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MATHEMATICAL ANALYSIS FOR QUADRILATERAL ROTATED $\mathcal{Q}_1$ ELEMENT II: POINCARÈ INEQUALITY AND TRACE INEQUALITY*1)

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Abstract

This is the second part of the paper for the mathematical study of nonconforming rotated $\mathcal{Q}_1$ element (NRQ$_1$ hereafter) on arbitrary quadrilateral meshes. Some *Poincarè Inequalities* are proved *without* assuming the quasi-uniformity of the mesh subdivision. A discrete trace inequality is also proved.

Key words: Quadrilateral rotated $\mathcal{Q}_1$ element, Poincarè inequality, Trace inequality.

1. Mesh Subdivision

Let $\mathcal{T}_h$ be a partition of $\bar{\Omega}$ by convex quadrilaterals K with the mesh size h_K and $h := \max_{K\in\mathcal{T}_h} h_K$. We assume that $\mathcal{T}_h$ is shape regular in the sense of Ciarlet-Raviart [3, p. 247]. We define a mesh condition which actually quantifies the deviation of a quadrilateral away from a parallelogram [10].

Definition 1.1. (1+α)-Section Condition ($0 \le \alpha \le 1$). *The distance d_K between the midpoints of two diagonals of $K \in \mathcal{T}_h$ is of $\mathcal{O}(h_K^{1+\alpha})$ uniformly for all elements K as $h \to 0$. In case of $\alpha = 0$, $\mathcal{T}_h$ is the trapezoid mesh, and in case of $\alpha = 1$, $\mathcal{T}_h$ satisfies the* Bi-Section Condition *[13].*

We define by $\mathcal{P}_k$, the space of polynomials of degrees no more than k, and by $\mathcal{Q}_k$, the space of degrees no more than k in each variable.

Let $\hat{K} = [-1,1]^2$ be the reference square, the coordinates of the its four vertices are denoted by $\{(\xi_i,\eta_i)\}_{i=1}^4$ which is labelled from the lower-left to the upper-left in a counterclockwise manner, the same rule applies to K, whose vertices are denoted by $\{(x_i,y_i)\}_{i=1}^4$. There exists a bilinear mapping $\boldsymbol{F}$ such that $\boldsymbol{F}(\hat{K}) = K$. Let $\boldsymbol{F} = (x^K, y^K)$, with

$$x^K := \frac{1}{4}\sum_{i=1}^4 (1+\xi_i\xi)(1+\eta_i\eta)x_i = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta,$$
$$y^K := \frac{1}{4}\sum_{i=1}^4 (1+\xi_i\xi)(1+\eta_i\eta)y_i = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta.$$

To each scalar function $\hat{v}$ defined on $\hat{K}$, we associate it a function v on K such that $v(\boldsymbol{x}) = v(\boldsymbol{F}(\hat{\boldsymbol{x}})) = \hat{v}(\hat{\boldsymbol{x}})$.

Before closing this section, we fix some notations. For any integer k, $H^k(\Omega)$ denotes the standard Sobolev spaces [5]. $⨍_\Omega u\,dx$ is defined as the integral average of u on Ω. Denote by V_h the NRQ$_1$ finite element space, and by V_h^a, V_h^p the corresponding finite element spaces

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with continuous edge integral mean or with continuous mid-point on each edge (see [12,9] for a definition). For any $v \in V_h$, we define a piecewise norm as

$$|v|_h := \Big(\sum_{K \in \mathcal{T}_h} \|\nabla v\|_{0,K}^2 \Big)^{1/2}.$$

Throughout this paper, the generic constant C is assumed to be independent of the mesh size h.

2. Poincarè Inequality

In this section, we present some versions of the *Poincaré inequality* for the nonconforming finite element spaces V_h^a and V_h^p [12]. We adopt all the notations appeared in [9].

In case the element K is a rectangular parallelepiped, the *Poincarè inequality* has been proved in [6]. A strengthened version of this inequality is presented in [7]. But both of them are suitable only for the homogeneous space $V_{0,h}$. Moreover, as to the strengthened *Poincarè Inequality*, the quasi-uniformity of the mesh subdivision is assumed. In this section, we will extend *Poincaré inequalities* appeared in [6, 7] to both homogeneous and nonhomogeneous spaces over arbitrary quadrilateral meshes *without* the quasi-uniformity assumption, which allows for the adaptive mesh subdivision.

Meanwhile, some generalized *Poincarè Inequalities* have been proved by Stummel in [14] by virtue of the compact argument. However, when it applies to the quadrilateral rotated $\mathcal{Q}_1$ element, we have to assume the quasi-uniformity of meshes and the closedness of the given finite element space via the generalized patch test. But as we have seen in [9] that the finite element space V_h^p does not pass the generalized patch test for arbitrary quadrilaterals. So, instead of the compact argument, we adopt Teman's approach [15] which avoids the generalized patch test.

There is also another approach appeared in [8, Chp.3] to prove the *Poincaré inequality* for nonconforming elements, which starts from the conforming "relative" of the relevant nonconforming element, then exploits the high order distance between the conforming "relative" and the nonconforming element to prove the desired inequality. This approach is very flexible which allows for very "rough" mesh. Recently the same approach is employed by Brenner [2] to prove the generalized *Poincaré-Friedrichs inequality* for piecewise H^1 functions.

Theorem 2.1. *Poincarè Inequality*

$$\|v\|_0 \le C|v|_h \qquad \forall v \in V_{0,h}. \tag{2.1}$$

$$\|v\|_0 \le C\Big(|v|_h + |\fint_\Omega v\,dx|\Big) \qquad \forall v \in V_h. \tag{2.2}$$

$$\|v\|_0 \le C(|v|_h + \|v\|_{0,\Gamma}) \qquad \forall v \in V_h^a. \tag{2.3}$$

$$\|v\|_0 \le C|v|_h + C\Big(\sum_{e \subset \Gamma \cap \mathcal{T}_h, M \in \mathcal{F}} |e||v(\mathcal{M})|^2 \Big)^{1/2} \qquad \forall v \in V_h^p. \tag{2.4}$$

Proof. We prove the above four inequalities one by one.

For any $\boldsymbol{\psi} \in [\mathcal{H}^1(\Omega)]^2$ and $v \in V_{0,h}$, an integration by parts gives

$$\int_\Omega \operatorname{div} \boldsymbol{\psi} v\,dx = \sum_{K \in \mathcal{T}_h} \Big(-\int_K \boldsymbol{\psi} \nabla v\,dx + \int_{\partial K} v \boldsymbol{\psi} \cdot \boldsymbol{n}\,ds \Big).$$

It can be shown that

$$|\sum_{K\in\mathcal{T}_h}\int_{\partial K} v\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds| \le C\|\boldsymbol{\psi}\|_1|v|_h. \tag{2.5}$$

In fact, for any $v\in V_{0,h}^a$ and $e\in\partial K$,

$$\int_e v\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds = \int_e (v - ⨍_e v)(\boldsymbol{\psi}\cdot\boldsymbol{n} - ⨍\boldsymbol{\psi}\cdot\boldsymbol{n})\,ds,$$

thus (2.5) follows from the above identity and the scaled trace inequality. As for $v\in V_{0,h}^p$, we decompose

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{\psi}\cdot\boldsymbol{n}v\,ds &= \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K, M\in e}\int_e \boldsymbol{\psi}\cdot\boldsymbol{n}(v - ⨍_e v)\,ds\\ &= \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K, M\in e}\int_e (\boldsymbol{\psi}\cdot\boldsymbol{n} - ⨍_K\boldsymbol{\psi}\cdot\boldsymbol{n}\,dx)(v - v(M)\,ds)\,ds\\ &+ \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K, M\in e}\int_e (v - v(M))\,ds ⨍_K\boldsymbol{\psi}\cdot\boldsymbol{n} = T^1 + T^2.\end{aligned} \tag{2.6}$$

Obviously, the first term T^1 can be bounded as $|T^1|\le Ch|\boldsymbol{\psi}|_1|v|_h$. The estimate of the second term T^2 is performed as follows:

$$\begin{aligned}|T^2| &\le \sum_{K\in\mathcal{T}_h}|\sum_{e\subset\partial K, M\in e}\int_e (v - v(M))\,ds ⨍_K\boldsymbol{\psi}\cdot\boldsymbol{n}|\\ &\le C\sum_{K\in\mathcal{T}_h} h_K^{1/2}|v|_{1,K}h_K^{-1/2}\|\boldsymbol{\psi}\|_{0,K} \le C|v|_h\|\boldsymbol{\psi}\|_0.\end{aligned}$$

Here we have used

$$\|v - v(M)\|_{0,e} \le Ch_K^{1/2}\|\nabla v\|_{0,K}, \tag{2.7}$$

and the inequality $\|⨍_K\boldsymbol{\psi}\cdot\boldsymbol{n}\|_{0,e}\le Ch_K^{-1/2}\|\boldsymbol{\psi}\|_{0,K}$. Therefore,

$$|\int_\Omega \operatorname{div}\boldsymbol{\psi}v\,dx| \le C\|\boldsymbol{\psi}\|_1|v|_h.$$

By virtue of a result in [1], there exists $\boldsymbol{\psi}\in[\mathcal{H}^1(\Omega)]^2$ such that

$$\operatorname{div}\boldsymbol{\psi} = v \quad\text{and}\quad \|\boldsymbol{\psi}\|_1\le C\|v\|_0,$$

which together with (2.5) gives (2.1).

As to (2.2), invoking [1, Theorem 3.3] once again, then there exists $\boldsymbol{\psi}\in[\mathcal{H}_0^1(\Omega)]^2$ such that

$$\operatorname{div}\boldsymbol{\psi} = v - ⨍_\Omega v\,dx \quad\text{and}\quad \|\boldsymbol{\psi}\|_1\le C\|v\|_0. \tag{2.8}$$

As before,

$$\begin{aligned}\int_\Omega \operatorname{div}\boldsymbol{\psi}(v-\fint_\Omega v)\,dx)\,dx &= \int_\Omega \operatorname{div}\boldsymbol{\psi} v\,dx\\ &=\sum_{K\in\mathcal{T}_h}(-\int_K \boldsymbol{\psi}\nabla v\,dx+\int_{\partial K} v\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds)\\ &\le C\|\boldsymbol{\psi}\|_1|v|_h. \end{aligned}\tag{2.9}$$

Combining (2.8), (2.9) and noticing

$$\|v-\fint_\Omega v\,dx\|_0\le C|v|_h,$$

then by use of the triangle inequality we thus get

$$\|v\|_0\le\|v-\fint_\Omega v\,dx\|_0+\|\fint_\Omega v\,dx\|_0\le C(|v|_h+|\fint_\Omega v\,dx|).$$

To prove (2.3) and(2.4), we proceed along the same line as the case for homogeneous spaces. In fact, we only need to estimate (2.9) for the case $e\subset\Gamma\cap\mathcal{T}_h$. For $v\in V_h^a$, we have

$$\begin{aligned}|\sum_{e\subset\Gamma\cap\mathcal{T}_h}\int_e v\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds| &= |\sum_{e\subset\Gamma\cap\mathcal{T}_h}\int_e(v-\fint_e v)\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds+\int_e\fint_e v\,ds\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds|\\ &\le\sum_{e\subset\Gamma\cap\mathcal{T}_h}Ch_K^{1/2}|v|_{1,K}(h_K^{-1/2}\|\boldsymbol{\psi}\|_{0,K}+h_K^{1/2}|\boldsymbol{\psi}|_{1,K})+\|v\|_{0,e}\|\boldsymbol{\psi}\|_{0,e}\\ &\le C(|v|_h+\|v\|_{0,\Gamma})\|\boldsymbol{\psi}\|_1.\end{aligned}\tag{2.10}$$

For $v\in V_h^p$, we have

$$\begin{aligned}|\sum_{e\subset\Gamma\cap\mathcal{T}_h}\int_e v\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds| &= \sum_{e\subset\Gamma\cap\mathcal{T}_h}\int_e(v-v(M))\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds+\int_e v(M)\boldsymbol{\psi}\cdot\boldsymbol{n}\,ds\\ &\le C\sum_{e\subset\Gamma\cap\mathcal{T}_h}h_K^{1/2}|v|_{1,K}(h_K^{-1/2}\|\boldsymbol{\psi}\|_0+h^{1/2}|\boldsymbol{\psi}|_{1,K})\\ &\quad+\sum_{e\subset\Gamma\cap\mathcal{T}_h}|e|^{1/2}|v(M)|\|\boldsymbol{\psi}\|_{0,e}\\ &\le C(|v|_h+\Big(\sum_{e\subset\Gamma\cap\mathcal{T}_h}|e||v(M)|^2\Big)^{1/2})\|\boldsymbol{\psi}\|_1.\end{aligned}\tag{2.11}$$

Note that in the second to the last step of both (2.10) and (2.11), we have used the trace inequality

$$\|\boldsymbol{\psi}\|_{0,\Gamma}\le C\|\boldsymbol{\psi}\|_1.$$

Thus (2.3) and (2.4) are direct consequences of the above two inequalities. The proof is completed.

Remark 2.2. Note that a Sobolev version of the inequality in this Theorem is proved in [8].

We will give a strengthened *Poincaré inequality* for the quadrilateral rotated $\mathcal{Q}_1$ elements, its parallelepiped counterpart has been presented in [7]. Such inequality reflects the rotation property of NRQ_1.

Theorem 2.3. *For any $\boldsymbol{\omega} \in R^2$ with $|\boldsymbol{\omega}| = 1$, if the* $(1+\alpha)-$Section Condition *holds, then*

$$\|v\|_0 \le C(\|\boldsymbol{\nabla} v \cdot \boldsymbol{\omega}\|_{0,h} + h^{\min(1/2,\alpha)}|v|_h) \quad \forall v \in V_{0,h}, \tag{2.12}$$

$$\|v\|_0 \le C(\|\boldsymbol{\nabla} v \cdot \boldsymbol{\omega}\|_{0,h} + h^{1/2}|v|_h + \|v\|_{0,\Gamma}) \quad \forall v \in V_h^a, \tag{2.13}$$

$$\|v\|_0 \le C(\|\boldsymbol{\nabla} v \cdot \boldsymbol{\omega}\|_{0,h} + h^{\min(1/2,\alpha)}|v|_h + \Big(\sum_{e\subset\Gamma\cap\mathcal{T}_h, M\in e} |e||v(M)|^2\Big)^{1/2} \quad \forall v \in V_h^p. \tag{2.14}$$

Proof. First we prove the homogeneous case (2.12). For any $\boldsymbol{\omega} \in \mathcal{R}^2$ with $|\boldsymbol{\omega}| = 1$, we have

$$\begin{aligned}
\int_\Omega |v|^2\,dx &= \int_\Omega |v|^2 \operatorname{div}(\boldsymbol{\omega}\cdot x)\boldsymbol{\omega}\,dx \\
&= -\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\nabla}|v|^2\cdot\boldsymbol{\omega}\ \boldsymbol{\omega}\cdot x\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K} |v|^2 \boldsymbol{\omega}\cdot x\boldsymbol{\omega}\cdot\boldsymbol{n}\,ds \\
&= I_1 + I_2.
\end{aligned}$$

We bound I_1 as follows:

$$\begin{aligned}
|I_1| &\le 2\max_{x\in\overline{\Omega}}|\boldsymbol{\omega}\cdot x|\Big(\sum_{K\in\mathcal{T}_h}\int_K |\boldsymbol{\nabla} v\cdot\boldsymbol{\omega}|^2\,dx\Big)^{1/2}\Big(\int_\Omega |v|^2\,dx\Big)^{1/2} \\
&\le \frac{1}{4}\int_\Omega |v|^2\,dx + 4\max_{x\in\overline{\Omega}}|\boldsymbol{\omega}\cdot x|^2 \sum_{K\in\mathcal{T}_h}\int_K |\boldsymbol{\nabla} v\cdot\boldsymbol{\omega}|^2.
\end{aligned}$$

As to the term I_2, we distinguished two cases. Let $f_{\boldsymbol{n}}(x) = \boldsymbol{\omega}\cdot x\boldsymbol{\omega}\cdot\boldsymbol{n}$. If $v \in V_{0,h}^a$, then

$$\begin{aligned}
|I_2| &\le \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\Big|\int_e |v - ⨍_e v\,ds + ⨍_e v\,ds|^2 f_{\boldsymbol{n}}(x)\,ds\Big| \\
&= \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\Big|\int_e (|v - ⨍_e v\,ds|^2 + |⨍_e v\,ds|^2) f_{\boldsymbol{n}}(x)\,ds\Big| \\
&\quad + 2\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\Big|\int_e (v - ⨍_e v\,ds)⨍_e v\,ds f_{\boldsymbol{n}}(x)\,ds\Big| \\
&= \sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\Big|\int_e |v - ⨍_e v\,ds|^2 f_{\boldsymbol{n}}(x)\Big|\,ds \\
&\quad + 2\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}\Big|\int_e (v - ⨍_e v\,ds)⨍_e v\,ds f_{\boldsymbol{n}}(x)\,ds\Big| \\
&= J_1^a + J_2^a.
\end{aligned} \tag{2.15}$$

Obviously, we have $J_1^a \le Ch|v|_h^2$. Observing $\fint_e v\,ds = 0$ for $e \subset \Gamma$, then

$$\begin{aligned}
J_2^a &\le 2\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}|\int_e \fint_e v\,ds(v-\fint_e v\,ds)(f_{\boldsymbol{n}}(x)-\fint_e f_{\boldsymbol{n}})|\,ds\\
&\le C\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K}|\fint_e v\,ds|\|v-\fint_e v\,ds\|_{0,e}\|f_{\boldsymbol{n}}(x)-\fint_e f_{\boldsymbol{n}}\|_{0,e}\\
&\le C\sum_{K\in\mathcal{T}_h} h_K^{-\frac{1}{2}}(h_K^{-\frac{1}{2}}\|v\|_{0,K}+h_K^{\frac{1}{2}}|v|_{1,K})h_K^2|v|_{1,K}\\
&\le C\sum_{K\in\mathcal{T}_h}(h_K\|v\|_{0,K}|v|_{1,K}+h_K^2|v|_{1,K})\\
&\le Ch\|v\|_0|v|_h+Ch^2|v|_h^2\\
&\le \frac{1}{4}\|v\|_0^2+Ch^2|v|_h^2.
\end{aligned}$$

Summing up all the above estimates we come to (2.12) for $v \in V_{0,h}^a$.

For $v \in V_{0,h}^p$, we only need to replace $\fint_e v\,ds$ in (2.15) by $T_{\mathcal{M}}^p(v)$ (for the definition, see [9]), and denote the right side of (2.15) by J_1^p and J_2^p. The estimate of J_1^p is the same as J_1^a. We estimate J_2^p as follows.

Note $f_{\boldsymbol{n}}(x) = \boldsymbol{\omega}\cdot x\boldsymbol{\omega}\cdot\boldsymbol{n}$, then $\hat{f}_{\boldsymbol{n}}(\hat{x}) = \hat{g}(\hat{x})\boldsymbol{\omega}\cdot\boldsymbol{n}$. For any quadrilateral element K, if we denote its west, south, east and north edges by e_1, e_2, e_3 and e_4, respectively, then we expand J_2^p as

$$\begin{aligned}
J_2^p &= 2\sum_{K\in\mathcal{T}_h}\sum_{e\subset\partial K, M\in e}\int_e v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds\\
&= 2\sum_{K\in\mathcal{T}_h}\Big(\int_{e_1} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds+\int_{e_3} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds\\
&\quad+\int_{e_2} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds+\int_{e_4} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds\Big).
\end{aligned}$$

In what follows, we shall consider the cancellation of the line integrals. Firstly we transform the integrals over e_1 and e_3 to the reference element as

$$\begin{aligned}
&\int_{e_1} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds\\
&=\int_{-1}^1 \hat{v}(-1,0)(\hat{v}(-1,\eta)-\hat{v}(-1,0))\hat{g}(-1,\eta)\omega_1(b_{12}-b_2)\,d\eta\\
&\quad+\int_{-1}^1 \hat{v}(-1,0)(\hat{v}(-1,\eta)-\hat{v}(-1,0))\hat{g}(-1,\eta)\omega_2(a_{12}-a_2)\,d\eta\\
&= I_1^1+I_1^2.
\end{aligned}\tag{2.16}$$

Similarly,

$$\begin{aligned}
&\int_{e_3} v(M)(v-v(M))f_{\boldsymbol{n}}(x)\,ds \\
&\quad = \int_{-1}^{1} \hat{v}(1,0)(\hat{v}(1,\eta)-\hat{v}(1,0))\hat{g}(1,\eta)\omega_1(b_{12}+b_2)\,d\eta \\
&\quad + \int_{-1}^{1} \hat{v}(1,0)(\hat{v}(1,\eta)-\hat{v}(1,0))\hat{g}(1,\eta)\omega_2(a_{12}+a_2)\,d\eta \\
&= I_3^1 + I_3^2.
\end{aligned} \tag{2.17}$$

Therefore,

$$\begin{aligned}
I_1^1 + I_3^1 = &\int_{-1}^{1} (\hat{v}(-1,\eta)-\hat{v}(-1,0))\hat{v}(-1,0)\hat{g}(-1,\eta)\omega_1(b_{12}-b_2) \\
&+ \int_{-1}^{1} (\hat{v}(-1,\eta)-\hat{v}(-1,0))\hat{v}(-1,0)\hat{g}(-1,\eta)\omega_1(b_{12}+b_2) \\
&- \int_{-1}^{1} (\hat{v}(-1,\eta)-\hat{v}(-1,0))\hat{v}(-1,0)\hat{g}(-1,\eta)\omega_1(b_{12}+b_2) \\
&+ \int_{-1}^{1} (\hat{v}(1,\eta)-\hat{v}(1,0))\hat{v}(1,0)\hat{g}(1,\eta)\omega_1(b_{12}+b_2),
\end{aligned}$$

which can be reshaped into

$$\begin{aligned}
I_1^1 + I_3^1 = &\ 2\omega_1 b_{12} \int_{-1}^{1} \hat{v}(-1,0)\hat{g}(-1,\eta) \int_0^{\eta} \frac{\partial}{\partial t}\hat{v}(-1,t)\ dtd\eta \\
&+ \omega_2(b_{12}+b_2) \int_{-1}^{1} \int_0^{\eta} \frac{\partial}{\partial t}\hat{v}(-1,t)\ dt \int_{-1}^{1} \frac{\partial}{\partial t}\hat{v}(t,0)\hat{g}(t,\eta)\ dt\ d\eta. \\
= &\ T^1 + T^2.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|T^1| &\le C d_K \|\hat{v}\|_{0,\hat{K}} \|\hat{g}\|_{0,\infty,\hat{K}} |\hat{v}|_{1,\hat{K}} \\
&\le C d_K \|\hat{v}\|_{0,\hat{K}} |\hat{v}|_{1,\hat{K}} \le C d_K/h_K \|v\|_{0,K} |v|_{1,K},
\end{aligned}$$

and

$$|T^2| \le C h_K |\hat{v}|_{1,\hat{K}} |\hat{v}\hat{g}|_{1,\infty,\hat{K}} \le C h_K |v|_{1,K} \|v\|_{0,K} + C h_K |v|_{1,K}^2.$$

A combination of the above two inequalities gives

$$|I_1^1 + I_3^1| \le C(d_K/h_K + h_K)|v|_{1,K}\|v\|_{0,K} + C h_K |v|_{1,K}^2.$$

Similarly,

$$|I_1^2 + I_3^2| \le C(d_K/h_K + h_K)|v|_{1,K}\|v\|_{0,K} + C h_K |v|_{1,K}^2.$$

The estimate of the second term on the right side of J_2^p is the same. Summing up all the previous estimates, we get (2.12) for $v \in V_{0,h}^p$. As to (2.13) and (2.14), we only need to estimate the following boundary terms that are not vanishing in (2.15):

$$\sum_{e \subset \Gamma \cap \mathcal{T}_h} \int_e |⨍_e v\,ds|^2 f_{\boldsymbol{n}}(x)\ ds \ \text{ and } \sum_{e \subset \Gamma \cap \mathcal{T}_h, M \in e} \int_e |v(M)|^2 f_{\boldsymbol{n}}(x)\ ds.$$

The estimates can be processed as that in (2.3) and (2.4). The proof is completed.

Remark 2.4. Comparing to the previous result [6], we have not used the quasi-uniformity assumption on the mesh, since the inverse inequality is not used in this proof.

Corollary 2.1. *If* $v \in V_{0,h}$*, then*

$$\|v\|_0 \leq C(\|\partial v/\partial x\|_0 + h^{\frac{1}{2}}|v|_h), \tag{2.18}$$

$$\|v\|_0 \leq C(\|\partial v/\partial y\|_0 + h^{\frac{1}{2}}|v|_h). \tag{2.19}$$

Proof. We only need to select $\omega = (1,0)$ and $\omega = (0,1)$ in (2.12), then (2.18) and (2.19) follow, respectively.

3. Trace Inequality

In [7], a global version of the trace inequality for the rotated $\mathcal{Q}_1$ element over parallelepipeds is presented, which is fundamental for the analysis of the martensitic problem. However, their proof heavily depends on the specific configuration, thus it seems difficult to be extended to general quadrilateral meshes. In the following theorem, we will give its quadrilateral counterpart, which maybe a basic vehicle for some 2-$\mathcal{D}$ martensitic problems.

Before state the trace theorem, we cite a lemma about the domain.

Lemma 3.1 [5, Lemma 1.5.1.9, pp. 40]. *Let* Ω *be a bounded open subset of* R^n *with the Lipschitz boundary* Γ*. Then there exist* $\delta > 0$ *and* $\boldsymbol{\mu} \in \mathcal{C}^\infty(\overline{\Omega})^2$ *such that*

$$\boldsymbol{\mu} \cdot \boldsymbol{n} \geq \delta \quad a.e. \quad on \quad \Gamma. \tag{3.1}$$

Theorem 3.2. *If* ω *is a subdomain of* Ω *with the Lipschitz boundary* $\partial\omega$*, which is a collection of elements* K*, then for* $u \in V_h^\alpha$*, if the* $(1+\alpha)-$Section Condition *holds, we have*

$$\begin{aligned}\sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K}\int_e |u|^2\,ds \quad &\leq \quad C\Big(\int_\omega |u|^2\,dx + h^\alpha \sum_{K\subset\omega}|u|_{1,K}^2 \\ &\quad + (\int_\omega |u|^2 dx)^{1/2}(\sum_{K\subset\omega}|u|_{1,K}^2)^{1/2}\Big),\end{aligned} \tag{3.2}$$

where C *depends on* δ *and* $\|\boldsymbol{\mu}\|_{\mathcal{C}^1(\overline{\omega})}$.

Proof. As ω is a open set with a Lipschitz boundary, then there exist $\delta > 0$ and $\boldsymbol{\mu} \in \mathcal{C}^\infty(\overline{\omega})^2$ such that

$$\boldsymbol{\mu} \cdot \boldsymbol{n} \geq \delta \quad a.e. \text{ on } \partial\omega. \tag{3.3}$$

Note the following integral identity:

$$\int_\omega \boldsymbol{\nabla}|u|^2 \cdot \boldsymbol{\mu}\,dx = \sum_{K\subset\omega}\int_K \boldsymbol{\nabla}|u|^2 \cdot \boldsymbol{\mu}\,dx = 2\sum_{K\subset\omega}\int_K u\boldsymbol{\nabla}|u| \cdot \boldsymbol{\mu}\,dx. \tag{3.4}$$

Using Green's formula [11], we have

$$\int_\omega \boldsymbol{\nabla}|u|^2 \cdot \boldsymbol{\mu}\,dx = -\sum_{K\subset\omega}\int_K u^2 \boldsymbol{\nabla}\cdot\boldsymbol{\mu}\,dx + \sum_{K\subset\omega,e\subset\partial K}\int_e u^2\boldsymbol{\mu}\cdot\boldsymbol{n}\ ds. \tag{3.5}$$

Thus

$$\begin{aligned}\sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K}\int_e u^2\boldsymbol{\mu}\cdot\boldsymbol{n}\,ds &= 2\sum_{K\subset\omega}\int_K u\boldsymbol{\nabla}|u|\cdot\boldsymbol{\mu}\,dx+\sum_{K\subset\omega}\int_K u^2\boldsymbol{\nabla}\cdot\boldsymbol{\mu}\,dx\\ &- \sum_{K\subset\omega,\partial K\not\subset\partial\omega}\sum_{e\subset\partial K}\int_e u^2\boldsymbol{\mu}\cdot\boldsymbol{n}\,ds\\ &= J_1+J_2+J_3.\end{aligned}$$

It is easily seen that

$$|J_1|\le C\sum_{K\subset\omega}\|u\|_{0,K}|u|_{1,K}$$

and

$$|J_2|\le C\int_\omega |u|^2dx.$$

J_3 can be estimated as the term I_2 in Theorem 2.3, i.e

$$|J_3|\le C(h\sum_{K\subset\omega}|u|^2_{1,K}+h\sum_{K\subset\omega}\|u\|_{0,K}|u|_{1,K})\quad\forall u\in V_h^a.$$

Combining the above three inequalities and Lemma 3.1, we get (3.2) for the case when $u\in V_h^a$.

Following the same line, we get (3.2) for $u\in V_h^p$. Noting that

$$\begin{aligned}\sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K,M\in e}|e|u(M)^2 &= \sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K,M\in e}\int_e u(M)^2ds\\ &\le 2\sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K,\mathcal{M}\in e}\int_e |u|^2ds\\ &+ 2\sum_{K\subset\omega,\partial K\subset\partial\omega}\sum_{e\subset\partial K,M\in e}\int_e |u-u(M)|^2ds,\end{aligned}$$

the second term on the right hand side of the above inequality can be bounded by

$$Ch\sum_{K\subset\omega}|u|^2_{1,K}.$$

This completes the proof.

Remark 3.3. Note all results in this paper are valid for some other NR $\mathcal{Q}_1$ elements proposed in [4].

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