projection and subsequently by applying (13), yielding

$$\begin{aligned}\|\hat\partial_j(\hat u-\hat\Pi_{\vec p}\hat u)\|_{\partial\hat\kappa_i} &\le \frac{\Phi_1(p_j,q_j)}{\sqrt{2p_j+1}}\|\hat\partial_j^{q_j+2}\hat u\|_{\hat\kappa}+(p_j+1)\Phi_1(p_i+1,r_i)\|\hat\partial_i^{r_i}\hat\partial_j\hat u\|_{\hat\kappa}\\ &\quad +(p_j+1)C^{L^2}_{p_j}\Phi_1(p_j,s_j)\|\hat\partial_j^{s_j+1}\hat u\|_{\hat\kappa},\end{aligned}\tag{32}$$

for $i=1$ and $j=2$, where $0\le t_i,q_i\le\min\{p_i,k-1\}$ and $0\le s_i\le\min\{p_i,k\}$ and $0\le r_i\le\min\{p_i+1,k\}$; the proof for $i=2$ and $j=1$ follows analogously. Now, choosing $t_i=s_i-1$, for $i=1,2$, in (30), using the relation

$$\frac{\Phi_1(p_j,s_j-1)}{\sqrt{2p_j+1}}\le\Big(\frac{15p_j}{28}\Big)^{\frac12}\Phi_1(p_j,s_j),$$

for $1\le s_j\le p_j$, and scaling back to $\tilde\kappa$ we obtain (24). Also, setting $q_i=s_i-1$, $r_i=s_i$ in (32), and scaling back to $\tilde\kappa$, (25) follows. □

Remark 7.10. *Setting $p_1=p_2$ and using Stirling's formula, as above, one can easily see that the bounds (18) and (24), (25) are optimal in the meshsize h but suboptimal in the polynomial degree p by half an order of p, and by a whole order of p, respectively. Similar bounds were derived in* [4], *for shape-regular domains.*

7.2. $\boldsymbol{H^1}$-Projection Operator. The p-suboptimality of the approximation bounds for the error of the L^2-projection noted in Remark 7.10 means that this projection operator is not particularly suitable for use in our error analysis of the hp-DGFEM when diffusion is present in the problem.

Motivated by [13], we shall also consider the H^1-projection operator which will be required in the error analysis of the hp-DGFEM for reaction-diffusion problems.

Definition 7.11. *In one dimension, we define the H^1-projection operator*

$$\hat{\lambda}_p : H^1(\hat{I}) \to \mathcal{P}_p(\hat{I}), \quad p \geq 1,$$

by setting, for $\hat{u} \in H^1(\hat{I})$,

$$(\hat{\lambda}_p \hat{u})(x) := \int_{-1}^{x} \hat{\pi}_{p-1}(\hat{u}')(\eta)\mathrm{d}\eta + \hat{u}(-1), \qquad x \in \hat{I} = (-1, 1),$$

with $\hat{\pi}_{p-1}$ the L^2-projection operator onto $\mathcal{P}_{p-1}(\hat{I})$. In analogy with Definition 7.4 we define the operators $\tilde{\Lambda}_{\vec{p}}$ and $\Lambda_{\vec{p}}$, both with the obvious meanings (see also Definition 6.10 in [13]*).*

As the corresponding bounds on the approximation error of the H^1-projection operator on the reference element and on the reference element boundary, along with their proofs, are included in [13] (Section 6.1), here we shall only present the bounds on general (possibly anisotropic) elements, thus extending the error estimates from Section 6.2 of [13] to the anisotropic setting. For the sake of brevity, for those of the results below which are straightforward extensions of the corresponding anisotropic result from [13] only a sketch of the proof is included.

Lemma 7.12. *Let κ be as above, let $u \in \tilde{H}^{k+1}(\kappa)$, for $k \geq 0$, and let Q_κ be a C^{k+2}-diffeomorphism; then, the following error estimates hold:*

$$\begin{aligned}(C_\kappa)^{-1}\|u - \Lambda_{\vec{p}} u\|_\kappa \quad \leq \quad \tilde{N}^0_\kappa &:= \sum_{i=1}^{2} \Phi_2(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i+1} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} \\ &+ \min_{i\in\{1,2\}} \left\{\Phi_2(p_i, 0)\Phi_2(p_j, s_j)\frac{h_i}{2}\left(\frac{h_j}{2}\right)^{s_j+1} \|\tilde{\partial}_j^{s_j+1}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}\right\}, \end{aligned} \tag{33}$$

and

$$\|\partial_i(u - \Lambda_{\vec{p}} u)\|_\kappa \leq C^1_{\kappa,i}\tilde{N}^1_{\kappa,i} + C^2_{\kappa,i}\tilde{N}^1_{\kappa,j}, \tag{34}$$

with

$$\tilde{N}^1_{\kappa,i} := \Phi_1(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \Phi_2(p_j, s_j)\left(\frac{h_j}{2}\right)^{s_j+1} \|\tilde{\partial}_j^{s_j+1}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}, \tag{35}$$

where $i, j = 1, 2$, $i \neq j$, $0 \leq s_i \leq \min\{p_i, k\}$, $p_i \geq 1$, for $i = 1, 2$, and C_κ, $C^1_{\kappa,i}, C^2_{\kappa,i}$ are as defined in Lemma 7.5.

If $u \in H^{k+1}(\kappa)$, for $k \geq 1$, then the bounds (33) and (34) hold, with

$$\begin{aligned} N^0_\kappa \quad &:= \quad \sum_{i=1}^{2} \Phi_2(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i+1} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} \\ &\quad + \min_{i\in\{1,2\}} \left\{\Phi_2(p_i, 0)\Phi_1(p_j, s_j)\frac{h_i}{2}\left(\frac{h_j}{2}\right)^{s_j} \|\tilde{\partial}_j^{s_j}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}\right\}, \end{aligned} \tag{36}$$

and

$$N^1_{\kappa,i} := \Phi_1(p_i, s_i)\left(\frac{h_i}{2}\right)^{s_i} \|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \Phi_1(p_j, s_j)\left(\frac{h_j}{2}\right)^{s_j} \|\tilde{\partial}_j^{s_j}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}, \tag{37}$$

replacing $\tilde{N}^0_\kappa$ and $\tilde{N}^1_{\kappa,i}$ respectively, where $1 \leq s_i \leq \min\{p_i, k\}$ and $p_i \geq 1$, for $i = 1, 2$.

Proof. The proof follows in a straightforward fashion combining the bounds in Lemma 6.7 of [13], together with the scaling argument presented in the proof of Lemma 7.5 above. □

Next, we present bounds on the approximation error on the boundary of κ.

Lemma 7.13. *Let $u \in \tilde{H}^{k+1}(\kappa)$, with $k \geq 0$, and let Q_κ be a C^{k+2}-diffeomorphism; then*

$$\|u - \Lambda_{\vec{p}} u\|_{\partial\kappa_i} \leq C^i_{\partial\kappa} \tilde{N}^0_{\partial\kappa,i}, \tag{38}$$

with

$$\tilde{N}^0_{\partial\kappa,i} := 2\Phi_2(p_i, s_i)\Big(\frac{h_i}{2}\Big)^{s_i+\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \frac{(h_1h_2)^{\frac{1}{2}}}{2}\|\tilde{\partial}_i^{s_i+1}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\bigg), \tag{39}$$

where $i, j = 1, 2$, $i \neq j$, $0 \leq s_i \leq \min\{p_i, k\}$, for $p_i \geq 1$, $i = 1, 2$.

Now, let $u \in H^{k+1}(\kappa)$, with $k \geq 1$; then (38) holds, with

$$N^0_{\partial\kappa,i} := \sqrt{2}p_i^{-\frac{1}{2}}\Phi_1(p_i, s_i)\Big(\frac{h_i}{2}\Big)^{s_i+\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \Big(\frac{h_j}{h_i}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\bigg), \tag{40}$$

replacing $\tilde{N}^0_{\partial\kappa,i}$, for $1 \leq s_i \leq \min\{p_i, k\}$, $p_i \geq 1$ for $i = 1, 2$.

Proof. The proof follows in a straightforward fashion combining the bounds in Lemma 6.8 of [13], together with the anisotropic scaling argument described above. □

Remark 7.14. *Making use of Stirling's formula, we can see that, working on augmented Sobolev spaces, we have* enhanced convergence *in the polynomial degree p by half an order of p. This is in line with our results in* [13] *for the case of shape-regular meshes.*

Finally, we present bounds for the H^1-approximation on the element boundary.

Lemma 7.15. *Let $u \in \tilde{H}^{k+1}(\kappa)$, $k \geq 1$, and let Q_κ be a C^{k+1}-diffeomorphism; then*

$$\|\partial_i(u - \Lambda_{\vec{p}} u)\|_{\partial\kappa_i} \leq C^{1,i}_{\partial\kappa,i}\tilde{N}^1_{\partial\kappa,i} + C^{2,i}_{\partial\kappa,i}\tilde{N}^2_{\partial\kappa,i}, \tag{41}$$

$$\|\partial_j(u - \Lambda_{\vec{p}} u)\|_{\partial\kappa_i} \leq C^{1,i}_{\partial\kappa,j}\tilde{N}^2_{\partial\kappa,i} + C^{2,i}_{\partial\kappa,j}\tilde{N}^1_{\partial\kappa,i}, \tag{42}$$

with

$$\begin{aligned}
\tilde{N}^1_{\partial\kappa,i} &:= 2\Phi_1(p_i, s_i)\Big(\frac{h_i}{2}\Big)^{s_i-\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \frac{(h_1h_2)^{\frac{1}{2}}}{2}\|\tilde{\partial}_i^{s_i+1}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\bigg),\\
\tilde{N}^2_{\partial\kappa,i} &:= \sqrt{2}p_i^{-\frac{1}{2}}\Phi_1(p_i, s_i)\Big(\frac{h_i}{2}\Big)^{s_i+\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}} + \Big(\frac{h_j}{h_i}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j^2\tilde{u}\|_{\tilde{\kappa}}\bigg)\\
&\quad + 2p_j^{\frac{1}{2}}\Phi_1(p_j, s_j)\Big(\frac{h_j}{2}\Big)^{s_j-\frac{1}{2}}\bigg(\|\tilde{\partial}_j^{s_j+1}\tilde{u}\|_{\tilde{\kappa}} + \frac{h_i}{2}\Phi_2(p_i, 0)\|\tilde{\partial}_j^{s_j+1}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}\bigg),
\end{aligned}$$

where $i, j = 1, 2$, $i \neq j$, $1 \leq s_i \leq \min\{p_i, k\}$, $p_i \geq 1$, $i = 1, 2$, and $C^{1,l}_{\partial\kappa,i}$ and $C^{2,l}_{\partial\kappa,i}$, $l = 1, 2$, are as in Lemma 7.9 above.

Now, let $u \in H^{k+1}(\kappa)$, with $k \geq 2$; then (41) and (42) hold with $\tilde{N}^1_{\partial\kappa,i}$ replaced by $N^1_{\partial\kappa,i}$ with

$$N^1_{\partial\kappa,i} = (2p_i)^{\frac{1}{2}}\Phi_1(p_i, s_i)\Big(\frac{h_i}{2}\Big)^{s_i-\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i+1}\tilde{u}\|_{\tilde{\kappa}} + \Big(\frac{h_j}{h_i}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}}\bigg), \tag{43}$$

with $i, j = 1, 2$, $i \neq j$, $1 \leq s_i \leq \min\{p_i, k\}$, $i = 1, 2$, and $\tilde{N}^2_{\partial\kappa,i}$ replaced by $N^2_{\partial\kappa,i}$

$$N^2_{\partial\kappa,i} = 2p_i^{\frac{1}{2}}\Phi_1(p_i,s_i)\Big(\frac{h_i}{2}\Big)^{s_i-\frac{1}{2}}\bigg(\Big(\frac{h_i}{h_j}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i}\tilde{\partial}_j\tilde{u}\|_{\tilde{\kappa}} + \Big(\frac{h_j}{h_i}\Big)^{\frac{1}{2}}\|\tilde{\partial}_i^{s_i-1}\tilde{\partial}_j^2\tilde{u}\|_{\tilde{\kappa}}\bigg)$$

$$+(2p_j)^{\frac{1}{2}}\Phi_1(p_j,s_j)\Big(\frac{h_j}{2}\Big)^{s_j-\frac{1}{2}}\bigg(\|\tilde{\partial}_j^{s_j+1}\tilde{u}\|_{\tilde{\kappa}} + 2p_j\frac{h_i}{h_j}\Phi_2(p_i,0)\|\tilde{\partial}_j^{s_j}\tilde{\partial}_i\tilde{u}\|_{\tilde{\kappa}}\bigg), \tag{44}$$

with $i,j=1,2,\ i\neq j,\ 2\leq s_i\leq\min\{p_i,k\},\ p_i\geq 2$ *for* $i=1,2$.

Proof. The proof follows in a straightforward fashion combining the bounds in Lemma 6.9 of [13], together with the scaling argument presented in the proof of Lemma 7.9 above. □

Remark 7.16. *The approximation estimates for the H^1-projection operator are optimal both in h and in p in each of the norms considered here.*

7.3. Exponential Convergence Estimates. With the aid of the following lemma, we shall be able to prove p-exponential bounds for the L^2- and H^1-projection operators.

Lemma 7.17. *Let $u:\tilde{\kappa}\to\mathbb{R}$ have an analytic extension to an open neighbourhood of $\bar{\tilde{\kappa}}$. Let, also, p,s be positive numbers such that $0\leq n\leq s:=\alpha p+n\leq p$, with $0<\alpha<1$; then the following bound holds:*

$$\Phi_1(p,s)\|\tilde{\partial}_i^{s+1}\tilde{\partial}_j^m\tilde{u}\|_{\tilde{\kappa}}\leq C_u p^{\frac{1}{2}\min\{3,n+\frac{5}{2}\}}\mathrm{e}^{-rp}\big(\operatorname{meas}_2(\tilde{\kappa})\big)^{\frac{1}{2}}, \tag{45}$$

where $r,C_u>0$ are constants that depend on n,u, $0\leq m\leq n$, with $i,j\in\{1,2\}$ for $i\neq j$, and $\operatorname{meas}_n(X)$ denotes the n-dimensional Lebesgue measure of the domain X.

For a proof, see [12], or Remark 3.9 in [17] for a similar argument.

7.4. Inverse Inequalities. In the error analysis in Section 8, we shall be interested in applying inverse inequalities to functions of the form $|\sqrt{a}\nabla v|$, where a denotes the diffusion tensor from (1) and $v\in\mathcal{Q}_{\vec{p}}(\kappa)$ with $\vec{p}=(p_1,p_2)$. We shall say that the tensor a has the *inverse property* if an inverse inequality of the form

$$\|\sqrt{a}\nabla v\|_{\partial\kappa_i}\leq C_{\mathrm{inv}}p_jh_j^{-\frac{1}{2}}\|\sqrt{a}\nabla v\|_{\kappa}, \tag{46}$$

holds, for all $v\in\mathcal{Q}_{\vec{p}}(\kappa)$, with $i,j=1,2$, $i\neq j$, $\vec{p}=(p_i,p_2)$, $p_1,p_2\geq 1$, where C_{inv} is a positive constant independent of v, p_j and h_j. We consider some examples. If $(\sqrt{a}\circ Q_\kappa)\in[\mathcal{Q}_{\vec{q}}(\kappa)]^{2\times 2}$ for every $\kappa\in\mathcal{T}$, for some (uniformly bounded, as we refine the mesh) composite polynomial degree $\mathbf{\vec{q}}:=(\vec{q}:\kappa\in\mathcal{T})$ then (46) holds. If, on axiparallel elements, a is of the form

$$a(x_1,x_2)=\begin{pmatrix}a_1(x_2)v_1(x_1) & 0\\ 0 & a_2(x_1)v_2(x_2)\end{pmatrix},$$

where a_1,a_2 are arbitrary element-wise bounded functions and v_1,v_2 are element-wise polynomial functions, then (46) holds. This family includes the Grušin-type operators considered below, whose diffusion tensor is of the form $a(x_1,x_2):=\operatorname{diag}(1,\lambda^2(x_1))$, with λ a bounded and Lipschitz-continuous function [14] (cf. also [24]). Finally, if a is positive definite and

$$0<c_a^{-1}\leq\|\,|\sqrt{a}|_F\|_{L^\infty(\partial\kappa)}\|\,|(\sqrt{a})^{-1}|_F\|_{L^\infty(\kappa)}\leq c_a,$$

uniformly on every $\kappa\in\mathcal{T}$, where by $|\cdot|_F$ we denote the Frobenius norm of a matrix, then (46) holds with $C_{\mathrm{inv}}=\sqrt{8}c_a$.

8. Error Analysis

We define the *energy norm* $|||\cdot|||$ by

$$|||w|||^2 \quad := \quad \sum_{\kappa\in\mathcal{T}} \bigg(\|\sqrt{a}\nabla w\|_\kappa^2 + \|c_0 w\|_\kappa^2 + \frac{1}{2}\|b_0 w^+\|^2_{\partial_-\kappa\cap(\Gamma_{\rm D}\cup\Gamma_-)} + \frac{1}{2}\|b_0 w^+\|^2_{\partial_+\kappa\cap\Gamma_\partial} + \frac{1}{2}\|b_0(w^+ - w^-)\|^2_{\partial_-\kappa\backslash\Gamma_\partial} \bigg) + \int_{\Gamma_{\rm D}} \sigma w^2 {\rm d}s + \int_{\Gamma_{\rm int}} \sigma [w]^2 {\rm d}s,$$

where $b_0 := \sqrt{|b\cdot\mu|}$, with μ on $\partial\kappa$ denoting the outward normal to $\partial\kappa$, $c_0(x) := (c(x) - \frac{1}{2}\nabla\cdot b(x))^{1/2}$ (whose radicand is, in a standard fashion, assumed to be nonnegative), and σ is a positive function on $\Gamma_{\rm D}\cup\Gamma_{\rm int}$.

We introduce some more notation. We decompose $\mathcal{E}_{\rm D}$ into two parts $\mathcal{E}_{\rm D}^1$ and $\mathcal{E}_{\rm D}^2$ containing those element edges that are images of the reference element edges parallel to the $\hat{x}_1$- and $\hat{x}_2$-axes respectively. Also, we shall denote the entries of a by $a_{11} = a_1$, $a_{22} = a_2$ and $a_{12} = a_{21} = a_3$, for brevity. For $\omega \subset \bar{\Omega}$, we define

$$a_i^\omega := \|a_i\|^{\frac{1}{2}}_{L^\infty(\omega)},\ i\in\{1,2,3\}, \quad c_\omega := \|c\|^{\frac{1}{2}}_{L^\infty(\omega)}, \text{ and } a^e_{i,\rm nor} = \|\sqrt{a}n\|_{L^\infty(e)}, \tag{47}$$

where n denotes the normal vector to the edge $e\in\mathcal{E}_{\rm D}^j\cup\mathcal{E}_{\rm int}^j$, for $i,j=1,2$, $i\neq j$. Note that $a^e_{i,\rm nor} = a_i^e$ for axiparallel elements. Also, we define

$$b_0^\omega := \|(b_0)^2\|^{\frac{1}{2}}_{L^\infty(\omega)}, \qquad b_i^\omega := \|(b_i)^2\|^{\frac{1}{2}}_{L^\infty(\omega)}, \quad i=1,2.$$

We assume that

$$b\cdot\nabla_{\mathcal{T}} v_h \in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q}) \quad \forall v_h\in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q}). \tag{48}$$

The necessity of the condition (48) will be commented on below.

Lemma 8.1. *Let Ω be a (curvilinear) polygonal domain, $\mathcal{T}$ a subdivision of Ω into (possibly anisotropic) elements, constructed as in Section 3, and assume that the diffusion tensor a admits the inverse property. We assign to every edge $e\in\mathcal{E}$ the positive real number σ_e such that*

$$\sigma_e \geq \begin{cases} 0, & \text{if } \theta = 1; \\ C_\sigma \langle \frac{(a^e_{j,\rm nor} p_j)^2}{h_j} \rangle_e, & \text{if } \theta = -1, \end{cases}$$

for C_σ sufficiently large constant, depending on the constants $C_{\rm inv}|_\kappa$ and $C_{\rm inv}|_{\kappa'}$ of the elements κ and κ' with $e\subset\bar{\kappa}\cap\bar{\kappa}'$, respectively (cf. (46)). Then, assuming that $w\in H^1(\Omega,\mathcal{T})$, we have

$$|||w||| \leq C_\theta B(w,w),$$

for $C_\theta := \min\{1, 1-\theta\}$, with $\theta = \{-1, 1\}$.

Proof. The proof is an extension of the corresponding proof for shape-regular elements [25, 18]. The main difference lies in the use of the anisotropic inverse inequality (46) (see [12] for details). □

The crucial importance in this choice of σ when $\theta = -1$ (symmetric version of the DGFEM) will be highlighted also in the numerical experiments.

We first present the general error bound for reaction-diffusion problems, i.e., when $b\equiv\vec{0}$. We decompose the error $u - u_{\rm DG}$, where u denotes the analytical solution, as $u - u_{\rm DG} = \eta + \xi$ where $\eta := u - \Lambda^{\mathcal{T}}_{\vec{\mathbf{p}}} u$, $\xi := \Lambda^{\mathcal{T}}_{\vec{\mathbf{p}}} u - u_{\rm DG}$, with the broken H^1-projection operator $\Lambda^{\mathcal{T}}_{\vec{\mathbf{p}}}$ defined element-wise by $(\Lambda^{\mathcal{T}}_{\vec{\mathbf{p}}} u)|_\kappa := \Lambda_{\vec{p}_\kappa}(u|_\kappa)$, with $\vec{\mathbf{p}}$ as in Definition 5.1 and $\Lambda_{\vec{p}_\kappa}$ denoting the H^1-projection operator on the element κ.

Theorem 8.2. *Let Ω be a (curvilinear) polygonal domain, $\mathcal{T}$ a subdivision of Ω into (possibly anisotropic) elements, constructed as in Section 3, and assume that the diffusion tensor a admits the inverse property. We assign to every edge $e \in \mathcal{E}$ the positive real number σ_e defined as in Lemma 8.1. Then, assuming that $u \in A \cap \tilde{H}^{\mathbf{k+1}}(\Omega,\mathcal{T})$, $\mathbf{k} := \big(k_\kappa : \kappa \in \mathcal{T}\big)$ with $k_\kappa \geq 1$, $\kappa \in \mathcal{T}$, the solution $u_{\mathrm{DG}} \in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q})$ obeys the error bound*

$$
\begin{aligned}
|||u-u_{\mathrm{DG}}||| \quad \leq \quad & C_\theta \sum_{\kappa\in\mathcal{T}} \sum_{i=1}^{2} \Bigg\{ \Big(2\sqrt{2}a_i^\kappa (C^1_{\kappa,i}\tilde{N}^1_{\kappa,i} + C^2_{\kappa,i}\tilde{N}^1_{\kappa,j}) + c_\kappa C_\kappa \tilde{N}^0_\kappa\Big) \\
& + \sum_{e\subset\partial\kappa_i} \Bigg\{ z_e\Bigg(\langle C_{\mathrm{inv}}\frac{(a_j^e p_j)^2}{h_j}\rangle_e^{\frac12} + 2\sigma_e^{\frac12}\Bigg) C^i_{\partial\kappa}\tilde{N}^0_{\partial\kappa,i} \\
& + \sqrt{2}\frac{\langle a^e_{j,\mathrm{nor}}\rangle_e}{\sigma_e^{\frac12}} \Big((\langle a_i^e\rangle_e C^{1,i}_{\partial\kappa,i} + \langle a_j^e\rangle_e C^{2,i}_{\partial\kappa,j})\tilde{N}^1_{\partial\kappa,i} \\
& + (\langle a_i^e\rangle_e C^{2,i}_{\partial\kappa,i} + \langle a_j^e\rangle_e C^{1,i}_{\partial\kappa,j})\tilde{N}^2_{\partial\kappa,i}\Big)\Bigg\}\Bigg\},
\end{aligned} \tag{49}
$$

where $\tilde{N}^0_\kappa, \tilde{N}^1_{\kappa,i}, \tilde{N}^0_{\partial\kappa,i}, \tilde{N}^1_{\partial\kappa,i}$ and $\tilde{N}^2_{\partial\kappa,i}$ are as in Section 7.2, with $i,j = 1,2,\ i \neq j$, and z_e is a "switch" taking the value 0 when the projection $\Lambda^{\mathcal{T}}_{\vec{\mathbf{p}}}u$ is continuous across the interface e, and 1 otherwise. We have also adopted the convention that the outer trace of a quantity on a boundary edge is equal to its inner trace.

Moreover, if $u \in A \cap H^{\mathbf{k+1}}(\Omega,\mathcal{T})$ with $k_\kappa \geq 2$, $\kappa \in \mathcal{T}$, the solution $u_{\mathrm{DG}} \in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q})$ obeys the error bound (49), with $N^0_\kappa, N^1_{\kappa,i}, N^0_{\partial\kappa,i}, N^1_{\partial\kappa,i}$ and $N^2_{\partial\kappa,i}$ replacing $\tilde{N}^0_\kappa, \tilde{N}^1_{\kappa,i}, \tilde{N}^0_{\partial\kappa,i}, \tilde{N}^1_{\partial\kappa,i}$ and $\tilde{N}^2_{\partial\kappa,i}$, respectively.

Proof. We have

$$|||u-u_{\mathrm{DG}}||| \leq |||\eta||| + |||\xi|||. \tag{50}$$

We now bound $|||\xi|||$ by a collection of norms of η. First, we observe that $\xi \in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q})$; the *Galerkin orthogonality property*

$$B(u-u_{\mathrm{DG}},\xi) = 0$$

then implies that

$$|||\xi|||^2 \leq C_\theta B(\xi,\xi) = B((u-u_{\mathrm{DG}})-\eta,\xi) = -B(\eta,\xi). \tag{51}$$

We shall now bound the terms appearing in (51), starting with

$$
\begin{aligned}
\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa a\nabla\eta\cdot\nabla\xi\mathrm{d}x\right| \quad \leq \quad & \left(\sum_{\kappa\in\mathcal{T}}\int_\kappa |\sqrt{a}\nabla\eta|^2\mathrm{d}x\right)^{\frac12}\left(\sum_{\kappa\in\mathcal{T}}\int_\kappa |\sqrt{a}\nabla\xi|^2\mathrm{d}x\right)^{\frac12} \\
\leq \quad & |||\xi|||\left(\sum_{\kappa\in\mathcal{T}}\|\sqrt{a}\nabla\eta\|^2_\kappa\right)^{\frac12} \\
\leq \quad & |||\xi|||\left(\sum_{i=1}^{2}\sum_{\kappa\in\mathcal{T}} 2(a_i^\kappa)^2\|\partial_i\eta\|^2_\kappa\right)^{\frac12},
\end{aligned} \tag{52}
$$

making use of the discrete and continuous versions of the Cauchy–Schwarz inequalities. Similarly,

$$\left|\sum_{\kappa\in\mathcal{T}}\int_\kappa c\eta\xi\mathrm{d}x\right| \leq |||\xi|||\left(\sum_{\kappa\in\mathcal{T}}\int_\kappa c\eta^2\mathrm{d}x\right)^{\frac12} \leq |||\xi|||\left(\sum_{\kappa\in\mathcal{T}}c^2_\kappa\|\eta\|^2_\kappa\right)^{\frac12}. \tag{53}$$

Also,

$$
\begin{aligned}
\left|\int_{\Gamma_{\mathrm{D}}^i} \eta((a\nabla\xi)\cdot\mu)\mathrm{d}s\right| &= \left|\sum_{e\in\mathcal{E}_{\mathrm{D}}^i}\int_e \sqrt{\gamma_e^i}(\eta\sqrt{a}\mu)\cdot(\sqrt{a}\nabla\xi)\frac{1}{\sqrt{\gamma_e^i}}\,\mathrm{d}s\right| \\
&\leq \sum_{e\in\mathcal{E}_{\mathrm{D}}^i}\left(\int_e \gamma_e^i|\eta\sqrt{a}\mu|^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_e|\sqrt{a}\nabla\xi|^2\frac{1}{\gamma_e^i}\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq C_{\mathrm{inv}}|||\xi|||\left(\sum_{\substack{e\in\mathcal{E}_{\mathrm{D}}^i\\ e\subset\bar{\kappa}}}\frac{(a_{j,\mathrm{nor}}^e p_j^\kappa)^2}{h_j^\kappa}\|\eta\|_e^2\right)^{\frac{1}{2}}, \qquad (54)
\end{aligned}
$$

with $i,j=1,2$, $i\neq j$, where, in the last inequality we have applied to the second integral the inverse inequality (46), and chosen $\gamma_e^i := (p_j^\kappa)^2/h_j^\kappa$, for $e\subset\bar{\kappa}$, with C_{inv} an appropriate constant. Next, we have

$$
\begin{aligned}
\left|\int_{\Gamma_{\mathrm{D}}}((a\nabla\eta)\cdot\mu)\xi\mathrm{d}s\right| &\leq \sum_{e\in\mathcal{E}_{\mathrm{D}}}\int_e\frac{1}{\sqrt{\sigma}}|\nabla\eta\cdot(a\mu)|\sqrt{\sigma}|\xi|\,\mathrm{d}s \\
&\leq \left(\sum_{e\in\mathcal{E}_{\mathrm{D}}}\sigma_e^{-1}\|(a\mu)\cdot\nabla\eta\|_e^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{\mathrm{D}}}\|\sqrt{\sigma}\xi\|_e^2\right)^{\frac{1}{2}} \\
&\leq |||\xi|||\left(\sum_{e\in\mathcal{E}_{\mathrm{D}}}\sigma_e^{-1}\|(a\mu)\cdot\nabla\eta\|_e^2\right)^{\frac{1}{2}}, \qquad (55)
\end{aligned}
$$

with $\sigma_e := \sigma|_e > 0$ to be defined below.

Similarly, we shall bound the terms involving integrals over the interior element edges. We have

$$
\begin{aligned}
\left|\int_{\Gamma_{\mathrm{int}}^i}[\eta]\langle(a\nabla\xi)\cdot\nu\rangle\mathrm{d}s\right| &\leq \frac{1}{2}\sum_{e\in\mathcal{E}_{\mathrm{int}}^i}\int_e|[\eta](\sqrt{a}\nu)|_\kappa|\sqrt{\tau_{\kappa,e}^i}\frac{1}{\sqrt{\tau_{\kappa,e}^i}}|(\sqrt{a}\nabla\xi)|_\kappa|\mathrm{d}s \\
&\quad+\frac{1}{2}\sum_{e\in\mathcal{E}_{\mathrm{int}}^i}\int_e|[\eta](\sqrt{a}\nu)|_{\kappa'}|\sqrt{\tau_{\kappa',e}^i}\frac{1}{\sqrt{\tau_{\kappa',e}^i}}|(\sqrt{a}\nabla\xi)|_{\kappa'}|\mathrm{d}s \\
&\leq \left(\sum_{\kappa\in\mathcal{T}}\|\sqrt{a}\nabla\xi\|_\kappa^2\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{\mathrm{int}}^i}\langle C_{\mathrm{inv}}\frac{(a_j^e p_j)^2}{h_j}\rangle_e\|[\eta]\|_e^2\right)^{\frac{1}{2}} \\
&\leq |||\xi|||\left(\sum_{e\in\mathcal{E}_{\mathrm{int}}^i}\langle C_{\mathrm{inv}}\frac{(a_j^e p_j)^2}{h_j}\rangle_e\|[\eta]\|_e^2\right)^{\frac{1}{2}}, \qquad (56)
\end{aligned}
$$

where by κ, κ' we denote the two (generic) elements sharing the side e, C_{inv} an appropriate constant, and we have chosen $\tau_{\kappa,e}^i := (p_j^\kappa)^2/h_j^\kappa$ and $\tau_{\kappa',e}^i := (p_j^{\kappa'})^2/h_j^{\kappa'}$, $i,j=1,2$, $i\neq j$. Next, we have

$$\left|\int_{\Gamma_{\rm int}}[\xi]\langle(a\nabla\eta)\cdot\nu\rangle\mathrm{d}s\right| = \left|\sum_{e\in\mathcal{E}_{\rm int}}\int_e\sqrt{\sigma}[\xi]\frac{1}{\sqrt{\sigma}}\langle(a\nu)\cdot\nabla\eta\rangle\mathrm{d}s\right|$$
$$\leq \left(\sum_{e\in\mathcal{E}_{\rm int}}\int_e\sigma[\xi]^2\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{\rm int}^1}\int_e\sigma^{-1}\langle(a\nu)\cdot\nabla\eta\rangle\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\leq |||\xi|||\left(\sum_{e\in\mathcal{E}_{\rm int}}\sigma_e^{-1}\|\langle(a\nu)\cdot\nabla\eta\rangle\|_e^2\right)^{\frac{1}{2}}. \tag{57}$$

Now, bounding the two remaining terms, we have

$$\left|\int_{\Gamma_{\rm D}}\sigma\eta\xi\mathrm{d}s+\int_{\Gamma_{\rm int}}\sigma[\eta][\xi]\mathrm{d}s\right|\leq|||\xi|||\left\{\Big(\sum_{e\in\mathcal{E}_{\rm D}}\sigma_e\|\eta\|_e^2\Big)^{\frac{1}{2}}+\Big(\sum_{e\in\mathcal{E}_{\rm int}}\sigma_e\|[\eta]\|_e^2\Big)^{\frac{1}{2}}\right\}.$$

Collecting the bounds established above, and applying the identity $\left(\sum_n\alpha_n^2\right)^{\frac{1}{2}}\leq\sum_n\alpha_n$ for $\alpha_n\geq 0$, we obtain

$$\begin{aligned}|||\xi||| \leq\ & C_\theta\sum_{i=1}^{2}\Bigg\{\sum_{\kappa\in\mathcal{T}}\left(\sqrt{2}a_i^\kappa\|\partial_i\eta\|_\kappa+\frac{c_\kappa}{2}\|\eta\|_\kappa\right)\\
&+\sum_{\substack{e\in\mathcal{E}_{\rm D}^i\\ e\subset\bar\kappa}}\left(\left(C_{\rm inv}\frac{(a_{j,\rm nor}^e p_j^\kappa)^2}{h_j^\kappa}\right)^{\frac{1}{2}}\|\eta\|_e+\sigma_e^{-\frac{1}{2}}\|(a\nu)\cdot\nabla\eta\|_e+\sigma_e^{\frac{1}{2}}\|\eta\|_e\right)\\
&+\sum_{e\in\mathcal{E}_{\rm int}^i}\left(\langle C_{\rm inv}\frac{(a_{j,\rm nor}^e p_j)^2}{h_j}\rangle_e^{\frac{1}{2}}\|[\eta]\|_e+\sigma_e^{-\frac{1}{2}}\|\langle(a\nu)\cdot\nabla\eta\rangle\|_e+\sigma_e^{\frac{1}{2}}\|[\eta]\|_e\right)\Bigg\},\end{aligned}\tag{58}$$

for $i,j=1,2,\ i\neq j$, where every $\sigma_e:=\sigma|_e>0$ is associated with the (inter)face $e\in\mathcal{E}$.

Finally, combining (58) with (50), and using the bound

$$\|\langle(a\nu)\cdot\nabla\eta\rangle\|_e\leq\|\langle|\sqrt{a}\nu||\sqrt{a}\nabla\eta|\rangle\|_e\leq\langle a_{j,\rm nor}^e\rangle_e\langle\|\sqrt{a}\nabla\eta\|_e\rangle_e,$$

the result follows. □

Remark 8.3. *The bound (58) is an extension to the anisotropic setting of the bound appearing in Lemma 4.3 of* [18], *and the argument here is analogous; special care had to be taken, however, to respect the direction-wise features of both the boundary-value problem itself (explicit representation of each term appearing in the diffusion tensor a, which is relevant for problems with anisotropic or degenerate diffusion), and the (possibly) anisotropic choice of the discretisation parameters. The explicit representation of these direction-wise features will be our main concern in the subsequent discussion.*

The error bound (49) does not add much to our understanding since it involves terms that may in general tend to infinity as $h_1^\kappa, h_2^\kappa\to 0$ and/or $p_1^\kappa,p_2^\kappa\to\infty$. Furthermore, the terms involving $[\eta]$ in (58) may introduce a coupling between the quantities h_1,h_2,p_1 and p_2 for neighbouring elements. Therefore, in order to obtain more helpful bounds, we shall make various assumptions on the quantities involved.

The utility of the "switch" z_e for $e\in\mathcal{E}$ will become more clear when we consider special cases of the above result. In particular, when the image of a function under the broken H^1-projection operator $\Lambda_{\mathbf{p}}^{\mathcal{T}}$ is continuous across an element interface e, we have that $[\eta]_e=0$ and, therefore, terms involving $[\eta]_e$ are not required to be further bounded; so we "switch them off". The image of a function under the

H^1-projection operator is continuous across an element interface e, if, for example, polynomials of the same degree, say p, are used (in the direction normal to the direction of e) in the finite element space and e does not contain any hanging nodes, and/or if e is a boundary face where Dirichlet boundary conditions are applied, with boundary datum that is an edge-wise polynomial of degree p or less (cf. Lemma 6.7 in [13]).

In the subsequent results, we have chosen to derive bounds for axiparallel anisotropic meshes, i.e., when the diffeomorphisms $Q_\kappa \equiv \mathrm{id}$, for $\kappa \in \mathcal{T}$, since the advantages of the use of anisotropic elements become quite apparent in this case, without obscuring the key ideas by a complicated notation. For general meshes, the bound in Theorem 8.2 cannot be further simplified, unless additional assumptions are made on the variation of the discretisation parameters involved. We stress however that, given such assumptions, general bounds are by all means possible to obtain.

We begin by setting up the admissible finite element spaces for our next result. First, we assume that the polynomial degrees are anisotropic, with a directionally bounded local variation condition, i.e., there exist ρ_i such that $\rho_i^{-1} \le p_i^\kappa / p_i^{\kappa'} \le \rho_i$, $i = 1, 2$, for all pairs of neighbouring elements $\kappa, \kappa' \in \mathcal{T}$. A similar condition is required for the meshsizes. In particular, we assume that there exist positive constants δ_i such that $\delta_i^{-1} \le h_i^\kappa / h_i^{\kappa'} \le \delta_i$, $i = 1, 2$, for all pairs of neighbouring elements $\kappa, \kappa' \in \mathcal{T}$.

Corollary 8.4. *Let Ω be an axiparallel polygonal domain, $\mathcal{T}$ a subdivision of Ω into axiparallel elements, satisfying the bounded local variation properties stated above. We assign to each edge $e \in \mathcal{E}$ the positive real number σ_e, defined by*

$$\sigma_e := C_{\sigma,i} \frac{(a_j^e)^2 (p_j^\kappa)^m}{h_j^\kappa} \text{ for } e \in \mathcal{E}_{\mathrm{D}}^i,\ e \subset \bar{\kappa}, \text{ and } \sigma_e := C_{\sigma,i} \langle \frac{(a_j^e)^2 (p_j)^m}{h_j} \rangle_e \text{ for } e \in \mathcal{E}_{\mathrm{int}}^i,$$

where $m \in \{1, 2\}$, $C_\sigma = 1$ if $\theta = 1$, and $m = 2$, $C_\sigma = C_\sigma(C_{\mathrm{inv}}, \rho_j, \delta_j)$ if $\theta = -1$ (see [12] for a detailed description of the dependence of C_σ on the parameters, or [16] for the corresponding argument on shape-regular meshes), for $i, j = 1, 2$, $i \neq j$. Then, assuming that $u \in A \cap \tilde{H}^{\mathbf{k+1}}(\Omega, \mathcal{T})$ with $k_\kappa \ge 1$, $\kappa \in \mathcal{T}$, the solution $u_{\mathrm{DG}} \in S^{\vec{\mathbf{p}}}(\Omega, \mathcal{T}, \mathbf{F}, \mathbf{Q})$ obeys the error bound

$$|||u - u_{\mathrm{DG}}||| \le \sum_{\kappa \in \mathcal{T}} \sum_{i=1}^{2} C_{\kappa,i} \Phi_1(p_i^\kappa, s_i^\kappa) \left(\frac{h_i^\kappa}{2} \right)^{s_i^\kappa} N_{\mathrm{aug},\kappa}^i |u|_{\tilde{H}^{s_\kappa+1}(\kappa), i}, \tag{59}$$

where

$$N_{\mathrm{aug},\kappa}^i := \alpha_i^\kappa + \alpha_j^\kappa \Big((p_i^\kappa)^{-1} + (p_i^\kappa (p_j^\kappa)^m)^{-\frac{1}{2}} + z_e \frac{p_j^\kappa}{p_i^\kappa} \frac{h_i^\kappa}{h_j^\kappa} \Big) + \frac{h_i^\kappa}{2 p_i^\kappa} c_\kappa$$

with $1 \le s_i^\kappa \le \min\{p_i^\kappa, k_\kappa\}$, α_i^κ being the average of a_i^κ on all the elements neighbouring κ, including κ itself, c_κ as in (47), $C_{\kappa,i} = C_{\kappa,i}(C_{\mathrm{inv}}, \rho_j, \delta_j)$ a generic constant, and z_e as above.

Moreover, if $u \in A \cap H^{\mathbf{k+1}}(\Omega, \mathcal{T})$ with $k_\kappa \ge 2$, $\kappa \in \mathcal{T}$, u_{DG} obeys the error bound

$$|||u - u_{\mathrm{DG}}||| \le \sum_{\kappa \in \mathcal{T}} \sum_{i=1}^{2} C_{\kappa,i} \Phi_1(p_i^\kappa, s_i^\kappa) \left(\frac{h_i^\kappa}{2} \right)^{s_i^\kappa} N_{\mathrm{sta},\kappa}^i |u|_{H^{s_\kappa+1}(\kappa), i}^*, \tag{60}$$

with $1 \le s_i^\kappa \le \min\{p_i^\kappa, k_\kappa\}$,

$$N_{\mathrm{sta},\kappa}^i := \alpha_i^\kappa \Big(1 + \Big(\frac{p_i^\kappa}{p_j^\kappa} \Big)^{\frac{1}{2}} (p_j^\kappa)^{\frac{1-m}{2}} \Big) + \alpha_j^\kappa \Big(z_e (p_j^\kappa)^{\frac{1}{2}} \Big(\frac{p_j^\kappa}{p_i^\kappa} \Big)^{\frac{1}{2}} \frac{h_i^\kappa}{h_j^\kappa} + \Big(\frac{p_i^\kappa}{p_j^\kappa} \Big)^{\frac{1}{2}} (p_j^\kappa)^{\frac{1-m}{2}} \Big) + \frac{h_i^\kappa}{2 p_i^\kappa} c_\kappa,$$

and

$$|u|^*_{H^s(\kappa),i} := \left(\|\partial_i^s u\|^2_\kappa + \max_{r=0,1,2}\left\{1, \left(\frac{h_j^\kappa}{h_i^\kappa}\right)^2\left(\frac{p_i^\kappa}{p_j^\kappa}\right)^r\right\}\|\partial_i^{s-1}\partial_j u\|^2_\kappa + \left(\frac{h_j^\kappa}{h_i^\kappa}\right)^2\|\partial_i^{s-2}\partial_j^2 u\|^2_\kappa\right)^{\frac{1}{2}}.$$

Proof. The proof is a direct consequence of the combination of the bound (49) with the approximation estimates presented in Section 7.2. □

Remark 8.5. *Making use of Stirling's formula, we can see that the bound (59) is hp-optimal due to the additional regularity offered by the use of augmented Sobolev spaces. This phenomenon was first described in* [13], *in the case of shape-regular quadrilateral elements. Half an order of p, however, is lost when the solution belongs element-wise to a standard Sobolev space, as (60) reveals.*

Another interesting special case, on axiparallel meshes, is when the finite element space involves a *directionally uniform polynomial degree*, i.e., we assume $p_1^\kappa = p_1$ and $p_2^\kappa = p_2$ constant for all $\kappa \in \mathcal{T}$; note that, unlike the previous theorem, we do *not* assume any bounded local variation in the meshsize.

Corollary 8.6. *Let Ω be an axiparallel polygonal domain, $\mathcal{T}$ a subdivision of Ω into axiparallel elements (possibly shape-irregular) with directionally uniform polynomial degree, not containing any hanging nodes. We assign to each edge $e \in \mathcal{E}$ the positive real number*

$$\sigma_e := C_{\sigma,i}\frac{(a_j^e)^2(p_j)^m}{h_j^\kappa}, e \in \mathcal{E}_{\mathrm{D}}^i, e \subset \bar{\kappa} \text{ and } \sigma_e := C_{\sigma,i}\frac{\langle a_j^e\rangle_e^2(p_j)^m}{\min\{h_j^\kappa, h_j^{\kappa'}\}}, e \in \mathcal{E}_{\mathrm{int}}^i, e \subset \bar{\kappa}\cap\bar{\kappa}',$$

where $m \in \{1,2\}$, $C_{\sigma,i} = 1$ if $\theta = 1$, and $m = 2$, $C_{\sigma,i} = C(C_{\mathrm{inv}}, \rho_j, \delta_j)$ if $\theta = -1$ (see [12] *for detailed description of the dependence of C_σ on the parameters, or* [16] *for the corresponding argument on shape-regular meshes), for $i, j = 1, 2$, $i \neq j$. Then, assuming that $u \in A$ and that it is analytic on an open neighbourhood of every element $\kappa \in \mathcal{T}$, the solution $u_{\mathrm{DG}} \in S^{(p_1,p_2)}(\Omega, \mathcal{T}, \mathbf{F})$ obeys the error bound*

$$|||u - u_{\mathrm{DG}}||| \leq \sum_{i=1}^{2}\sum_{\substack{\kappa\in\mathcal{T}\\ e\subset\partial\kappa_i\cap\Gamma_{\mathrm{D}}}} C_u^\kappa \mathrm{e}^{-(r_i^\kappa-\frac{3}{2})p_i}\left(\frac{h_i^\kappa}{2}\right)^{s_i^\kappa} M^i_{\infty,\kappa,e}(\mathrm{meas}_2(\kappa))^{\frac{1}{2}}, \tag{61}$$

with

$$M^i_{\infty,\kappa,e} := \alpha_i^\kappa + \frac{h_i^\kappa}{2p_i}\left(\alpha_j^\kappa(1 + z_e\frac{p_j}{h_j^\kappa}) + c_\kappa\right),$$

where $p_i \geq 1$, α_i^κ and c_κ as in Corollary 8.4, C_u^κ, r_i^κ, for $\kappa \in \mathcal{T}$, are constants depending on u and z_e as above (note $z_e \equiv 0$ if $e \subset \Gamma_{\mathrm{int}}$). If, additionally, there exist $d_i^\kappa > 1$, $i = 1, 2$, such that

$$\|\partial_1^m\partial_2^n u\|_{L^\infty(\kappa)} \leq C(d_1^\kappa)^m(d_2^\kappa)^n, \quad m, n = 0, 1, 2, \ldots, \tag{62}$$

then the error bound can be improved to

$$|||u - u_{\mathrm{DG}}|||^2 \leq C\sum_{\kappa\in\mathcal{T}}\sum_{i=1}^{2}\left(\frac{h_i^\kappa d_i^\kappa \mathrm{e}}{4p_i}\right)^{p_i}\mathcal{M}^i_{\infty,\kappa,e}(\mathrm{meas}_2(\kappa))^{\frac{1}{2}}, \tag{63}$$

with

$$\begin{aligned}\mathcal{M}^i_{\infty,\kappa,e} \quad &:= \quad \alpha_i^\kappa d_i^\kappa\left(1 + \frac{h_j^\kappa d_j^\kappa}{2p_i^{m/2}}\right) + \alpha_j^\kappa d_j^\kappa\left(\frac{h_i^\kappa d_i^\kappa}{2p_i} + z_e\frac{p_j h_i^\kappa d_i^\kappa}{p_i h_j^\kappa d_j^\kappa}\left(1 + \frac{h_j^\kappa d_j^\kappa}{2}\right)\right)\\ &\quad + c_\kappa\frac{h_i^\kappa d_i^\kappa}{2p_i}\left(1 + \frac{h_j^\kappa d_j^\kappa}{2p_j}\right).\end{aligned} \tag{64}$$

Proof. For (61), we insert the bounds presented in Section 7.2 (exploiting the additional regularity) into (49) and we apply Lemma 7.17.

For (63), we apply Stirling's formula to $\Phi_1(p_i, p_i)$, to obtain

$$\begin{aligned}\Phi_1(p_i,p_i) &= (\Gamma(2p_i+1))^{-1} \le (2\pi)^{-\frac14}(2p_i)^{-p_i-\frac14}\mathrm{e}^{p_i}\\ &= (4\pi)^{-\frac14}(2p_i)^{-p_i}\left(\mathrm{e}^{1-\frac{\log p_i}{4p_i}}\right)^{p_i} \le (4\pi)^{-\frac14}(2p_i)^{-p_i}\mathrm{e}^{p_i}.\end{aligned}$$

On substituting the last relation into (49) and using (62), the result follows. $\square$

Remark 8.7. *The above result can be applied when the solution to the boundary-value problem exhibits boundary and/or interior layers. Indeed, asymptotic analysis indicates (and proves in certain cases, see* [21, 26] *and the references therein, for details) that solutions to such problems satisfy assumption (62).*

Now, we present the corresponding results for equations (1) with non-negative characteristic form, admitting $b \neq \vec{0}$ in general.

Theorem 8.8. *Consider the setting of Theorem 8.2; then,*

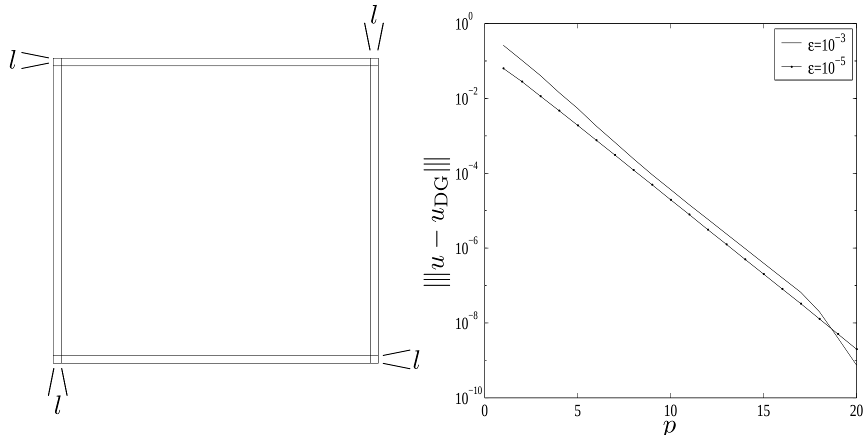
$$\begin{aligned}|||u-u_{\rm DG}||| \le\ & C_\theta \sum_{\kappa\in\mathcal{T}}\sum_{i=1}^{2}\Bigg\{\left(2\sqrt{2}a_i^\kappa(C^1_{\kappa,i}M^1_{\kappa,i}+C^2_{\kappa,i}M^1_{\kappa,j})+\frac12 c_2^\kappa C_\kappa M^0_\kappa\right)\\ &+\sum_{e\subset\partial\kappa_i}\Bigg\{\left(z_e\left(\langle C_{\rm inv}\frac{(a_j^e p_j)^2}{h_j}\rangle_e^{\frac12}+2\sigma_e^{\frac12}\right)+\left(1+\frac{\sqrt2}{2}\right)b_0^e\right)C^i_{\partial\kappa}M^0_{\partial\kappa,i}\\ &+\sqrt2\frac{\langle a^e_{j,\rm nor}\rangle_e}{\sigma_e^{\frac12}}\Big((\langle a_i^e\rangle_e C^{1,i}_{\partial\kappa,i}+\langle a_j^e\rangle_e C^{2,i}_{\partial\kappa,j})M^1_{\partial\kappa,i}\\ &+(\langle a_i^e\rangle_e C^{2,i}_{\partial\kappa,i}+\langle a_j^e\rangle_e C^{1,i}_{\partial\kappa,j})M^2_{\partial\kappa,i}\Big)\Bigg\}\Bigg\},\end{aligned}\tag{65}$$

with $M^0_\kappa, M^1_{\kappa,i}, M^0_{\partial\kappa,i}, M^1_{\partial\kappa,i}$, $M^2_{\partial\kappa,i}$ *as in Section 7.1, and* $c_2^\kappa := (1+\|(c-\nabla\cdot b)/(c_0)^2\|^{\frac12}_{L^\infty(\kappa)})\|(c_0)^2\|^{\frac12}_{L^\infty(\kappa)}$ *when* $c_0|_\kappa>0$ *and* $c_2^\kappa := c_\kappa$ *when* $c_0|_\kappa = 0$.

The complete proof can be found in [12]. For the diffusion part, the proof is entirely analogous to the proof of Theorem 8.2. For the advection part, the argument is an extension of the one presented in [18] to the anisotropic setting. As in [18], we have made use of the L^2-projection operator as interpolant in this case, as it yields hp-optimal bounds for the convection part of the discretisation. Note, however, that the bound for the component of the error due to the discretisation of diffusion is suboptimal in p by half an order of p (cf. the corresponding results on shape-regular elements in [18]).

Corollary 8.9. *Let Ω be an axiparallel polygonal domain, $\mathcal{T}$ a subdivision of Ω into axiparallel elements, satisfying the bounded local variation properties discussed above. We assign to each edge $e\in\mathcal{E}$ the positive real number σ_e, defined as in Corollary 8.4, with $m=2$. Then, assuming that $u\in A\cap H^{\mathbf{k+1}}(\Omega,\mathcal{T})$ with $k_\kappa\ge1$, $\kappa\in\mathcal{T}$, $u_{\rm DG}\in S^{\vec{\mathbf{p}}}(\Omega,\mathcal{T},\mathbf{F},\mathbf{Q})$ obeys the error bound*

$$|||u-u_{\rm DG}||| \le \sum_{\kappa\in\mathcal{T}}\sum_{i=1}^{2} C_{\kappa,i}\Phi_1(p_i^\kappa,s_i^\kappa)\left(\frac{h_i^\kappa}{2}\right)^{s_i^\kappa} N^i_{L^2,\kappa}|u|^*_{H^{s_\kappa+1}(\kappa),i},\tag{66}$$



(a) The 9-element mesh. (b) hp-convergence for $\lambda = 0.9$.

FIGURE 2. Example 1. The mesh and hp-convergence history. In figure (a) $l = \min\{1/2, \lambda p \epsilon\}$.

with $1 \leq s_i^\kappa \leq \min\{p_i^\kappa, k_\kappa\}$, $C_{\kappa,i}$ *as before, and*

$$\begin{aligned} N^i_{L^2,\kappa} \quad &:= \quad \alpha_i^\kappa \left((p_i^\kappa)^{\frac{1}{2}} + \frac{p_i^\kappa}{p_j^\kappa} \right) + \alpha_j^\kappa \left((p_i^\kappa)^{\frac{1}{2}} \frac{h_i^\kappa}{h_j^\kappa} + (p_i^\kappa)^{-\frac{1}{2}} \right) \\ &\qquad + \left(\frac{h_i^\kappa}{2p_i^\kappa} \right)^{\frac{1}{2}} \left(b_i^\kappa + b_j^\kappa \left(\frac{h_i^\kappa}{h_j^\kappa} \right)^{\frac{1}{2}} \right) + \frac{h_i^\kappa}{2p_i^\kappa} c_2^\kappa . \end{aligned}$$

If the assumption (48) is violated, then we can still obtain error bounds, at the cost of losing another half an order of p in the convergence rates (see [12] for details).

Remark 8.10. *We present an application of the above result to the (standard) advection-diffusion problem*

$$-\epsilon \Delta u + b \cdot \nabla u = f,$$

with (possibly mixed) boundary conditions of the form (3). For simplicity of the presentation, assume that $s_i^\kappa = k_\kappa$ *for* $i = 1, 2$ *and that* $p_1^\kappa = p_2^\kappa = p_\kappa$. *Making use of Stirling's formula and simplifying the constant terms, we obtain the error bound*

$$\begin{aligned} |||u - u_{\mathrm{DG}}||| \quad &\leq \quad C \sum_{\kappa \in \mathcal{T}} \sum_{i=1}^{2} \left(\frac{h_i^\kappa}{p_\kappa} \right)^{k_\kappa} \left(\epsilon p_\kappa^{\frac{1}{2}} \left(1 + \frac{h_i^\kappa}{h_j^\kappa} \right) \right. \\ &\qquad \left. + \left(\frac{h_i^\kappa}{p_\kappa} \right)^{\frac{1}{2}} \left(b_i^\kappa + b_j^\kappa \left(\frac{h_i^\kappa}{h_j^\kappa} \right)^{\frac{1}{2}} \right) \right) |u|^*_{H^{s_\kappa+1}(\kappa),i}, \end{aligned} \tag{67}$$

indicating that, for an efficient and accurate approximation, appropriate balance of the data of the equation and of the hp-mesh parameters should be sought.

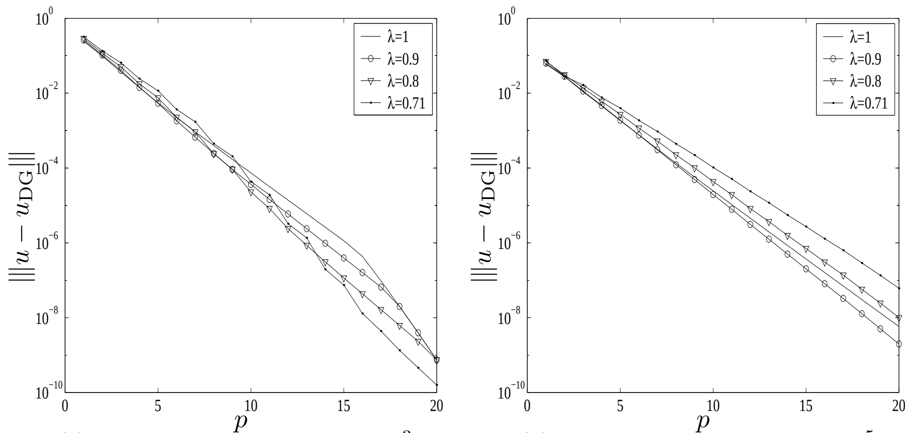
9. Numerical Experiments

9.1. Example 1. Let $\Omega := (-1, 1)^2$ and consider the equation

$$-\epsilon \Delta u + u = f \qquad \text{in } \Omega, \tag{68}$$

subject to a homogeneous Dirichlet boundary condition; f is chosen so that

$$u(x_1, x_2) = \left(1 - \frac{\cosh(x_1/\sqrt{\epsilon})}{\cosh(1/\sqrt{\epsilon})} \right) \left(1 - \frac{\cosh(x_2/\sqrt{\epsilon})}{\cosh(1/\sqrt{\epsilon})} \right). \tag{69}$$



(a) Sensitivity w.r.t. λ for $\epsilon = 10^{-3}$. (b) Sensitivity w.r.t. λ for $\epsilon = 10^{-5}$.

FIGURE 3. Example 3. Sensitivity with respect to the choice of λ for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$.

The solution exhibits a boundary-layer of thickness $O(\sqrt{\epsilon})$ near $\partial\Omega$. In order to resolve this behaviour we shall use a 9-element mesh, as shown in Figure 9.1(a), where the small element edges have length $l := \min\{1/2, \lambda p \epsilon\}$, λ is a user-defined parameter and p is the polynomial degree used in the elemental basis (cf. [28, 21]).

We observe robust exponential convergence, when applying the hp-DGFEM (with $\theta = 1$) on the 9-element mesh, as the polynomial degree is increased; in Figure 9.1(b) we can see the error measured in the DG-norm plotted against the polynomial degree p for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$ with $\lambda = 0.9$.

In order to test the sensitivity of the method with respect to the choice of the parameter λ, we solved the problem for $\epsilon = 10^{-3}$ and $\epsilon = 10^{-5}$ and various values of λ. The results, shown in Figure 3, indicate that the method is fairly insensitive to the choice of λ as long as λ is chosen to be near 1. We achieve the best results when λ is around 0.9, as opposed to the results presented in [28] for the conforming hp-finite element method, where $\lambda = 0.71$ was the best choice.

Let us now see how this example fits into the analysis presented in the previous section. The analytical solution u satisfies (62) on every element of the considered 9-element-mesh. A simple calculation reveals that $\mathcal{M}_{\infty,\kappa,e} \leq C$, with C independent of ϵ, for all 9 elements κ. Therefore, using the 9-element-mesh the bound (63) is independent of ϵ, which explains the robustness of the exponential convergence observed.

9.2. Example 2. We consider the Dirichlet boundary-value problem

$$\begin{aligned} -u_{x_1x_1} - 16x_1^6 u_{x_2x_2} &= f \quad \text{in } \Omega \equiv (-1,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{70}$$

with f is chosen so that the analytical solution is

$$u(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)(|x_1|^8 + x_2^2)^{1/4}. \tag{71}$$

This Grušin-type model problem is due to Franchi & Tesi [11], where for $p = 1$ fixed a slow sub-linear algebraic convergence rate was reported as $h \to 0$. The purpose of this example is to show that the pessimistic scenario of [11] can be considerably improved upon. Note that the analytical solution does not belong (globally) to

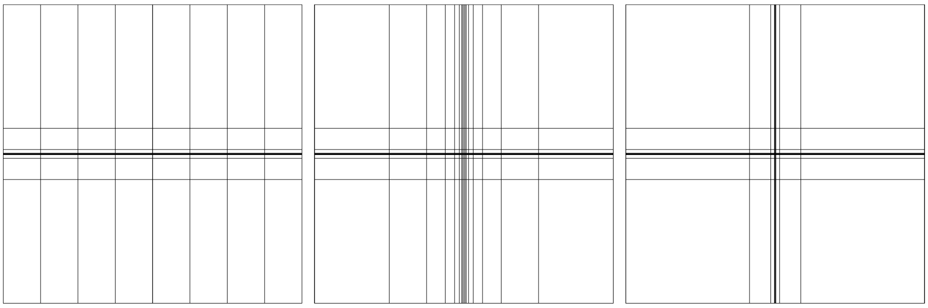


FIGURE 4. Example 2. The three meshes used in the numerical experiment.

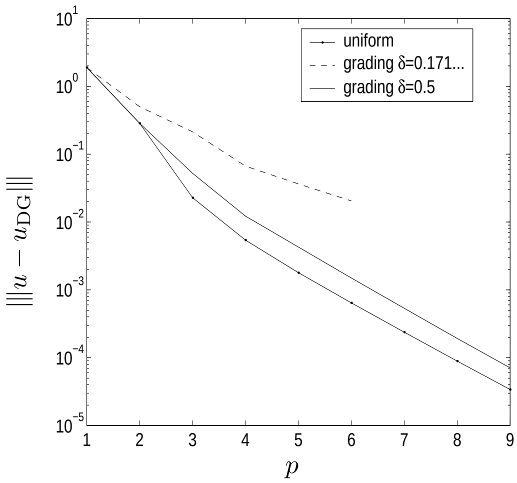


FIGURE 5. Example 2. Convergence under hp-refinement using the three meshes.

$H^1(\Omega)$ due to a singularity of the gradient at the origin; nevertheless, it is analytic in $\bar{\Omega}\backslash\{(0,0)\}$.

We shall employ a mesh sequence, inspired by one-dimensional problems with solutions that are analytic everywhere except at one point. We use a geometrically graded mesh, grading towards the singularity and we increase appropriately the polynomial degree on every element; this is described in [15].

Therefore, a geometrically graded mesh accumulating towards the line $x_2 = 0$ on both sides is employed in the x_2-direction. The grading factor δ is chosen to be the optimal one for the one-dimensional problem and has the value $\delta = (\sqrt{2}-1)^2 = 0.171\ldots$. For the grading in the x_1-direction three different choices are considered:

(1) uniform refinement;
(2) geometrically graded refinement towards $x_1 = 0$ from both sides with grading factor $\delta = 0.5$;
(3) geometrically graded refinement towards $x_1 = 0$ from both sides with grading factor $\delta = (\sqrt{2}-1)^2 = 0.171\ldots$.

The resulting meshes for these three choices are shown in Figure 4.

The hp-refinement path is the following: We start with 2 elements in each direction and polynomial degree $p = 1$. The next step is to add 2 elements in each direction and increase the polynomial degree to $p = 2$ uniformly; we then add 2 more elements in each coordinate direction and raise the polynomial degree to $p = 3$ uniformly, and so on. In Figure 5 we can see the convergence history (for $\theta = 1$) of this refinement strategy. The convergence appears to be exponential, at least for the cases of uniform refinement in the x_1-direction and a geometrically graded mesh with grading factor $\delta = 0.5$ in the x_1-direction, as they appear to be straight lines in a linear-log coordinate system. If the mesh in the x_1-direction is refined geometrically with grading factor $\delta = 0.171\ldots$, convergence is slower as, in this case, the main contributions to the error come from the large elements situated in the corners, which are larger than the corresponding elements in the other two refinement strategies.

Comparing our results with the numerical experiments described in [11] we can see that it is indeed possible to attain substantially faster convergence rates than the ones proved and observed numerically by using continuous piecewise linear finite elements therein. Motivated by our experimental observations, we conjecture that the convergence using any of the refinement strategies described above will be exponential.

9.3. Example 3. For $b = (1,1)^T$ and $0 < \epsilon \ll 1$, we consider the singularly perturbed convection-diffusion equation

$$-\epsilon\Delta u + b \cdot \nabla u = f \quad \text{for} \quad (x_1, x_2) \in (0,1)^2,$$

subject to a Dirichlet boundary condition, which, along with the forcing function f, is chosen so that the analytical solution is

$$u(x_1, x_2) = x_1 + x_2(1 - x_1) + \frac{\mathrm{e}^{-\frac{1}{\epsilon}} - \mathrm{e}^{-\frac{(1-x_1)(1-x_2)}{\epsilon}}}{1 - \mathrm{e}^{-\frac{1}{\epsilon}}}.$$

This problem was considered in [18] (Example 3) and it is inspired by a one-dimensional problem taken from [22]. Note that the theory developed above includes this case, as here $c_0 = 0$ on $\bar{\Omega}$.

In this numerical experiment, the stability, the accuracy, and the robustness with respect to ϵ of the hp-DGFEM are tested. The solution exhibits boundary layer behaviour along $x_1 = 1$ and $x_2 = 1$, and the layers become steeper as $\epsilon \to 0$. Motivated by this behaviour, the meshes are constructed by geometrical refinement towards the boundary layers, with grading factor $\delta = 0.5$, and are parametrized by n_ϵ denoting the number of (mesh-)points in the x_1- and x_2-directions (cf. Figure 6.8 in [18]). Following [18] we perform numerical experiments for $\epsilon = 10^{-1}, 10^{-3}, 10^{-5}$, with $n_\epsilon = 9, 15, 21$, respectively.

The numerical results in [18], are performed with a choice of the discontinuity-penalisation parameter σ that came from the error analysis for shape-regular elements presented in that paper. In particular, σ was chosen in [18] as

$$\tilde{\sigma}(m) := \frac{\langle a_e p_i^m \rangle_e}{h_i},$$

for $e \in \mathcal{E}^i$, $i, m = 1, 2$.

Here, we reproduce the numerical results of [18] and we perform further experiments with σ chosen as in our analysis, namely

$$\sigma(m) = \frac{\langle a_e p_j^m \rangle_e}{h_j},$$

ϵ	p	$\sigma(2)\,,\theta=-1$	It.	$\tilde{\sigma}(2)\,,\theta=-1$	It.	$\sigma(2),\,\theta=1$	It.	$\tilde{\sigma}(2),\,\theta=1$	It.
10^{-5}	1	0.10144	110	0.46401	2460	0.10151	40	0.10418	40
	2	0.1643E-01	97	failed	$> 10^4$	0.1581E-01	57	0.2050E-01	56
	3	0.2551E-02	122	failed	$> 10^4$	0.2514E-02	64	0.2598E-02	60
	4	0.4178E-03	119	failed	$> 10^4$	0.4124E-03	71	0.5013E-03	70
	5	0.6765E-04	133	failed	$> 10^4$	0.6673E-04	71	0.6920E-04	77
	6	0.1130E-04	135	failed	$> 10^4$	0.1122E-04	76	0.1299E-04	87
	7	0.1875E-05	137	failed	$> 10^4$	0.1855E-05	70	0.1939E-05	93
	8	0.3107E-06	142	failed	$> 10^4$	0.3088E-06	74	0.3463E-06	98
10^{-3}	1	0.10088	68	failed	$> 10^4$	0.10088	27	0.10299	29
	2	0.1646E-01	80	failed	$> 10^4$	0.1581E-01	37	0.2035E-01	39
	3	0.2551E-02	97	failed	$> 10^4$	0.2514E-02	42	0.2587E-02	47
	4	0.4207E-03	96	failed	$> 10^4$	0.4143E-03	49	0.4990E-03	55
	5	0.6772E-04	110	failed	$> 10^4$	0.6684E-04	52	0.6914E-04	61
	6	0.1118E-04	106	failed	$> 10^4$	0.1108E-04	56	0.1272E-04	69
	7	0.1847E-05	116	failed	$> 10^4$	0.1827E-05	58	0.1889E-05	74
	8	0.3133E-06	112	failed	$> 10^4$	0.3112E-06	60	0.3449E-06	81
10^{-1}	1	0.9909E-01	50	0.10484	113	0.9976E-01	18	0.9962E-01	21
	2	0.1544E-01	65	0.1534E-01	831	0.1517E-01	29	0.1645E-01	31
	3	0.2380E-02	77	0.2383E-02	238	0.2337E-02	35	0.2352E-02	38
	4	0.3627E-03	83	0.3626E-03	119	0.3637E-03	38	0.3735E-03	44
	5	0.5642E-04	92	0.5628E-04	118	0.5615E-04	42	0.5642E-04	47
	6	0.8458E-05	92	0.8433E-05	117	0.8517E-05	44	0.8608E-05	51
	7	0.1215E-05	98	0.1210E-05	131	0.1219E-05	49	0.1224E-05	52
	8	0.1605E-06	96	0.1598E-06	210	0.1616E-06	51	0.1624E-06	55

TABLE 1. Example 3. Convergence rates in the $\|\cdot\|_*$-norm under p-enrichment for various values of ϵ and different choices of penalisation σ.

for $e \in \mathcal{E}^i$, $i, j, m = 1, 2$, $i \neq j$. The results from these experiments for $m = 2$ are listed in Table 1: the error is measured in the following (σ-independent) norm

$$\|v\|_* = \left(\sum_{\kappa\in\mathcal{T}} \|\sqrt{a}\nabla v\|_\kappa^2 + \|c_0 v\|_\kappa^2\right)^{\frac{1}{2}},$$

$\sigma(2)$ and $\tilde{\sigma}(2)$ stand for the choice of discontinuity-penalisation parameter, $\theta = -1$ and $\theta = 1$ denote the symmetric and the non-symmetric versions DGFEM, respectively, and "It." stands for the number of GMRES(20) iterations needed in the solution of the linear system, using a block-Jacobi preconditioner.

For the symmetric version DGFEM ($\theta = -1$), the choice of the discontinuity-penalisation parameter as $\sigma(2)$ is crucial for the stability of the method. Indeed, when the discontinuity-penalisation parameter is chosen as $\tilde{\sigma}(2)$, the stiffness matrix appears to be very ill conditioned or even singular. Hence, the importance of the new recipe for the discontinuity-penalisation parameter is not only theoretical (as it enables us to prove coercivity), but is also manifested numerically.

On the other hand, the choice of discontinuity-penalisation parameter is irrelevant for the stability of the non-symmetric version DGFEM ($\theta = 1$), as the method is coercive for any $\sigma \geq 0$. Nevertheless, we observe that the results produced using $\sigma(2)$ are slightly better compared to the ones using $\tilde{\sigma}$. This is still true when the error in measured in the energy norm or when $m = 1$ is used; for brevity, these results are omitted (see [12] for details). When $\epsilon = 10^{-1}$ and $\epsilon = 10^{-3}$ the number of GMRES iterations and the approximation error are consistently smaller with $\sigma(2)$ than with $\tilde{\sigma}(2)$. The same is true for $\epsilon = 10^{-5}$ for $p \geq 5$; for $1 \leq p \leq 4$ the number of GMRES iterations with $\sigma(2)$ is either the same or marginally larger than with $\tilde{\sigma}(2)$, but then the resulting approximation error is always smaller.

Hence, choosing the discontinuity-penalisation parameter as advocated in this work, we observe a crucial improvement on the stability of the symmetric version DGFEM on anisotropic *hp*-meshes and a slight reduction in the computational cost at no loss in accuracy for the non-symmetric version DGFEM. Indeed, in most cases we looked at, we observed improved accuracy as well as reduction in the computational cost with the choice of the discontinuity-penalisation parameter proposed herein, when compared to the choice that was made in [18] where shape-regular meshes were assumed.

The results of the calculations indicate exponential convergence (cf. also Figure 6.9 in [18]). Furthermore, with this mesh sequence, the rate of exponential convergence is independent of ϵ.

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