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THE COUPLING OF NATURAL BOUNDARY ELEMENT AND FINITE ELEMENT METHOD FOR 2D HYPERBOLIC EQUATIONS *1)

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Abstract

In this paper, we investigate the coupling of natural boundary element and finite element methods of exterior initial boundary value problems for hyperbolic equations. The governing equation is first discretized in time, leading to a time-step scheme, where an exterior elliptic problem has to be solved in each time step. Second, a circular artificial boundary Γ_R consisting of a circle of radius R is introduced, the original problem in an unbounded domain is transformed into the nonlocal boundary value problem in a bounded subdomain. And the natural integral equation and the Poisson integral formula are obtained in the infinite domain Ω_2 outside circle of radius R. The coupled variational formulation is given. Only the function itself, not its normal derivative at artificial boundary Γ_R, appears in the variational equation, so that the unknown numbers are reduced and the boundary element stiffness matrix has a few different elements. Such a coupled method is superior to the one based on direct boundary element method. This paper discusses finite element discretization for variational problem and its corresponding numerical technique, and the convergence for the numerical solutions. Finally, the numerical example is presented to illustrate feasibility and efficiency of this method.

Key words: Hyperbolic equation, Natural boundary reduction, Finite element, Coupling, Exterior problem.

1. Introduction

In many fields of scientific and engineering computing, problems in unbounded spatial domains are encountered frequently, such as acoustic waves, electromagnetics wave guides, aerodynamics, and meteorology, and so on. Such problems pose a unique challenge to computation, since their domains are unbounded. Although we can apply classical boundary element methods (BEM) or boundary integral methods (BIM) to solve these problems in unbounded domains, in practice a great many singular integrations usually need to be calculated. At the same time, we make the integrations about time interval for the time-independent problems while the problems are discretized in time. Therefore, it takes a lot of time to deal with the original problem. The natural boundary element method initiated and developed by K Feng and D Yu (see $[1-6]$) has some distinctive advantages comparing with classical boundary element methods. One of them is fully compatible with finite element method, and it can be coupled with finite element method naturally and directly. The coupling of natural boundary element and finite element method for the elliptic problems, we can refer to [6,7,8]. Mathematical theory

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of natural boundary element method for the elliptic problems is being perfected (see [8]). At present, authors have made some developments in natural boundary element method for the parabolic problems(see [9]), and been investigating into the problems related to this.

In this paper, the coupling of natural boundary element and finite element method for hyperbolic exterior initial boundary value problems in R^2. A circular artificial common boundary Γ_R, which consists of a circle of radius R large enough, is introduced. It divides the domain into two subregions, a bounded inner region Ω_1 (bounded annular region by Γ_0 and Γ_R) and a regular unbounded region Ω_2 (unbounded domain outside circle Γ_R). We obtain the natural integral equation on boundary Γ_R and corresponding Poisson integral formula of the subproblem over unbounded domain Ω_2 by the natural boundary reduction. Only the function itself, not its normal derivative at the common boundary Γ_R, appears in the variational equation obtained. Not only the boundary element stiffness matrix is symmetric, but also its elements have explicit forms. We only calculate a few different elements, and can get the boundary element stiffness matrix. It is easy to be implemented on calculation and storage.

Following is the outline of this paper. In section 2 we first state the problem under consideration . Second we discretize the problem in time, leading to a time-stepping scheme, where an exterior elliptic problem has to be solved in each time step. In section 3 we introduce a circular common boundary Γ_R, and obtain natural integral equation in the unbounded domain outside circle Γ_R. The problem in an unbounded domain is transformed into the nonlocal boundary value problem in a bounded domain. We then introduce the corresponding variational formulation and show that the variational problem has a unique solution. We also give some properties of the natural integral operator $\mathcal{K}^c_\lambda$, and the bilinear form $\hat{D}_2(\cdot,\cdot)$ obtained by operator $\mathcal{K}^c_\lambda$. The finite element discretization is employed to solve the variational problem in section 4. The convergence for the numerical solution is taken into account. In section 5 we present some numerical results to illustrate feasibility and efficiency of our method.

2. Statement of the Problem and Time Discretization

Let Γ_0 be a closed curve in plane, and Ω be an exterior domain with Γ_0 as its boundary. For any fixed positive number T, writing $J := (0,T]$. Consider the following initial boundary problems:

$$\begin{cases} u_{tt} - a^2\Delta u = f(x,t), & (x,t) \in \Omega \times J; \\ \dfrac{\partial u(x,t)}{\partial n} = g(x,t), & (x,t) \in \Gamma_0 \times J; \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), & x \in \Omega. \end{cases} \tag{2.1}$$

where $u(x,t)$ is the unknown function, u_t and u_{tt} denote the first derivative and the second derivative with respect to time t, respectively. a is a given positive constant (It is usually wave speed). $f(x,t)$, $g(x,t)$, $\varphi(x)$ and $\psi(x)$ are all given functions. $\frac{\partial}{\partial n}$ is the normal derivative operator on Γ_0 (Here n is the outer unit normal vector on boundary Γ_0 of domain Ω toward the interior domain with Γ_0 as its boundary). Moreover, we assume the function $u(x,t)$ is bounded at infinity. However, there is no need in a "radiation condition" at infinity to complete the statement of the problem (see [10]).

Let τ be the time-step, and write $t_k = k\cdot\tau$, $u^k(x) = u(x,t_k)$, $z^k(x) = u_t(x,t_k)$, $w^k(x) = u_{tt}(x,t_k)$.

$$\begin{aligned} &\lambda := (\tau a\sqrt{\alpha})^{-1}, \qquad \tilde{u}^{k+1} := u^k + \tau\cdot z^k + \frac{\tau^2}{2}(1-2\alpha)w^k, \\ &\tilde{z}^{k+1} := z^k + (1-\beta)\tau w^k, \qquad \tilde{f}^{k+1} := -\tilde{u}^{k+1} - \alpha\tau^2 f^{k+1} \end{aligned}$$

Here $\alpha \in (0,\frac{1}{2}]$ and $\beta \in [0,1]$, $\quad k = 0,1,2,\cdots,[T/\tau]-1$. Then the original problem (2.1) can be reduced to the following one (discrete problem in time t):

I) Find $\tilde{u}^{k+1}$,$\tilde{z}^{k+1}$ and $\tilde{f}^{k+1}$ $(k = 0, 1, 2, \cdots, [T/\tau] - 1)$

$$\tilde{u}^{k+1} = u^k + \tau z^k + \frac{\tau^2}{2}(1 - 2\alpha)w^k \tag{2.2}$$

$$\tilde{z}^{k+1} = z^k + (1 - \beta)\tau w^k \tag{2.3}$$

$$\tilde{f}^{k+1} = -\tilde{u}^{k+1} - \alpha\tau^2 f^{k+1} \tag{2.4}$$

II) Solve the boundary value problem

$$\begin{cases} \Delta u^{k+1} - \lambda^2 u^{k+1} = \lambda^2 \tilde{f}^{k+1}, & x \in \Omega; \\ \dfrac{\partial u^{k+1}}{\partial n} = g^{k+1}(x), & x \in \Gamma_0; \\ |u^{k+1}| < +\infty, & |x| \longrightarrow +\infty. \end{cases} \tag{2.5}$$

III) Update w^{k+1} and z^{k+1}, preformed in the following relations:

$$w^{k+1} = a^2\lambda^2(u^{k+1} - \tilde{u}^{k+1}) \tag{2.6}$$

$$z^{k+1} = \tilde{z}^{k+1} + \beta\tau w^{k+1} \tag{2.7}$$

As stated above, we discretize the problem (2.1) with respect to time t, and obtain a time-stepping scheme (2.5) at each time step τ. For each time level, we first find $\tilde{u}^{k+1}$, $\tilde{z}^{k+1}$ and $\tilde{f}^{k+1}$ by (2.2) − (2.4) in turn. Second we mainly solve the boundary value problem (2.5), and get u^{k+1}. Then we update w^{k+1} and z^{k+1} by (2.6) − (2.7) in order to solve the problem in next time level. We repeat the solution procedure, and can obtain u^k at each time t_k.

3. Coupled Problem and its Variational Formulation

We truncate the exterior domain by introducing an artificial common boundary Γ_R consisting of a circle of radius R large enough to contain the interior domain with Γ_0 as its boundary. Γ_R divides the unbounded domain Ω into two subregions : a bounded annular region Ω_1 by Γ_0 and Γ_R

$$\Omega_1 := \{(x_1, x_2) \mid (x_1, x_2) \in \Omega,\ x_1^2 + x_2^2 < R^2\},$$

and a regular unbounded outer region Ω_2

$$\Omega_2 := \{(x_1, x_2) \mid (x_1, x_2) \in \Omega,\ x_1^2 + x_2^2 > R^2\}.$$

Referring to [9], the solution of the unbounded problem for Ω_2 is given by the following Poisson integral formula:

$$u^{k+1}(r, \theta) - F(\lambda, R; \tilde{f}^{k+1}, r, \theta) = \mathcal{P}_\lambda^c \gamma_0 u^{k+1}, \qquad r > R \tag{3.1}$$

where

$$\mathcal{P}_\lambda^c \gamma_0 u^{k+1} = \frac{1}{2\pi} \sum_{n=0}^{+\infty} \varepsilon_n \frac{K_n(\lambda r)}{K_n(\lambda R)} \int_0^{2\pi} \cos n(\theta - \theta') \cdot u^{k+1}(R, \theta') d\theta' \tag{3.2}$$

$$F(\lambda, R; \tilde{f}^{k+1}, r, \theta) = \frac{\lambda^2}{2} \sum_{n=0}^{+\infty} \varepsilon_n \int_R^{+\infty} \sigma^2 G_n(r, \sigma) \cdot \left[\tilde{f}_n^{k+1,c}(\sigma)\cos n\theta + \tilde{f}_n^{k+1,s}(\sigma)\sin n\theta\right] d\sigma \tag{3.3}$$

$$\tilde{f}_n^{k+1,c}(\sigma) = \frac{1}{\pi} \int_0^{2\pi} \cos n\theta \cdot \tilde{f}^{k+1}(\sigma, \theta) d\theta, \qquad n = 0, 1, 2, \cdots$$

$$\tilde{f}_n^{k+1,s}(\sigma) = \frac{1}{\pi} \int_0^{2\pi} \sin n\theta \cdot \tilde{f}^{k+1}(\sigma, \theta) d\theta, \qquad n = 1, 2, \cdots$$

$$\sigma^2 G_n(r,\sigma) = \begin{cases} K_n(\lambda\sigma)\psi_n(r)/E_n(\sigma), & r \le \sigma \\ \psi_n(\sigma)K_n(\lambda r)/E_n(\sigma), & r \ge \sigma \end{cases}$$

$$\psi_n(\sigma) = I_n(\lambda\sigma)K_n(\lambda R) - K_n(\lambda\sigma)I_n(\lambda R)$$

$$E_n(\sigma) = K_n(\lambda R)\big(I_n(\lambda\sigma)\big)^2 \frac{d}{d\sigma}\Big[\frac{K_n(\lambda\sigma)}{I_n(\lambda\sigma)}\Big]$$

where $\varepsilon_n = 1, n = 0; \varepsilon_n = 2, n > 0$. And there is a natural integral equation on Γ_R (where γ_0 is the trace operator, i.e., $\gamma_0 u = u|_{\Gamma_R}$)

$$\frac{\partial u^{k+1}(R,\theta)}{\partial n} + G(\lambda, R; \tilde{f}^{k+1}, \theta) = \mathcal{K}_\lambda^c \gamma_0 u^{k+1} \tag{3.4}$$

where

$$\mathcal{K}_\lambda^c \gamma_0 u^{k+1} = -\frac{\lambda}{2\pi} \sum_{n=0}^{+\infty} \varepsilon_n \frac{K_n'(\lambda R)}{K_n(\lambda R)} \int_0^{2\pi} \cos n(\theta - \theta') \cdot u^{k+1}(R,\theta') d\theta' \tag{3.5}$$

$$G(\lambda, R; \tilde{f}^{k+1}, \theta) = \frac{\lambda^2}{2} \sum_{n=0}^{+\infty} \varepsilon_n \int_R^{+\infty} \overline{G}_n(\lambda, R; \sigma) \cdot \big[\tilde{f}_n^{k+1,c}(\sigma)\cos n\theta + \tilde{f}_n^{k+1,s}(\sigma)\sin n\theta\big] d\sigma \tag{3.6}$$

$$\overline{G}_n(\lambda, R; \sigma) = -\frac{K_n(\lambda\sigma)}{K_n(\lambda R)} \cdot \frac{\sigma}{R}, \qquad n = 0, 1, 2, \cdots\cdots \tag{3.7}$$

where $I_n(x)$ and $K_n(x)$, which appear in the expressions above, are the modified Bessel functions of the first and second kind, respectively. The expressions (3.1) and (3.4) are usually known as the Poisson integral formula and the natural integral equation, respectively. The Poisson integral operator $\mathcal{P}_\lambda^c$ and natural integral operator $\mathcal{K}_\lambda^c$ are respectively expressed by (3.2) and (3.5). From this, the problem can be reduced to the following problem:

$$\begin{cases} \Delta u^{k+1} - \lambda^2 u^{k+1} = \lambda^2 \tilde{f}^{k+1}, & x \in \Omega_1; \\ \dfrac{\partial u^{k+1}}{\partial n} = g^{k+1}(x), & x \in \Gamma_0; \\ \dfrac{\partial u^{k+1}(R,\theta)}{\partial n} + G(\lambda, R; \tilde{f}^{k+1}, \theta) = \mathcal{K}_\lambda^c \gamma_0 u^{k+1}, & \text{on } \Gamma_R. \end{cases} \tag{3.8}$$

Using the conventional method, it is not difficult to obtain the following variational formulation associated with the problem (3.8):

$$\begin{cases} \text{find } u^{k+1} \in H^1(\Omega_1), \text{ such that} \\ D_1(u^{k+1}, v^{k+1}) + \hat{D}_2(u^{k+1}, v^{k+1}) = \hat{f}(v^{k+1}) \end{cases} \tag{3.9}$$

for all $v^{k+1} \in H^1(\Omega_1)$. Where

$$D_1(u,v) := \int_{\Omega_1} (\nabla u \cdot \nabla v + \lambda^2 u \cdot v) dx, \qquad (u,v)_1 := -\int_{\Omega_1} \lambda^2 u \cdot v dx,$$

$$\hat{D}_2(u,v) := < \mathcal{K}_\lambda^c \gamma_0 u, \gamma_0 v >_{\Gamma_R} \equiv \int_{\Gamma_R} \mathcal{K}_\lambda^c \gamma_0 u \cdot \gamma_0 v dS, \qquad < u, v >_* := \int_* u \cdot v dS,$$

$$\hat{f}(v) := < G(\lambda, R; \tilde{f}^{k+1}, \theta), v >_{\Gamma_R} + < g^{k+1}, v >_{\Gamma_0} + (\tilde{f}^{k+1}, v)_1.$$

For any non-negative p, we introduce Sobolev space as follows

$$H^p(\Gamma_R) := \{F : F \in L^2(\Gamma_R), \|F\|_{p,\Gamma_R} < +\infty\} \tag{3.10}$$

where the $H^p(\Gamma_R)$-norm of F is given by

$$\|F\|_{p,\Gamma_R}^2 := \{\sum_{n=-\infty}^{+\infty} (1+n^2)^p |F_n|^2\} \tag{3.11}$$

$$F_n = \frac{1}{2\pi}\int_0^{2\pi} F(R,\theta)e^{-in\theta}d\theta, \qquad \overline{F_n} = F_{-n} \tag{3.12}$$

The space $H^{-p}(\Gamma_R)$ is the dual space of $H^p(\Gamma_R)$.

Lemma 3.1[9]**.** *For the modified Bessel function $K_n(x)$, and $x > b$ (where b is arbitrary finite positive real number), the following assertion holds*

$$\left|\frac{K_n'(x)}{K_n(x)}\right|^2 \cdot \frac{1}{1+n^2} = O(1), \qquad n \longrightarrow +\infty \tag{3.13}$$

Corollary 3.1. *For all natural number n, $\left|\dfrac{K_n'(\lambda R)}{K_n(\lambda R)}\right| \cdot \dfrac{1}{\sqrt{1+n^2}}$ is bounded.*

Proof. By Lemma 3.1, there exist a positive integer N and two positive constants c_1 and c_2 such that $c_1 < |K_n'(\lambda R)/K_n(\lambda R)|/\sqrt{1+n^2} < c_2$ as $n > N$ and we can prove the assertion with

$$M_{min} := \min\left\{c_1,\ \min_{0\le n\le N}\left\{\left|\frac{K_n'(\lambda R)}{K_n(\lambda R)}\right| \cdot \frac{1}{\sqrt{1+n^2}}\right\}\right\}$$

$$M_{max} := \max\left\{c_2,\ \max_{0\le n\le N}\left\{\left|\frac{K_n'(\lambda R)}{K_n(\lambda R)}\right| \frac{1}{\sqrt{1+n^2}}\right\}\right\}$$

Theorem 3.1. *The natural integral operator $\mathcal{K}_\lambda^c$ has the following properties:*

(i) *The natural integral operator $\mathcal{K}_\lambda^c$ is a linear operator from $H^{p+1/2}(\Gamma_R)$ to $H^{p-1/2}(\Gamma_R)$;*

(ii) *The operator $\mathcal{K}_\lambda^c$ has the following explicit form*

$$\mathcal{K}_\lambda^c v = -\lambda \sum_{n=-\infty}^{+\infty} v_n \cdot \frac{K_n'(\lambda R)}{K_n(\lambda R)} \cdot e^{in\theta}. \tag{3.14}$$

where $\{v_n\}_{n=-\infty}^{+\infty}$ are the Fourier coefficients of function v.

(iii) *For all $f \in H^{p+1/2}(\Gamma_R)$, the following inequality holds*

$$\|\mathcal{K}_\lambda^c f\|_{p-1/2,\Gamma_R} \le C_\lambda \cdot \|f\|_{p+1/2,\Gamma_R} \tag{3.15}$$

where p is any non-negative integer, and C_λ is a positive constant independent of f.

Proof. (i) It is obvious that $\mathcal{K}_\lambda^c$ is a linear operator.

(ii) Setting

$$\overline{K}_n(\lambda, R; \theta-\theta') = -\sum_{n=0}^{+\infty} \varepsilon_n \frac{K_n'(\lambda R)}{K_n(\lambda R)} \cos n(\theta-\theta')$$

Since $e^{in\phi} + e^{-in\phi} = 2\cos n\phi$, and $K_{-n}(x) = K_n(x)$ (see [11]), we have

$$\overline{K}_n(\lambda, R; \theta-\theta') = -\sum_{n=-\infty}^{+\infty} \frac{K_n'(\lambda R)}{K_n(\lambda R)} \cdot e^{in(\theta-\theta')}$$

Thus, from (3.5) we get

$$\begin{aligned}
\mathcal{K}_\lambda^c v &= \frac{\lambda}{2\pi}\int_0^{2\pi}\left(-\sum_{n=-\infty}^{+\infty}\frac{K_n'(\lambda R)}{K_n(\lambda R)} \cdot e^{in(\theta-\theta')}\right) \cdot \left(\sum_{n=-\infty}^{+\infty} v_n e^{in\theta'}\right) d\theta' \\
&= \frac{\lambda}{2\pi}\sum_{n=-\infty}^{+\infty}\left(-2\pi \cdot \frac{K_n'(\lambda R)}{K_n(\lambda R)} \cdot v_n\right) e^{in\theta} \\
&= -\lambda \sum_{n=-\infty}^{+\infty} v_n \cdot \frac{K_n'(\lambda R)}{K_n(\lambda R)} e^{in\theta}
\end{aligned}$$

(iii) For all non-negative integer p, given $f \in H^{p+1/2}(\Gamma_R)$, we must show that its image $F := \mathcal{K}_\lambda^c f$ lies in $H^{p-1/2}(\Gamma_R)$. Let $h_\ell(x) := \dfrac{K'_\ell(x)}{K_\ell(\lambda R)}$. From (ii), we find out that

$$F(R,\theta) = -\lambda \sum_{n=-\infty}^{+\infty} f_n \cdot h_n(\lambda R) e^{in\theta} \equiv \sum_{n=-\infty}^{+\infty} F_n \cdot e^{in\theta}$$

From corollary 3.1, we have

$$\begin{aligned}
\|\mathcal{K}_\lambda^c f\|^2_{p-1/2,\Gamma_R} &= \|F\|^2_{p-1/2,\Gamma_R} = \sum_{n-\infty}^{+\infty} (1+n^2)^{p-1/2} \cdot |F_n|^2 \\
&= \lambda^2 \sum_{n=-\infty}^{+\infty} \cdot (1+n^2)^{p+1/2} |f_n|^2 \cdot \big(|h_n(\lambda R)|^2 \cdot \frac{1}{1+n^2}\big) \\
&= C\lambda^2 \sum_{n=-\infty}^{+\infty} (1+n^2)^{p+1/2} |f_n|^2 \\
&\leq C \cdot \lambda^2 \cdot \|f\|^2_{p+1/2,\Gamma_R}
\end{aligned}$$

Theorem 3.2. *The bilinear form $\hat{D}_2(u,v)$ obtained by natural integral operator $\mathcal{K}_\lambda^c$ is symmetric and continuous on $H^{1/2}(\Gamma_R) \times H^{1/2}(\Gamma_R)$ and positive definite on $H^{1/2}(\Gamma_R)$, i.e.*

(i) $\hat{D}_2(u,v) = \hat{D}_2(v,u), \qquad \forall\, u,v \in H^{1/2}(\Gamma_R)$ (3.16)

(ii) *There exists a positive constant C_0 such that*

$\hat{D}_2(u,v) \leq C_0 \cdot \|u\|_{1/2,\Gamma_R} \cdot \|v\|_{1/2,\Gamma_R}, \qquad \forall\, u,v \in H^{1/2}(\Gamma_R)$ (3.17)

(iii) *There exists a positive constant C_1 such that*

$\hat{D}_2(u,u) \geq C_1 \|u\|^2_{1/2,\Gamma_R} \qquad \forall\, u \in H^{1/2}(\Gamma_R)$ (3.18)

Proof. From (3.14), for all $u \in H^{1/2}(\Gamma_R)$

$$\mathcal{K}_\lambda^c u = -\lambda \sum_{n=-\infty}^{+\infty} u_n \cdot h_n(\lambda R) \cdot e^{in\theta}$$

(i) For all $u,v \in H^{1/2}(\Gamma_R)$, we have

$$\begin{aligned}
\hat{D}_2(u,v) &= <\mathcal{K}_\lambda^c \gamma u, \gamma v>_{\Gamma_R} \\
&= \int_{\Gamma_R} \Big(\sum_{n=-\infty}^{+\infty} v_n \cdot e^{in\theta}\Big) \cdot \Big(-\lambda \sum_{m=-\infty}^{+\infty} u_m \cdot h_m(\lambda R) \cdot e^{im\theta}\Big) dS \\
&= -\lambda R \int_0^{2\pi} \Big(\sum_{n=-\infty}^{+\infty} v_n \cdot e^{in\theta}\Big) \cdot \Big(\sum_{m=-\infty}^{+\infty} u_m \cdot h_m(\lambda R) \cdot e^{im\theta}\Big) d\theta \\
&= -\lambda R \cdot 2\pi \sum_{n=-\infty}^{+\infty} h_n(\lambda R) \cdot u_n \cdot \overline{v_n}
\end{aligned}$$

Thus

$$\hat{D}_2(v,u) = -\lambda R \cdot 2\pi \sum_{n=-\infty}^{+\infty} h_n(\lambda R) \cdot v_n \cdot \overline{u_n} = \overline{\hat{D}_2(u,v)} = \hat{D}_2(u,v).$$

The symmetry has been proved.

(ii) From Theorem 3.1, for all $u,v \in H^{1/2}(\Gamma_R)$, we get

$$\hat{D}_2(u,v) = <\mathcal{K}_\lambda^c \gamma u, \gamma v>_{\Gamma_R} \leq \|\mathcal{K}_\lambda^c u\|_{-1/2,\Gamma_R} \cdot \|v\|_{1/2,\Gamma_R} \leq C_\lambda \|u\|_{1/2,\Gamma_R} \cdot \|v\|_{1/2,\Gamma_R}$$

Inequality (3.17) is proved.

(iii) From [11], for the modified Bessel function $K_n(x)$, there is the following recursive formula

$$K_n'(x) = -\frac{1}{2}\big[K_{n+1}(x) + K_{n-1}(x)\big], \quad (n \geq 1); \qquad K_0'(x) = -K_1(x)$$

From the fact that $K_n(\lambda R) > 0$, implying $K_n'(\lambda R) < 0$. Therefore $K_n'(\lambda R)/K_n(\lambda R) < 0$. we have $-K_n'(\lambda R)/K_n(\lambda R) = |K_n'(\lambda R)/K_n(\lambda R)|$. For all $u \in H^{1/2}(\Gamma_R)$, taking $u \equiv v$. From (i) and Corollary 3.1, we can obtain

$$\begin{aligned}
\hat{D}_2(u,u) &= \lambda R \cdot 2\pi \sum_{n=-\infty}^{+\infty} \big(-\frac{K_n'(\lambda R)}{K_n(\lambda R)}\big) \cdot |u_n|^2 \\
&= \lambda R \cdot 2\pi \sum_{n=-\infty}^{+\infty} (1+n^2)^{1/2} |u_n|^2 \cdot \left[-\frac{K_n'(\lambda R)}{K_n(\lambda R)} \cdot \frac{1}{\sqrt{1+n^2}} \right] \\
&\geq \lambda R \cdot 2\pi c \sum_{n=-\infty}^{+\infty} (1+n^2)^{1/2} |u_n|^2 \\
&= C_1 \cdot \|u\|_{1/2,\Gamma_R}^2
\end{aligned}$$

which proves (3.18).

If we define $\mathcal{A}(u,v) := D_1(u,v) + \hat{D}_2(u,v)$, we have the following results:

Theorem 3.3. *The bilinear form $\mathcal{A}(\cdot,\cdot)$ is symmetric and continuous on $H^1(\Omega_1) \times H^1(\Omega_1)$ and $H^1(\Omega_1)-$coercive in the sense that exist two positive constants C_2 and C_3 such that*

(i) $\mathcal{A}(u,v) = \mathcal{A}(v,u), \qquad \forall\ u,v \in H^1(\Omega_1)$

(ii) $\mathcal{A}(u,v) \leq C_2 \cdot \|u\|_{1,\Omega_1} \cdot \|v\|_{1,\Omega_1}, \qquad \forall\ u,v \in H^1(\Omega_1)$

(iii) $\mathcal{A}(u,u) \geq C_3 \cdot \|u\|_{1,\Omega_1}^2, \qquad \forall\ u \in H^1(\Omega_1)$

Proof. (i) It follows from (3.16) and $D_1(\cdot,\cdot)$ is symmetric.

(ii) The Cauchy-Schwarz inequality, (3.17) and the trace theorem give, successively,

$$\begin{aligned}
\mathcal{A}(u,v) &\leq |\mathcal{A}(u,v)| \leq |D_1(u,v)| + |\hat{D}_2(u,v)| \\
&\leq \max\{1,\lambda^2\} \|u\|_{1,\Omega_1} \cdot \|v\|_{1,\Omega_1} + C_\lambda \cdot \|u\|_{1/2,\Gamma_R} \cdot \|v\|_{1/2,\Gamma_R} \\
&\leq C_2 \cdot \|u\|_{1,\Omega_1} \cdot \|v\|_{1,\Omega_1}
\end{aligned}$$

(iii) Let $C_3 := \min\{1,\lambda^2\}$ and by (3.18), we can obtain

$$\mathcal{A}(u,u) \geq D_1(u,u) \geq \min\{1,\lambda^2\} \|u\|_{1,\Omega_1}^2 = C_3 \cdot \|u\|_{1,\Omega_1}^2.$$

From Theorem 3.3 and the closed graph theorem, we have

Theorem 3.4. *Let $\tilde{f}^{k+1} \in L^2(\Omega_1)$, $g^{k+1} \in H^{-1/2}(\Gamma_0)$, the variational problem (3.9) has a unique solution u^{k+1}. Moreover, if Γ_0 is regular enough, $g^{k+1} \in H^{1/2}(\Gamma_0)$ then $u^{k+1} \in H^2(\Omega_1)$ and there exists a positive constant C such that*

$$\|u^{k+1}\|_{2,\Omega_1} \leq C\big\{\|\tilde{f}^{k+1}\|_{0,\Omega_1} + \|g^{k+1}\|_{1/2,\Gamma_0} + \|G\|_{1/2,\Gamma_R}\big\} \tag{3.19}$$

4. Finite Element Discretization

We now consider the finite dimensional version of the variational problem (3.9). To this end, we impose a subdivision on domain Ω_1 into $\{e_i\}$ of which longest element has diameter h. We divide the artificial common boundary Γ_R into N parts (It is equivalent to divide the interval $[0, 2\pi]$ into N parts). In addition, it is assumed the N_1 and N_2 nodes are taken, respectively, in Ω_1 and on Γ_0, and the nodes on Γ_R coincide with the nodes on $\partial\overline{\Omega_1} \cap \Omega$. All the nodes on Γ_R, in Ω_1 and on Γ_0 are numbered from 1 to N to M_1 to M_2 in turn. Where $M_1 = N + N_1$ and

$M_2 = M_1 + N_2$. And let $W_h(\Omega_1)$ be a finite element subspace of space $H^1(\Omega_1)$. So the discrete variational problem corresponding to (3.9) is

$$\begin{cases} \text{Find } \; u_h^{k+1} \in W_h(\Omega_1), \text{ such that} \\ D_1(u_h^{k+1}, v_h^{k+1}) + \hat{D}_2(u_h^{k+1}, v_h^{k+1}) = \hat{f}_h(v_h^{k+1}), \qquad v_h^{k+1} \in W_h(\Omega_1) \end{cases} \tag{4.1}$$

Where $\hat{f}_h(v_h^{k+1}) :=< G(\lambda, R; \tilde{f}_h^{k+1}, \theta), v_h^{k+1} >_{\Gamma_R} + < g^{k+1}, v_h^{k+1} >_{\Gamma_0} + (\tilde{f}_h^{k+1}, v_h^{k+1})_1$, and $\tilde{f}_h^{k+1} := -\tilde{u}_h^{k+1} - \alpha\tau^2 f^{k+1}$. Since $W_h(\Omega_1) \subset H^1(\Omega_1)$ and the existence and uniqueness of solution to the problem (3.9), the problem (4.1) has a unique solution u_h^{k+1} in $W_h(\Omega_1)$.

Let $\{\phi_i(x)\}_{i=1}^{M_2}$ be a basis function system of $W_h(\Omega_1)$, then finite element discretization of problem (4.1) leads to the system of algebraic equations

$$K \cdot U^{k+1} = b^{k+1} \tag{4.2}$$

where $K := K^{(1)} + \begin{bmatrix} K_{11}^{(2)} & 0 \\ 0 & 0 \end{bmatrix}$, $K^{(1)} := \Big(D_1(\phi_i, \phi_j)\Big)_{M_2 \times M_2}$, $K_{11}^{(2)} := \Big(\hat{D}(\phi_i, \phi_j)\Big)_{N \times N}$, $U^{k+1} := \Big(u_1^{k+1}, u_2^{k+1}, \cdots, u_{N+1}^{k+1}, \cdots, u_{M_2}^{k+1}\Big)^T$, $b^{k+1} := \Big(b_1^{k+1}, b_2^{k+1}, \cdots, b_N^{k+1}, b_{N+1}^{k+1}, \cdots, b_{M_2}^{k+1}\Big)^T$,
$b_i^{k+1} := \hat{f}_h(\phi_i) = \int_0^{2\pi} G(\lambda, R; \tilde{f}_h^{k+1}, \theta) \cdot \phi_i(\theta) \cdot R d\theta + \int_{\Gamma_0} g^{k+1} \cdot \phi_i(\theta) dS - \lambda^2 \int_{\Omega_1} \tilde{f}_h^{k+1} \cdot \phi_i(x) dx.$

According to Theorems 3.2 and 3.3, we know that matrices $K^{(1)}$, $K_{11}^{(2)}$ and K are all symmetric and positive definite, we can obtain

Theorem 4.1. *The system of linear equations (4.2) has a unique solution.*

Once we solve the system of linear equations (4.2), and get U^{k+1}, then the approximate solution of the original problem can be expressed by

$$u_h^{k+1} = \begin{cases} \displaystyle\sum_{j=1}^{M_2} u_j^{k+1} \cdot \phi_j(x), & x \in \overline{\Omega_1} \\ \displaystyle\sum_{j=1}^{N} u_j^{k+1} \cdot \mathcal{P}_\lambda^c \cdot \gamma_0 \phi_j(x) + F(\lambda, R; \tilde{f}_h^{k+1}, r, \theta), & x \in \Omega_2 \end{cases} \tag{4.3}$$

Since

$$\begin{aligned} \hat{D}_2(\phi_i, \phi_j) &= \int_0^{2\pi} \phi_j(\theta) \cdot \mathcal{K}_\lambda^c \phi_i(\theta) \cdot R d\theta \\ &= -\frac{\lambda R}{2\pi} \sum_{n=0}^{+\infty} \varepsilon_n \int_0^{2\pi} \int_0^{2\pi} h_n(\lambda R) \cos n(\theta - \theta') \phi_i(\theta) \cdot \phi_j(\theta') d\theta d\theta' \end{aligned}$$

If Γ_R is divided into N equal parts and $\phi_i(x)$ $(i = 1, 2, \cdots, N)$ are piecewise linear functions, and setting

$$a_m = -\frac{2\pi\lambda R}{N^2} \Big\{ h_0(\lambda R) + \frac{2N^4}{\pi^4} \sum_{n=1}^{+\infty} \frac{1}{n^4} h_n(\lambda R) \cdot \sin^4\big(\frac{n\pi}{N}\big) \cdot \cos\big(\frac{nm}{N} \cdot 2\pi\big) \Big\} \tag{4.4}$$
$$m = 0, 1, 2, \cdots, N-1$$

Thus

$$\hat{D}_2(\phi_i, \phi_j) = a_{|i-j|} = \hat{D}_2(\phi_j, \phi_i), \quad i, j = 1, 2, \cdots, N. \tag{4.5}$$

$$K_{11}^{(2)} = (\hat{D}_2(\phi_i, \phi_j))_{N \times N} = \Big((a_0, a_1, \cdots, a_{N-1})\Big) \tag{4.6}$$

From (4.6), we know that the matrix $K_{11}^{(2)}$ is a cyclical matrix produced by $a_0, a_1, \cdots, a_{N-1}$. We can get the matrix $K_{11}^{(2)}$ by only calculating $[\frac{N}{2}] + 1$ numbers $a_0, a_1, \cdots, a_{[\frac{N}{2}]}$ owing to $a_i = a_{N-i}$ $(i = 0, 1, 2, \cdots, N-1)$. So it is easy to be implemented on calculation and storage.

From Lemma 3.1, the series a_m is convergent, and absolute convergence. Under the statements above, we have

$$\mathcal{P}_\lambda^c \cdot \gamma_0 \phi_j = \frac{1}{N}\left\{\frac{K_0(\lambda r)}{K_0(\lambda R)} + \frac{2N^2}{\pi^2}\sum_{n=1}^{+\infty}\left[\frac{1}{n^2}\frac{K_n(\lambda r)}{K_n(\lambda R)}\left(\sin\frac{n\pi}{N}\right)^2 \cdot \cos n\left(\theta - j\frac{2\pi}{N}\right)\right]\right\} \tag{4.7}$$

Referring to [8], we have

Theorem 4.2. *Let u^{k+1} and u_h^{k+1} be the solutions to problems (3.9) and (4.1), respectively. Then the following hold*

$$\lim_{h\longrightarrow 0}\|u^{k+1} - u_h^{k+1}\|_1 = 0, \qquad \lim_{h\longrightarrow 0}\|u^{k+1} - u_h^{k+1}\|_2 = 0. \tag{4.8}$$

where $\|\cdot\|_1 := \sqrt{D_1(\cdot,\cdot)}$ and $\|\cdot\|_2 := \sqrt{\hat{D}_2(\cdot,\cdot)}$.

5. Numerical Examples

We consider the following problem in square exterior domain Ω, i.e. $\Omega := \{(x_1,x_2) \mid (x_1,x_2) \in R^2;\ |x_i| > 1,\ i = 1,2\}$. $f(x,t) = e^{-\pi t}\{(\frac{2\pi^2}{r} - \frac{3}{r^3} + \pi^2)\sin\pi x_1 \cdot \sin\pi x_2 + \frac{2\pi}{r^3}(x_1\cos\pi x_1 \sin\pi x_2 + x_2 \sin\pi x_1 \cos\pi x_2)\}$, where $r = \sqrt{x_1^2 + x_2^2}$. $a = 1$, $\varphi(x) = \frac{1}{r}\sin\pi x_1 \cdot \sin\pi x_2$, $\psi(x) = -\frac{\pi}{r}\sin\pi x_1 \cdot \sin\pi x_2$. $\Gamma_R := \{(x_1,x_2) \mid (x_1,x_2) \in \mathbf{R}^2; r = 2\}$.

$$g(1,x_2,t) = \frac{\pi}{\sqrt{1+x_2^2}} \cdot e^{-\pi t} \cdot \sin\pi x_2, \qquad |x_2| \le 1;$$

$$g(x_1,1,t) = \frac{\pi}{\sqrt{1+x_1^2}} \cdot e^{-\pi t} \cdot \sin\pi x_1, \qquad |x_1| < 1;$$

$$g(-1,x_2,t) = -\frac{\pi}{\sqrt{1+x_2^2}} \cdot e^{-\pi t} \cdot \sin\pi x_2, \qquad |x_2| \le 1;$$

$$g(x_1,-1,t) = -\frac{\pi}{\sqrt{1+x_1^2}} \cdot e^{-\pi t} \cdot \sin\pi x_1, \qquad |x_1| < 1.$$

The expressions for $F(\lambda, R; \tilde{f}^{k+1}, r, \theta)$ and $G(\lambda, R; \tilde{f}^{k+1}, \theta)$ (see (3.3) and (3.6)) involve infinite series. In practice all the infinite sums are truncated after a finite number of terms, M. So are the expressions for a_m and $\mathcal{P}_\lambda^c \cdot \gamma_0\phi_j$ (see (4.4) and (4.7)). E_{max} denotes the maximum of the relative errors for all the nodes on Γ_R, in Ω_1 and on Γ_0. The results are shown in the following Tables 5.1 and 5.2.

Table 5.1: $\alpha = 1/6, \beta = 1/2,\ t = 0.1$

N	M_2	M	$E_{max}(\tau = 0.05)$	$E_{max}(\tau = 0.025)$	$E_{max}(\tau = 0.0125)$
8	28	20	5.973551E-1	2.131856E-1	4.125776E-2
16	66	40	2.178516E-1	5.925028E-2	1.178099E-2
32	232	80	7.015094E-2	1.910536E-2	3.480178E-3
64	784	120	2.231752E-2	5.753432E-3	1.197058E-3

Table 5.2: $\alpha = 1/6, \beta = 1/2,\ t = 1$

N	M_2	M	$E_{max}(\tau = 0.05)$	$E_{max}(\tau = 0.025)$	$E_{max}(\tau = 0.0125)$
8	28	20	2.381972E-1	7.155211E-2	1.701513E-2
16	66	40	8.019251E-2	2.283867E-2	4.659449E-3
32	232	80	2.746892E-2	6.310132E-3	1.417132E-3
64	784	120	7.914087E-3	2.096349E-3	3.908435E-4

As shown in the Tables above, the coupling of natural boundary element and finite element method is efficient.

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