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ROBUSTNESS OF AN UPWIND FINITE DIFFERENCE SCHEME FOR SEMILINEAR CONVECTION-DIFFUSION PROBLEMS WITH BOUNDARY TURNING POINTS *

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Abstract

We consider a singularly perturbed semilinear convection-diffusion problem with a boundary layer of attractive turning-point type. It is shown that its solution can be decomposed into a regular solution component and a layer component. This decomposition is used to analyse the convergence of an upwinded finite difference scheme on Shishkin meshes.

Key words: Convection-diffusion, Singular perturbation, Solution decomposition, Shishkin mesh.

1. Introduction

We consider the singularly perturbed semilinear convection-diffusion problem

$$\mathcal{T}u(x) := -\varepsilon u''(x) - x^p a(x)u'(x) + x^p b(x,u(x)) = 0 \quad \text{for} \quad x \in (0,1), \tag{1a}$$

$$u(0) = \gamma_0,\ u(1) = \gamma_1, \tag{1b}$$

where $0 < \varepsilon \ll 1$ is a small constant, $p > 0$, $a(x) > \alpha > 0$, $b_u \geq 0$ for $x \in [0,1]$, $a \in C^1[0,1]$ and $b \in C^1([0,1] \times I\!R)$. Its solution u typically has a boundary layer of width $\mathcal{O}\big(\varepsilon^{1/(p+1)} \ln \varepsilon\big)$ at $x = 0$. Numerical schemes for the case when $p = 0$ have been extensively studied in the literature; see [6] for a survey.

The class of problems considered includes

$$-\varepsilon u'' - xu' + xu = 0, \quad \text{for} \quad x \in (0,1), \quad u(0) = \gamma_0,\ u(1) = \gamma_1,$$

which models heat flow and mass transport near oceanic rises [1]. Multiple boundary turning points ($p > 1$) are also covered by (1) and they too arise in applications [7].

We are aware of four publications that analyse numerical methods for (1) with $p = 1$. Liseikin [2] constructs a special transformation and solves the transformed problem on a uniform mesh. The method obtained is proven to be first-order uniformly convergent in the discrete maximum norm. Vulanović [8] studies an upwind-difference scheme on a layer-adapted Bakhvalov-type mesh and proves convergence in a discrete ℓ_1 norm. This result is generalized in [9] for quasilinear problems. In [3] the authors establish almost first-order convergence in the discrete ℓ_∞ norm for an upwind difference scheme on a Shishkin mesh. There are also a number of papers that consider problems of the type

$$-\varepsilon u''(x) - x^p a(x)u'(x) + c(x,u(x)) = 0 \quad \text{in} \quad (0,1)$$

with Dirichlet boundary conditions and $c_u(0,u(0)) \geq \gamma > 0$. In this case, however, the behaviour is dominated by the relation between the diffusion term and the reaction term. The layer structure is like that of reaction-diffusion problems and is different from the layer occurring in (1). We are not aware of any publication that considers numerical methods for (1) with general $p > 0$.

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The main purpose of the present paper is to derive a decomposition of the solution of (1) into a regular solution component and a boundary layer component, with sharp estimates for their derivatives up to the third order (Section 2). In Section 3 we shall show how this decomposition can be used to analyse the convergence of an upwinded difference scheme for the approximate solution of (1). We prove that the scheme on a Shishkin mesh is almost first-order convergent in the discrete maximum norm, no matter how small the perturbation parameter ε may be. This error analysis is based on a hybrid stability inequality derived in [3] which implies that the error in the ℓ_∞ norm is bounded by a specially weighted ℓ_1 norm of the truncation error.

Notation. By C we denote throughout the paper a generic positive constant that is independent of ε and of N, the number of mesh nodes used.

2. Solution Decomposition

Theorem 1. *Let $a \in C^1[0,1]$ and $b \in C^1([0,1] \times \mathbb{R})$. Then (1) has a unique solution $u \in C^3[0,1]$ and this solution can be decomposed as $u = v + w$, where the regular solution component v satisfies*

$$\mathcal{T}v = 0, \quad \left|v'(x)\right| + \left|v''(x)\right| \le C \quad \text{and} \quad \varepsilon\left|v'''(x)\right| \le Cx^p \quad \text{for} \quad x \in (0,1),$$

while the boundary layer component w satisfies

$$\tilde{\mathcal{T}}w := -\varepsilon w'' - x^p a w' + x^p \tilde{b}(x,w) = 0, \quad \tilde{b}(x,w) = b(x, v+w) - b(x,v)$$

and

$$\left|w^{(i)}(x)\right| \le C\mu^{-i} \exp\left(-\frac{\alpha x^{p+1}}{\varepsilon(p+1)}\right) \quad \text{for} \quad i = 0,1,2,3, \quad x \in (0,1)$$

with $\mu = \varepsilon^{1/(p+1)}$.

Proof. The decomposition is constructed as follows. We define v and w to be the solution of the boundary-value problems

$$\mathcal{T}v = 0 \quad \text{for} \quad x \in (0,1), \quad a(0)v'(0) = b(0, v(0)), \ v(1) = \gamma_1 \tag{2a}$$

and

$$\tilde{\mathcal{T}}w = 0 \quad \text{for} \quad x \in (0,1), \quad w(0) = \gamma_0 - v(0), \ w(1) = 0. \tag{2b}$$

The bounds for v and w and their derivatives will be given in Sections 2.2 and 2.3.

2.1. Preliminaries

Let

$$A(x) := \frac{1}{\varepsilon}\int_0^x s^p a(s) ds$$

and choose α^* to satisfy $a(x) \ge \alpha^* > 0$. For our analysis we need bounds for a number of integral expressions involving A. First of all we have

$$-A(x) \le -\frac{\alpha^*}{\varepsilon}\frac{x^{p+1}}{p+1} \quad \text{and} \quad A(s) - A(x) \le \frac{\alpha^*}{\varepsilon}\frac{s^{p+1} - x^{p+1}}{p+1} \quad \text{for} \quad 0 \le s \le x \le 1. \tag{3}$$

From this, for arbitrary $q \ge 0$ we get

$$\frac{\alpha^*}{\varepsilon}\int_0^x s^{(p+q)} \exp(A(s) - A(x)) ds \le \frac{\alpha^*}{\varepsilon}\int_0^x s^p \exp\left(\frac{\alpha^*}{\varepsilon}\frac{s^{p+1} - x^{p+1}}{p+1}\right) ds \le 1. \tag{4}$$

We shall also use

$$\begin{aligned}\int_0^1 \exp(-A(s)) ds &\ge \int_0^1 \exp\left(-\frac{\|a\|_\infty s^{p+1}}{(p+1)\varepsilon}\right) ds = \mu \int_0^{1/\mu} \exp\left(-\frac{\|a\|_\infty t^{p+1}}{(p+1)}\right) dt \\ &\ge \mu \int_0^1 \exp\left(-\frac{\|a\|_\infty t^{p+1}}{(p+1)}\right) dt = C\mu.\end{aligned} \tag{5}$$

Lemma 1. *For arbitrary $p > 0$ there exists a constant $C = C(p)$ such that*

$$\frac{x^p}{\varepsilon}\int_0^x \exp\left(\frac{\alpha^*}{\varepsilon}\frac{s^{p+1}-x^{p+1}}{p+1}\right) ds \le C \quad \text{for all} \quad x \ge 0,\ \varepsilon > 0.$$

Proof. Using the transformations

$$x = (\varepsilon(p+1)t/\alpha^*)^{1/(p+1)} \quad \text{and} \quad s = (\varepsilon(p+1)\sigma/\alpha^*)^{1/(p+1)},$$

we see that

$$\frac{\alpha^* x^p}{\varepsilon}\int_0^x \exp\left(\frac{\alpha^*}{\varepsilon}\frac{s^{p+1}-x^{p+1}}{p+1}\right) ds = e^{-t}t^{p/(p+1)}\int_0^t e^{\sigma}\sigma^{-p/(p+1)}\, d\sigma := F_p(t).$$

Clearly $F_p \in C^0[0,\infty)$ and $F_p(0) = 0$ for $p > 0$. On the other hand we have $\lim_{t\to\infty} F_p(t) = 1$. Thus there exists a constant $C(p) > 0$ with $\max_{t\in[0,\infty)} F_p(t) \le C(p)$.

2.2. The Regular Solution Component

In (2a) we have defined v as the solution of

$$\mathcal{T}v(x) = 0 \quad \text{for} \quad x \in (0,1), \quad \mathcal{B}_0 v := -a(0)v'(0) + b(0, v(0)) = 0,\ v(1) = \gamma_1.$$

The operator $\mathcal{T}$ with these boundary conditions satisfies a comparison principle [5], which ensures the existence of a unique solution: if two functions $\check{u}$ and $\hat{u}$ satisfy $\mathcal{T}\check{u}(x) \le \mathcal{T}\hat{u}(x)$ in $(0,1)$, $\mathcal{B}_0\check{u} \le \mathcal{B}_0\hat{u}$ and $\check{u}(1) \le \hat{u}(1)$, then $\check{u}(x) \le \hat{u}(x)$ on $[0,1]$. Using this comparison principle with

$$v^{\pm} = \pm\left(\frac{1-x}{\alpha}\max_x |b(x,0)| + |\gamma_1|\right),$$

we get

$$\left|v(x)\right| \le C \quad \text{for} \quad x \in (0,1).$$

Now let us bound the derivatives of v. It is easily verified that

$$v(x) = \int_x^1 \vartheta_v(s)ds - \frac{b(0,v(0)}{a(0)}\int_x^1 \exp(-A(s))ds + \gamma_1,$$

where

$$\vartheta_v(x) = -\frac{1}{\varepsilon}\int_0^x s^p b(s,v(s))\exp(A(s)-A(x))ds.$$

From this representation we immediately get

$$v'(x) = \frac{1}{\varepsilon}\int_0^x s^p b(s,v(s))\exp(A(s)-A(x))ds + \frac{b(0,v(0))}{a(0)}\exp(-A(x)). \tag{6}$$

This gives

$$\left|v'(x)\right| \le C \quad \text{for} \quad x \in (0,1),$$

because of (4).

Differentiating (6) once and using integration by parts, we get

$$v''(x) = \frac{x^p a(x)}{\varepsilon}\int_0^x \left(\frac{b(\cdot,v)}{a}\right)'(s)\exp(A(s)-A(x))ds.$$

Therefore

$$\left|v''(x)\right| \le C\frac{x^p}{\varepsilon}\int_0^x \exp(A(s)-A(x))ds \le C\frac{x^p}{\varepsilon}\int_0^x \exp\left(\frac{\alpha}{\varepsilon}\frac{s^{p+1}-x^{p+1}}{p+1}\right)ds$$

and

$$\left|v''(x)\right| \le C \quad \text{for} \quad x \in (0,1)$$

by Lemma 1.

A bound for the third-order derivative is obtained from the differential equation and the bounds on v' and v'':

$$-\varepsilon v''' = x^p\big(av - b(\cdot,v)\big)' + px^{p-1}\big(av' - b(\cdot,v)\big).$$

Let $F(x) := av' - b(\cdot,v)$. Equation (2a) implies $F(0)=0$. On the other hand we have

$$|F'(x)| = \left|\big(av' - b(\cdot,v)\big)'(x)\right| \le C,$$

by our earlier bounds for v, v' and v''. Thus $\left|F(x)\right| \le Cx$. We get

$$\varepsilon\left|v'''(x)\right| \le Cx^p \quad \text{for} \quad x \in (0,1).$$

This completes our analysis of the regular part of u.

2.3. The Boundary Layer Component

Let α_i be arbitrary but fixed with $\min_{x\in[0,1]} a(x) = \alpha_1 > \alpha_2 > \alpha_3 > \alpha$. Recall that the layer component solves

$$\tilde{\mathcal{T}} w(x) = 0 \quad \text{for} \quad x \in (0,1), \quad w(0) = \gamma_0 - v(0), \; w(1) = 0.$$

The operator $\tilde{\mathcal{T}}$ with Dirichlet boundary conditions also satisfies a comparison principle [5]: if two functions $\check{u}$ and $\hat{u}$ satisfy $\tilde{\mathcal{T}}\check{u}(x) \le \tilde{\mathcal{T}}\hat{u}(x)$ in $(0,1)$ and $\check{u}(x) \le \hat{u}(x)$ for $x = 0,1$, then $\check{u}(x) \le \hat{u}(x)$ on $[0,1]$. This comparison principle guarantees the existence of a unique solution. Using the barrier functions

$$w^{\pm} = \pm|\gamma_0 - v(0)| \exp\left(-\frac{\alpha_1}{\varepsilon}\frac{x^{p+1}}{p+1}\right),$$

we obtain

$$\left|w(x)\right| \le C \exp\left(-\frac{\alpha_1}{\varepsilon}\frac{x^{p+1}}{p+1}\right) \quad \text{for} \quad x \in (0,1). \tag{7}$$

To bound the derivatives of w we use

$$w(x) = \int_x^1 \vartheta_w(s)ds - \frac{v(0) - \gamma_0 + \displaystyle\int_0^1 \vartheta_w(s)ds}{\displaystyle\int_0^1 \exp(-A(s))ds} \int_x^1 \exp(-A(s))ds,$$

where

$$\vartheta_w(x) = -\frac{1}{\varepsilon}\int_0^x s^p \tilde{b}(s,w(s)) \exp(A(s) - A(x))ds.$$

Thus

$$w'(x) = -\vartheta_w(x) + \frac{v(0) - \gamma_0 + \displaystyle\int_0^1 \vartheta_w(s)ds}{\displaystyle\int_0^1 \exp(-A(s))ds} \exp(-A(x)). \tag{8}$$

We have

$$\left|\tilde{b}(s,w(s))\right| = \left|b(s,v(s)+w(s)) - b(s,v(s))\right| \le C|w(s)| \le C\exp\left(-\frac{\alpha_1}{\varepsilon}\frac{s^{p+1}}{p+1}\right),$$

by (7). Using this bound and (3) with $\alpha^* = \alpha_1$, we obtain

$$\left|\vartheta_w(x)\right| \le C\frac{x^{p+1}}{\varepsilon}\exp\left(-\frac{\alpha_1}{\varepsilon}\frac{x^{p+1}}{p+1}\right) \le C\exp\left(-\frac{\alpha_2}{\varepsilon}\frac{x^{p+1}}{p+1}\right) \quad \text{for} \quad x \in (0,1). \tag{9}$$

From (5), (8) and (9) we get

$$\left|w'(x)\right| \le C\mu^{-1} \exp\left(-\frac{\alpha_2 x^{p+1}}{\varepsilon(p+1)}\right) \quad \text{for} \quad x \in (0,1).$$

Use the differential equation and the estimates for w and w' to get

$$\left|w''(x)\right| \le C\mu^{-2} \exp\left(-\frac{\alpha_3 x^{p+1}}{\varepsilon(p+1)}\right) \quad \text{for} \quad x \in (0,1).$$

We differentiate (2b) and apply our bounds for w, w' and w'' to get the desired bound for w'''. This completes the proof of Theorem 1.

3. Error Analysis of a First-order Upwind Scheme

Let N, our discretization parameter, be a positive integer. Let $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh and set $h_i = x_i - x_{i-1}$ for $i = 1, \ldots, N$. We discretize (1) using the following simple upwind scheme:

$$\left[TU\right]_i = 0 \quad \text{for } i = 1, \ldots, N-1, \; U_0 = \gamma_0, \; U_N = \gamma_1, \tag{10}$$

where

$$\left[TU\right]_i := -\varepsilon\left[D^+D^-U\right]_i - x_i^p a_i \left[D^+U\right]_i + x_i^p b(x_i, U_i),$$

$$\left[D^+U\right]_i := \frac{U_{i+1} - U_i}{h_{i+1}} \quad \text{and} \quad \left[D^-U\right]_i := \frac{U_i - U_{i-1}}{h_i}.$$

In order to achieve uniform convergence, i.e., convergence that is independent of the perturbation parameter ε, we use generalized Shishkin meshes [4]. Let

$$\lambda = \left(\lambda_0 \frac{\varepsilon(p+1)}{\alpha} \ln N\right)^{1/(p+1)}$$

with a constant $\lambda_0 \ge 2$. Also, let $J = qN$ be a positive integer such that $q < 1$ and $q^{-1} \le C$. We assume that $\lambda \le q$, since N is unreasonably large otherwise. Then we form the Shishkin mesh by dividing the interval $[0, \lambda]$ into J equidistant subintervals and the interval $[\lambda, 1]$ into $N - J$ equidistant subintervals. Note that $x_J = \lambda$. We denote by $h = \lambda/J$ and $H = (1-\lambda)/(N-J) \le C/N$ the local mesh sizes on the fine and coarse parts of the mesh.

In [3] stability properties of the discrete operator T were studied. It was established that (10) possesses a unique solution on arbitrary meshes and that for any mesh functions V and W with $V_0 = W_0$ and $V_N = W_N$, one has

$$\|V - W\|_{\omega,\infty} := \max_{j=0,\ldots,N} |(V-W)_j| \le \alpha^{-1} \|TV - TW\|_{\omega,1} \tag{11}$$

where

$$\|V\|_{\omega,1} := \sum_{j=1}^{N-1} h_{j+1} Q_j \, |V_j|$$

with

$$Q_N = 0 \quad \text{and} \quad Q_{j-1} = \left(1 + \frac{x_{j-1}^p h_j}{\varepsilon}\right)^{-1} \left(Q_j + \frac{h_j}{\varepsilon}\right) \quad \text{for} \quad j = 1, \ldots, N.$$

Using induction it was proved that

$$0 \le Q_j \le x_j^{-p} \quad \text{for} \quad j = 1, \ldots, N-1, \tag{12}$$

see [3]. However for our analysis we need a sharper bound on Q_j for $j = 1, \ldots, J-1$. On our Shishkin mesh we have

$$Q_{j-1} \le Q_j + \frac{\lambda}{\varepsilon J} \quad \text{for} \quad j = 1, \ldots, J,$$

since $h_j = \lambda/J$ for $j \le J$. Thus

$$Q_j \le Q_J + \frac{(J-j)\lambda}{\varepsilon J} \le \frac{1}{\lambda^p} + \frac{\lambda}{\varepsilon} \le C\frac{\lambda}{\varepsilon} \quad \text{for} \quad j = 1, \dots, J-1, \tag{13}$$

by (12).

Theorem 2. *Let u be the solution of problem* (1). *Then the following ε-uniform convergence result holds true for the solution U of the discrete problem* (10) *on the Shishkin mesh:*

$$\left\| u - U \right\|_{\omega,\infty} \le C\left(\ln N\right)^{2/(p+1)} N^{-1}.$$

Proof. We have

$$\left\| u - U \right\|_{\omega,\infty} \le \alpha^{-1} \left\| Tu \right\|_{\omega,1},$$

by (10) and (11). Using the solution decomposition of Theorem 1 it is easily verified that

$$Tu = Lv - \mathcal{L}v + Lw - \mathcal{L}w \quad \text{in the mesh points},$$

where L and $\mathcal{L}$ are the linear parts of T and $\mathcal{T}$, i. e., $L = -\varepsilon D^+ D^- - aD^+$ and $\mathcal{L} = -\varepsilon \frac{d^2}{dx^2} - a\frac{d}{dx}$. In Sections 3.1 and 3.2 we shall show that

$$\left\| Lv - \mathcal{L}v \right\|_{\omega,1} \le CN^{-1} \quad \text{and} \quad \left\| Lw - \mathcal{L}w \right\|_{\omega,1} \le C\left(\ln N\right)^{2/(p+1)} N^{-1},$$

which together with a triangle inequality yields the desired result.

Remark 1. Numerical experiments suggest that the estimate of (13) can be improved to

$$Q_j \le C\mu/\varepsilon \quad \text{for} \quad j = 1, \dots, J-1.$$

This would imply the better convergence result

$$\left\| u - U \right\|_{\omega,\infty} \le C\left(\ln N\right)^{1/(p+1)} N^{-1},$$

but we do not yet have a rigorous proof of this.

3.1. Regular Component of the Error

Let $\tau^v = Tv - \mathcal{T}v$. Then in view of (12) we have

$$\|\tau^v\|_{\omega,1} \le \sum_{j=1}^{N-1} \frac{h_{j+1}}{x_j^p} \left|\tau_j^v\right| \tag{14}$$

When studying $|\tau_j^v|$ we shall distinguish three cases: $j < J$, $j = J$ and $j > J$.

Layer region. For $j = 1, \dots, J-1$ use a Taylor expansion to get

$$\left|\tau_j^v\right| \le Ch\left(\max_{[x_{j-1},x_{j+1}]} |\varepsilon v'''| + x_j^p \max_{[x_j,x_{j+1}]} |v''| \right). \tag{15}$$

Thus

$$\frac{h_{j+1}}{x_j^p}\left|\tau_j^v\right| \le Ch^2 \left\{ \left(\frac{x_{j+1}}{x_j}\right)^p + 1 \right\} \le Ch^2, \tag{16}$$

because $x_{j+1}/x_j = (j+1)/j \le 2$ for $j < J$.

Transition point. For $j = J$ a Taylor expansion gives

$$\left|\tau_j^v\right| \le C\left(\max_{[x_{j-1},x_{j+1}]} |\varepsilon v''| + Hx_j^p \max_{[x_j,x_{j+1}]} |v''| \right).$$

Thus

$$\frac{h_{j+1}}{x_j^p}\left|\tau_j^v\right| \le C\left(\frac{H\varepsilon}{\lambda^p}+H^2\right) \le C\left(H\mu+H^2\right) \le CH. \tag{17}$$

Coarse mesh region. Similarly to (15), for $j = J+1, \dots, N-1$ we have

$$\left|\tau_j^v\right| \le CH\left(\max_{[x_{j-1},x_{j+1}]}|\varepsilon v'''| + x_j^p \max_{[x_j,x_{j+1}]}|v''|\right).$$

Thus

$$\frac{h_{j+1}}{x_j^p}\left|\tau_j^v\right| \le CH^2\left\{\left(\frac{\lambda+(j+1-J)H}{\lambda+(j-J)H}\right)^p+1\right\} \le CH^2, \tag{18}$$

because $\lambda+(j+1-J)H \le 2\left(\lambda+(j-J)H\right)$ for $j > J$.

Combining (14) with (16)–(18), we obtain

$$\|Tv-\mathcal{T}v\|_{\omega,1} \le CN^{-1}.$$

3.2. Layer Component of the Error

Let $\tau_j^w = Tw - \mathcal{T}w$. We have

$$\|\tau^w\|_{\omega,1} \le \sum_{j=1}^{N-1} Q_j h_{j+1}\left|\tau_j^w\right|. \tag{19}$$

We shall distinguish two cases: $j < J$ and $j \ge J$.

Layer region. For $j = 1, \dots, J-1$ we have (cf. (15))

$$\left|\tau_j^w\right| \le Ch\left(\max_{[x_{j-1},x_{j+1}]}|\varepsilon w'''| + x_j^p \max_{[x_j,x_{j+1}]}|w''|\right) \le Ch\mu^{-2}\left(\varepsilon\mu^{-1}+x_j^p\right)\exp\left(-\frac{\alpha x_{j-1}^p}{\varepsilon(p+1)}\right),$$

by Theorem 1. This inequality, (12) and (13) give

$$Q_j h_{j+1}\left|\tau_j^w\right| \le C\frac{\Lambda^3}{J^2}\exp\left(-\frac{\alpha x_{j-1}^p}{\varepsilon(p+1)}\right),$$

where $\Lambda := (\ln N)^{1/(p+1)}$. For any $m > 0$ there exists a constant $\bar{C} = \bar{C}(m)$ such that

$$\exp\left(-\frac{\alpha x^p}{\varepsilon(p+1)}\right) \le \bar{C}\exp\left(-m\frac{x}{\mu}\right).$$

This yields

$$Q_j h_{j+1}\left|\tau_j^w\right| \le C\frac{\Lambda^3}{J^2}\exp\left(-\frac{\Lambda}{J}\right)^{j-1}.$$

Thus

$$\sum_{j=1}^{J-1} Q_j h_{j+1}\left|\tau_j^w\right| \le C\frac{\Lambda^3}{J^2}\frac{1}{1-\exp\left(-\frac{\Lambda}{J}\right)} \le C\frac{\Lambda^2}{J} \le C\frac{\Lambda^2}{N}, \tag{20}$$

since $\lim_{z\to 0} z/\big(1-\exp(-z)\big) = 1$ and $\lim_{N\to\infty} \Lambda/J = 0$.

Transition point and coarse mesh region. Here we use the fact that for $j = J, \dots, N-1$ one has

$$\left|\frac{w_j - w_{j-1}}{h_j}\right| \le \max_{[x_{j-1},x_j]}|w'(x)| \le C\mu^{-1}\exp\left(-\frac{\alpha x_{J-1}^{p+1}}{\varepsilon(p+1)}\right)$$

and

$$|w(x)| \le C \exp\left(-\frac{\alpha x_J^{p+1}}{\varepsilon(p+1)}\right)$$

by Theorem 1. Thus

$$\begin{aligned} Q_j h_{j+1} \left|\tau_j^w\right| &\le \frac{h_{j+1}}{x_j^p} \left|\tau_j^w\right| \le C\left(\frac{\varepsilon}{\lambda^p \mu} + 1 + H\right) \exp\left(-\frac{\alpha x_{J-1}^{p+1}}{\varepsilon(p+1)}\right) \\ &\le C \exp\left(-\frac{\alpha x_J^{p+1}}{\varepsilon(p+1)}\right) \exp\left(\frac{\alpha\left(x_J^{p+1} - x_{J-1}^{p+1}\right)}{\varepsilon(p+1)}\right) \\ &\le C \exp\left(-\frac{\alpha x_J^{p+1}}{\varepsilon(p+1)}\right) \exp\left(\lambda_0 (p+1)\frac{\ln N}{J}\right) \le C N^{-\lambda_0}, \end{aligned}$$

since $\ln N/J \le C$. We get

$$\sum_{j=1}^{J-1} Q_j h_{j+1} \left|\tau_j^w\right| \le C N^{-1} \tag{21}$$

Now $\|Tw - \mathcal{T}w\|_{\omega,1} \le C\Lambda^2 N^{-1}$ follows from (19), (20) and (21).

4. Numerical Results

In this section we verify experimentally our convergence result. Our test problem is the semilinear problem

$$-\varepsilon u'' - x^p(2-x)u' + x^p e^u = 0 \quad \text{for} \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{22}$$

The exact solution of this problem is unavailable. We therefore estimate the accuracy of the numerical solution by comparing it with the numerical solution on a finer mesh. For our tests we take $\alpha = 1$ and $q = 1/2$.

Indicating by U_ε^N that the numerical approximation of (22) depends on both N and ε, we estimate the uniform error by

$$\eta^N := \max_{\varepsilon=1,10^{-1},\ldots,10^{-12}} \left\|U_\varepsilon^N - \tilde{U}_\varepsilon^{8N}\right\|_\infty,$$

where $\tilde{U}_\varepsilon^{8N}$ is the approximate solution of the first-order scheme on a mesh obtained by bisecting the original mesh three times, i. e. a mesh that is eight times finer. The rates of convergence are computed using the standard formula $r^N = \ln\left(\eta^N/\eta^{2N}\right)/\ln 2$.

The results of our test computations are given in Tables 1. They are clear illustrations of the almost first-order convergence proved in Theorem 2.

5. Interior Turning Point Problems

Let us now briefly discuss the case of interior turning points. For this purpose we consider the boundary-value problem

$$\mathcal{T}u(x) := -\varepsilon u''(x) - x|x|^{p-1}a(x)u'(x) + |x|^p b(x, u(x)) = 0 \quad \text{for} \quad x \in (-1,1), \tag{23a}$$

$$u(-1) = \gamma_{-1}, \; u(1) = \gamma_1. \tag{23b}$$

Again, we assume that $0 < \varepsilon \ll 1$ is a small constant, $p > 0$, $a(x) > \alpha > 0$, $b_u \ge 0$ for $x \in [-1,1]$, $a \in C^1[0,1]$ and $b \in C^1([-1,1] \times I\!R)$. Because the convection coefficient changes sign at an interior point of the domain, u has an interior layer.

Table 1: Upwind scheme on Shishkin meshes.

	$p=1/2$		$p=2$		$p=3$	
N	error	rate	error	rate	error	rate
16	6.630e-2	0.76	5.405e-2	0.90	5.298e-2	0.92
32	3.912e-2	0.82	2.902e-2	0.93	2.797e-2	0.95
64	2.214e-2	0.86	1.520e-2	0.95	1.451e-2	0.97
128	1.222e-2	0.89	7.850e-3	0.96	7.421e-3	0.98
256	6.610e-3	0.91	4.022e-3	0.97	3.771e-3	0.98
512	3.529e-3	0.92	2.050e-3	0.98	1.906e-3	0.99
1024	1.867e-3	0.93	1.042e-3	0.98	9.596e-4	0.99
2048	9.806e-4	0.94	5.280e-4	0.98	4.822e-4	0.99
4096	5.127e-4	0.94	2.671e-4	0.99	2.422e-4	0.99
8192	2.670e-4	0.95	1.349e-4	0.99	1.218e-4	0.99
16384	1.387e-4	—	6.806e-5	—	6.128e-5	—

The operator $\mathcal{T}$ enjoys a comparison principle which can be used to show that $|u(x)| \le C$ for $x \in (-1,1)$. Then $u^+ := u|_{[0,1]}$ and $u^- := u|_{[-1,0]}$ solve

$$\mathcal{T}u^+ = 0 \ \text{ in } \ (0,1), \quad u^+(0) = u(0), \quad u^+(1) = \gamma_1$$

and

$$\mathcal{T}u^- = 0 \ \text{ in } \ (-1,0), \quad u^-(-1) = \gamma_{-1}, \quad u^-(0) = u(0).$$

Hence u^+ and u^- can be regarded as solutions of boundary-turning point problems of the type considered in Section 2. This immediately gives us bounds for the derivatives of u and a decomposition into regular and layer components.

The generalization of the simple upwind scheme (10) for (23) on the mesh $\omega : -1 = x_0 < x_1 < \cdots < x_N = 1$ is

$$[TU]_i = 0 \ \text{ for } \ i = 1, \ldots, N-1, \ U_0 = \gamma_{-1}, \ U_N = \gamma_1,$$

where

$$[TU]_i := \begin{cases} -\varepsilon\big[D^+D^-U\big]_i - x_i^p a_i \big[D^+U\big]_i + x_i^p b(x_i, U_i) & \text{if } x_i \ge 0 \\ -\varepsilon\big[D^-D^+U\big]_i - x_i^p a_i \big[D^-U\big]_i + x_i^p b(x_i, U_i) & \text{if } x_i < 0. \end{cases}$$

The technique from [3] can be used to prove that for any mesh functions V and W with $V_0 = W_0$ and $V_N = W_N$, one has

$$\|V - W\|_{\omega,\infty} \le \alpha^{-1} \|TV - TW\|_{\omega,1}$$

where

$$\|V\|_{\omega,1} := \sum_{j=1}^{N-1} \hbar_j Q_j \, |V_j| \quad \text{with} \quad \hbar_j := \begin{cases} h_{j+1} & \text{if } x_j \ge 0, \\ h_j & \text{otherwise,} \end{cases}$$

$$Q_N = 0, \quad Q_{j-1} = \left(1 + \frac{x_{j-1}^p h_j}{\varepsilon}\right)^{-1} \left(Q_j + \frac{h_j}{\varepsilon}\right) \quad \text{for} \quad x_{j-1} \ge 0,$$

and

$$Q_0 = 0, \quad Q_j = \left(1 + \frac{|x_j|^p h_j}{\varepsilon}\right)^{-1} \left(Q_{j-1} + \frac{h_j}{\varepsilon}\right) \quad \text{for} \quad x_j < 0.$$

The convergence analysis then follows along the lines of Section 3.

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