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UNIFORM CONVERGENCE OF HERMITE INTERPOLATION OF HIGHER ORDER*1)

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Abstract

In this paper the uniform convergence of Hermite-Fejér interpolation and Grünwald type theorem of higher order on an arbitrary system of nodes are presented.

Key words: Hermite-Fejér interpolation, convergence, Hermit interpolation.

1. Introduction

Let us assume n, $n \geq 2$, m_{kn}, $k = 1, 2, 3, ..., n$, be integers and triangular matrix $X = \{x_{1n}, x_{2n}, ..., x_{nn}\}$, where

$$1 = x_{0n} \geq x_{1n} > x_{2n} > ... > x_{nn} \geq x_{n+1,n} = -1.$$

Let $\mathcal{N}_n = \sum_{k=1}^{n} m_{kn} - 1$, $m = \max_{n\geq 2, 1\leq k\leq n} m_{kn} < +\infty$, $\mathbf{N}_1=\{1, 3, 5, ...\}$, $\mathbf{N}_2=\{2, 4, 6, ...\}$ and $\mathbf{N}_0=\mathbf{N}_2 \bigcup\{0\}$. For simplicity we denote $\mathcal{N}_n$ as $\mathcal{N}$. In the following discussion we replace $x_{kn}, m_{kn}, k = 1, 2, ..., n$ with $x_k, m_k, k = 1, 2, ..., n$. Denoted by $\boldsymbol{P}_{\mathcal{N}}$ the set of polynomials of degree at most $\mathcal{N}$ and by A_{jk} the fundamental polynomials for Hermite interpolation of higher order, then we have $A_{jk} \in \boldsymbol{P}_{\mathcal{N}}$ satisfy

$$A_{jk}^{(p)}(x_q) = \delta_{jp}\delta_{kq}, \quad p = 0, 1, ..., m_q - 1, \quad j = 0, 1, ..., m_k - 1, \quad q, k = 1, 2, ..., n. \tag{1.1}$$

For $f \in C^r[-1, 1]$, $0 \leq r \leq m - 1$, the unique truncated Hermite interpolatory polynomial is given by

$$H_{nmr}(f, x) = \sum_{i=0}^{r}\sum_{k=1}^{n} f^{(i)}(x_k)A_{ik}(x), \tag{1.2}$$

here let $A_{jk} = 0$ if $j \geq m_k$. In particular, when $r = 0$ and $r = m - 1$ H_{nm0} and $H_{nm,m-1}$ are denoted by H_{nm} and H^*_{nm} respectively. We recognize that H_{n1} is the classical Lagrange interpolation and H_{n2} the classical Hermite-Fejér interpolation. H_{nmr} is called Lagrange type interpolation for odd m_k, $k = 1, 2, ..., n$, and Hermite-Fejér type interpolation for even m_k, $k = 1, 2, ..., n$, respectively.

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For giving the explicit expression of $A_{jk}(x)$ we let

$$L_k(x)=\prod_{q=1,q\neq k}^{n}\left(\frac{x-x_q}{x_k-x_q}\right)^{m_q},\quad k=1,2,...,n,$$

$$b_{vk}=\frac{1}{v!}\left[\frac{1}{L_k(x)}\right]^{(v)}_{x=x_k},\quad v=0,1,...,m_k-1,\quad k=1,2,...,n,$$

$$B_{jk}(x)=\sum_{v=0}^{m_k-j-1}b_{vk}(x-x_k)^v,\quad j=0,1,...,m_k-1,\quad k=1,2,...,n.$$

Then by [6] it has

$$A_{jk}(x)=\frac{1}{j!}(x-x_k)^jB_{jk}(x)L_k(x),\quad 0\leq j\leq m_k,\quad 1\leq k\leq n. \tag{1.3}$$

More let

$$d_k=\max\{|x_k-x_{k+1}|,|x_k-x_{k-1}|\},\quad k=1,2,...,n,$$
$$D_n=\max_{1\leq k\leq n}d_k,\quad \|P\|_j:=\max_{1\leq l\leq j}\|P^{(l)}\|,$$
$$R_{nm}(f,x):=|H_{nm}(f,x)-f(x)|,$$
$$r_{nm}(x):=R_{nm}(f_1,x)+R_{nm}(f_2,x),\quad f_i=x^i,i=0,1,2,...$$
$$S_{nm}(x):=\sum_{k=1}^{n}|(x-x_k)A_{0k}(x)|\,.$$

In what follows we denote by c, c_1, ..., positive constants independent of variables and indices, unless otherwise indicated; their value may be different occurrences even in subsequent formulas.

2. Main Result

In [4], Y. G. Shi has proved an important theorem about fundamental polynomials A_{jk}, B_{jk} as following:

Theorem A ([4, Theorem 2.1]). *If for a fixed n, m_k-j is odd and $j<i\leq m_k-1$ then*

$$B_{jk}(x)\geq c\left|\frac{x-x_k}{d_k}\right|^{i-j}|B_{ik}(x)|,\quad x\in\Re,\quad 1\leq k\leq n, \tag{2.1}$$

and

$$|A_{ik}(x)|\leq c_1d_k^{i-j}|A_{jk}(x)|,\quad x\in\Re,\quad 1\leq k\leq n, \tag{2.2}$$

hold, where c and c_1 are positive constants depending only on m.

More the estimate of $R_{nm}(P,x)$ for all polynomials for the case of $m_k\equiv m$, $k=1,2,...,n$ is given, that is the following theorem:

Theorem B ([4, Theorem 4.1]). *Let $m_k\equiv m$ be an even integer. Then for any $P\in P_{\mathcal{N}}$*

$$R_{nm}(P,x)\leq c\|P\|_m\left\{r_{nm}(x)+\frac{\|r_{nm}\|\ln^{10}[n(1+\|r_{nm}\|)]}{n}\right\}, \tag{2.3}$$

where c depends only on m. Further more, if

$$\|H_{nm}\|=\left\|\sum_{k=1}^{n}|A_{0k}|\right\|=O(1) \tag{2.4}$$

holds and

$$\lim_{n\to\infty} \|H_{nm}(f) - f\| = 0 \tag{2.5}$$

holds for $f = f_i$, $i = 1, 2$, *then* (2.5) *holds for every* $f \in C[-1, 1]$.

Theorem C ([5, Lemma 2.1]). *Assume that* $m_k \geq 3$ *then we have*

$$B_{m_k-3,k}(x) \geq \frac{1}{2} B_{m_k-1,k}(x) = \frac{1}{2}, \quad x \in \Re, \quad k = 1, 2, ..., n, \tag{2.6}$$

$$B_{m_k-3,k}(x) \geq |B_{m_k-2,k}(x)|, \quad x \in \Re, \quad k = 1, 2, ..., n, \tag{2.7}$$

and

$$|A_{m_k-1,k}(x)| \leq c|(x - x_k)^2 A_{m_k-3,k}(x)|, \tag{2.8}$$

$$|A_{m_k-2,k}(x)| \leq c|(x - x_k) A_{m_k-3,k}(x)|. \tag{2.9}$$

Developing and modifying their ideas we can give the condition of uniform convergence for even integers m_k, $k = 1, 2, ..., n$. The main result in this section is the following theorem.

Theorem 1. *Let* m_k, $k = 1, 2, ..., n$, $m \geq 4$, *be even integers. Then for any* $P \in P_{\mathcal{N}}$

$$R_{nm}(P, x) \leq c\|P\|_m \left\{ \sum_{i=1}^{m-1} R_{nm}(f_i, x) \right\}. \tag{2.10}$$

Further more , if (2.4) *holds and* (2.5) *holds for* $f = f_i$, $i = 1, 2, ..., m-1$, *then* (2.5) *holds for every* $f \in C[-1, 1]$, *where* c *depends only on* m.

First let us prove some needed lemmas. Similarly as [4 , Lemma 2.1] we have following lemma.

Lemma 1. *For* $A_{jk}(x)$ *we have*

$$i! \sum_{k=1}^{n} \sum_{j=1}^{m_k-1} \frac{(-1)^{j+1}}{(i-j)!} (x - x_k)^{i-j} A_{jk}(x) = \sum_{k=1}^{n} (x - x_k)^i A_{0k}(x). \tag{2.11}$$

Lemma 2. *Let* m_k, $1 \leq k \leq n$, *be even integers, then for* $A_{0k}(x)$, *it has*

$$\left| \sum_{k=1}^{n} (x - x_k)^i A_{0k}(x) \right| \leq C \sum_{k=1}^{i} R_{nm}(f_k, x), \tag{2.12}$$

where C *is a positive const depending only on* m.

Proof. By means of $R_{nm}(f, x)$ and [4, (4.12)] it has

$$\sum_{k=1}^{n} A_{0k} = 1,$$

$$\left| \sum_{k=1}^{n} (x - x_k) A_{0k}(x) \right| = \left| x - \sum_{k=1}^{n} x_k A_{0k}(x) \right| = R_{nm}(f_1, x),$$

$$\left| \sum_{k=1}^{n} (x - x_k)^2 A_{0k}(x) \right| \leq 2R_{nm}(f_1, x) + R_{nm}(f_2, x).$$

By $(x-x_k)^i=\sum_{j=0}^{i}(-1)^j\binom{i}{j}x^{i-j}x_k^j=\sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}x^{i-j}(x^j-x_k^j)$, we have

$$\left|\sum_{k=1}^{n}(x-x_k)^iA_{0k}(x)\right|=\left|\sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}x^{i-j}\sum_{k=1}^{n}(x^j-x_k^j)A_{0k}(x)\right|$$

$$\leq\sum_{j=1}^{i}\binom{i}{j}|x^{i-j}|R_{nm}(f_j,x)\leq c\sum_{j=1}^{i}R_{nm}(f_j,x).\quad\square$$

Lemma 3. *Let m_k, $1\leq k\leq n$, be even integers, then for $A_{jk}(x)$*

$$\left|\sum_{i=1}^{j-1}\sum_{k=1}^{n}\frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x)\right|\leq c\sum_{k=1}^{n}(x-x_k)A_{1k}(x)\tag{2.13}$$

holds, where c is a positive constant depending only on m.

Proof. By Theorem A it has

$$\sum_{k=1}^{n}|(x-x_k)^lA_{jk}(x)|\leq 2^{l-1}\sum_{k=1}^{n}|(x-x_k)A_{jk}(x)|$$
$$=2^{l-j-2}\sum_{k=1}^{n}(x-x_k)A_{1k}(x),$$

thus

$$\left|\sum_{i=1}^{j-1}\sum_{k=1}^{n}\frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x)\right|\leq\sum_{i=1}^{j-1}\left|\sum_{k=1}^{n}\frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x)\right|$$
$$\leq\sum_{i=1}^{j-1}\frac{2^{j-2}}{(j-i)!}\sum_{k=1}^{n}(x-x_k)A_{1k}(x)\leq c\sum_{k=1}^{n}(x-x_k)A_{1k}(x).\quad\square$$

Now Let us prove the Theorem 1. By means of definition of $R_{nm}(f,x)$ we have

$$R_{nm}(P,x)=\left|\sum_{k=1}^{n}\sum_{j=1}^{m_k-1}P^{(j)}(x_k)A_{jk}(x)\right|,$$

so $R_{nm}(P,x)\leq\sum_{j=1}^{m-1}|\sum_{k=1}^{n}P^{(j)}(x_k)A_{jk}(x)|:=\sum_{j=1}^{m-1}S_j$.

Now let us estimate term S_j for $j<m-1$ and $j=m-1$, respectively. In the first case by mean theorem of derivatives it has

$$S_j=\left|\sum_{k=1}^{n}P^{(j)}(x_k)A_{jk}(x)\right|$$
$$=\left|\sum_{k=1}^{n}\left\{P^{(j)}(x)-(P^{(j)}(x)-P^{(j)}(x_k))\right\}A_{jk}(x)\right|$$
$$\leq\left|\sum_{k=1}^{n}P^{(j)}(x)A_{jk}(x)\right|+\left|\sum_{k=1}^{n}P^{(j+1)}(\xi_k)(x-x_k)A_{jk}(x)\right|$$
$$\leq\|P\|_m\left|\sum_{k=1}^{n}A_{jk}(x)\right|+\left|\sum_{k=1}^{n}P^{(j+1)}(\xi_k)(x-x_k)A_{jk}(x)\right|.$$

For term $|\sum_{k=1}^{n} A_{jk}(x)|$, by Lemma 1

$$\left|\sum_{k=1}^{n} A_{jk}(x)\right| = \left|\sum_{i=0}^{j-1} \frac{(-1)^i}{(j-i)!}(x-x_k)^{j-i}A_{ik}(x)\right|,$$

thus by Lemma 2 and Lemma 3 we have

$$\left|\sum_{k=1}^{n} A_{jk}(x)\right| \le c\left\{\sum_{k=1}^{n} |(x-x_k)A_{1k}(x)| + \sum_{i=0}^{j} R_{nm}(f_i,x)\right\}. \tag{2.14}$$

In the second case by Theorem A and Theorem C we have

$$\left|\sum_{k=1}^{n} P^{(m-1)}(x_k)A_{m-1,k}(x)\right| \le c\|P\|_m \sum_{k=1}^{n} |(x-x_k)^2 A_{m-3,k}(x)| \le c\|P\|_m r_{nm}(x).$$

Then by Banach Theorem we have the result . □

3. Grünwald Type Theorem

In [7] P. Vértesi has proved a theorem of Grünwald type for Hermite-Fejér interpolation of higher order , which is a generalization of [1] given by G. Grünwald for $m_k \equiv m$.

Theroem D ([7, Theorem 2.3]). *Let $m_k \equiv m$ be even integers, I_{1n} and I_{2n} be two disjoint subsets of the set $\{1,2,3,...,n\}$ with $I_{1n} \bigcup I_{2n}$=\{1,2,...,n\}. If for fixed positive numbers ρ_2 and n_0*

$$B_{0k}(x) \ge \rho_2 |B_{jk}(x)|, \quad |x| \le 1, \quad k \in I_{1n}, \quad j=1,2,...,m-1, \quad n \ge n_0,$$

$$\lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |x-x_k|^\delta |A_{0k}(x)| \right\| = 0, \quad \delta > 0,$$

$$\left\| \sum_{k\in I_{2n}} |A_{0k}(x)| \right\| \le C < \infty,$$

$$\lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |A_{jk}(x)| \right\| = 0, \quad j=1,2,...,m-1,$$

$$\left\| \sum_{k\in I_{2n}} |B_{jk}(x) l_k^m(x)| \right\| \le C, \quad j=1,2,...,m-1,$$

then (2.5) *holds for all $f \in C[-1,1]$, here $l_k(x) = \prod_{q=1,q\ne k}^{n} \frac{x-x_q}{x_k-x_q}$.*

The main aim of this section is to improve the above theorem as follows

Theorem 2. *Let m_k, $1 \le k \le n$, be even integers, I_{1n} and I_{2n} be two disjoint subsets of* *the set* $\{1,2,3,...,n\}$ *with* $I_{1n}\bigcup I_{2n}=\{1,2,...,n\}$. *If for fixed positive numbers* ρ *and* n_0,

$$B_{0k}(x)\geq\rho|B_{jk}(x)|,\quad |x|\leq 1,\quad k\in I_{1n},\quad j=1,2,...,m_k-1,\quad n\geq n_0, \tag{3.1}$$

$$\lim_{n\to\infty}\left\|\sum_{k\in I_{2n}}|(x-x_k)A_{0k}(x)|\right\|=0, \tag{3.2}$$

$$\left\|\sum_{k\in I_{2n}}|A_{0k}(x)|\right\|\leq C_1<\infty, \tag{3.3}$$

$$\lim_{n\to\infty}\left\|\sum_{k\in I_{2n}}|A_{1k}(x)|\right\|=0, \tag{3.4}$$

$$\left\|\sum_{k\in I_{2n}}|B_{jk}(x)L_k(x)|\right\|\leq C_2<+\infty,\quad j=1,2,...,m_k-1, \tag{3.5}$$

then (2.5) *holds for all* $f\in C[-1,1]$.

Before proving this theorem we give some lemmas and statements.

Lemma 4. *Let* m_k, $1\leq k\leq n$, *be even integers. If*

$$\left\|\sum_{k=1}^{n}\sum_{j=0}^{m_k-1}|A_{jk}|\right\|=O(1), \tag{3.6}$$

holds, then

$$\lim_{n\to+\infty}\|H^*_{nm}(f,X)-f\|=0 \tag{3.7}$$

holds for every $f\in C^{m-1}[-1,1]$.

Proof. The proof is same as [7]. $\forall\varepsilon$ there exists a polynomial $P(x)$ for which

$$|f(x)-P(x)|<\varepsilon,\quad |f^{'}(x)-P^{'}(x)|<\varepsilon,\ \cdots,\quad |f^{(m-1)}(x)-P^{(m-1)}(x)|<\varepsilon,$$

and

$$P(x)=\sum_{k=1}^{n}\sum_{j=0}^{m_k-1}P^{(j)}(x_k)A_{jk}(x),$$

then it has

$$\begin{aligned}|H^*_{nm}(f)-f|&=|H^*_{nm}(f-P)+P-f|\\&\leq|H^*_{nm}(f-P)|+\varepsilon\\&\leq\sum_{k=1}^{n}\sum_{j=0}^{m_k-1}|f^{(j)}(x_k)-P^{(j)}(x_k)||A_{jk}(x)|+\varepsilon\\&\leq O(1)\varepsilon\end{aligned}$$

and we obtain the result. □

More let us verify a very simple relation as [7, Theorem 1].

Lemma 5. *If*

$$\left\|\sum_{k=1}^{n}|A_{0k}(x)|\right\|=C=O(1),\quad n=1,2,..., \tag{3.8}$$

and

$$\left\|\sum_{k=1}^{n}\sum_{j=1}^{m_k-1}|A_{jk}(x)|\right\| = o(1), \quad n = 1, 2, ..., \tag{3.9}$$

hold, then (2.5) *holds for all* $f \in C[-1, 1]$.

Proof. If $P(x)$ is a fixed polynomial with $\|f - P\| < \varepsilon/C$ and $\|P^{(j)}\| \le M$, $1 \le j \le m - 1$, then

$$\begin{aligned}|H_{nm}(f, x) - f(x)| &\le |H_{nm}(f, x) - P(x)| + |P(x) - f(x)|\\ &\le \sum_{k=1}^{n}|f(x_k) - P(x_k)||A_{0k}(x)| + \sum_{k=1}^{n}\sum_{j=1}^{m_k-1}|P^{(j)}(x_k)||A_{jk}(x)| + \varepsilon/C\\ &\le \varepsilon/C\sum_{k=1}^{n}|A_{0k}(x)| + M\sum_{k=1}^{n}\sum_{j=1}^{m_k-1}|A_{jk}(x)| + \varepsilon \le 3\varepsilon,\end{aligned}$$

if n is big enough . □

Statement 1. *Let* $V_k(\eta; x) = \sum_{j=0}^{m_k-1}(-1)^j\binom{\eta}{j}B_{jk}(x)$. *For* $k \in I_{1n}$, *Let us choose* η, *where* $0 < \eta < 1$ *such that* $|\binom{\eta}{j}| < \frac{\rho}{2(m-1)}$, $\quad j = 1, 2, ..., m - 1$. *Then*

$$V_k(\eta; x) \ge \frac{1}{2}\rho|B_{jk}(x)|. \tag{3.10}$$

Proof. By (3.1)

$$\begin{aligned}V_k(\eta; x) &\ge B_{0k}(x) - \sum_{j=1}^{m_k-1}\left|\binom{\eta}{j}B_{jk}(x)\right|\\ &\ge B_{0k}(x) - \frac{1}{2(m-1)}\sum_{j=1}^{m_k-1}\rho|B_{jk}(x)|\\ &\ge B_{0k}(x) - \frac{1}{2}B_{0k}(x) \ge \frac{1}{2}\rho|B_{jk}(x)|.\end{aligned}$$

Statement 2. *For this matrix X it has*

$$\sum_{k\in I_{1n}} A_{0k}(x) \le C_1 + 1 < +\infty, \tag{3.11}$$

$$S_0 = \sum_{k=1}^{n}|A_{0k}(x)| \le 2C_1 + 1 < +\infty. \tag{3.12}$$

Proof. By

$$1 = \sum_{k=1}^{n} A_{0k}(x), \quad \textit{but}$$

$$\sum_{k=1}^{n}|A_{0k}(x)| \le \sum_{k=1}^{n} A_{0k}(x) + 2\sum_{k\in I_{2n}}|A_{0k}(x)|,$$

we have $\left|\sum_{k\in I_{1n}} A_{0k}(x)\right| \le C_1 + 1$, and the other is obvious.

Statement 3. *For this matrix X it has*

$$T := \sum_{k=1}^{n}\sum_{j=0}^{m_k-1} |A_{jk}(x)| = O(1). \tag{3.13}$$

Proof. For $A_{1k}(x)$, by Theorem A,

$$\sum_{k\in I_{1n}} |A_{1k}(x)| \le \frac{2}{\rho}\sum_{k\in I_{1n}} |A_{0k}(x)| \le \frac{2C_1+2}{\rho},$$

Let $T_1 := \sum_{k=1}^{n} |A_{0k}(x)|$ then we have

$$\begin{aligned}
\sum_{k=1}^{n}\sum_{j=0}^{m_k-1} |A_{jk}(x)| &= \sum_{k\in I_{1n}}\sum_{j=0}^{m_k-1} |A_{jk}(x)| + \sum_{k\in I_{2n}}\sum_{j=0}^{m_k-1} |A_{jk}(x)| \\
&\le T_1 + \sum_{k\in I_{1n}}\sum_{j=1}^{m_k-1} |A_{jk}(x)| + \sum_{k\in I_{2n}}\sum_{j=1}^{m_k-1} |A_{jk}(x)| \\
&\le T_1 + \sum_{k\in I_{1n}}\sum_{j=1}^{m_k-1} C^* d_k^{j-1} |A_{1k}(x)| + \sum_{k\in I_{2n}}\sum_{j=1}^{m_k-1} C^* d_k^{j-1} |A_{1k}(x)| \\
&\le T_1 + C^* 2^m \sum_{k\in I_{1n}} |A_{1k}(x)| + C^* 2^m \sum_{k\in I_{2n}} |A_{1k}(x)| \\
&\le T_1 + C^* 2^m \frac{2C_1+1}{\rho} = O(1).
\end{aligned}$$

Now let us prove the Theorem 2.

Let $n \ge n_0$ and a be an arbitrary point in $[-1,1]$ such that $a \ne x_k, \quad k = 1,2,...,n$, put

$$g(x) = \begin{cases} 0, & \text{if} \quad x \in [-1,a], \\ (x-a)^\eta, & \text{if} \quad x \in [a,1], \quad \eta \in (0,1]. \end{cases} \tag{3.14}$$

Additional we define following function for $t = 1,2,...,n$,

$$g_t(x) = \begin{cases} 0, & \text{if} \quad x \in [-1,a], \\ \sum_{j=0}^{m-1} C_{jt}(x-a)^{\eta+m-1+j}, & \text{if} \quad x \in [a, a+\frac{1}{t}], \\ (x-a)^\eta, & \text{if} \quad x \in [a+\frac{1}{t}, 1], \quad \eta \in (0,1]. \end{cases} \tag{3.15}$$

Where C_{jt} are choosen so that $g_t \in C^{m-1}[-1,1]$ and this yields

$$\sum_{j=0}^{m-1} C_{jt}\binom{\eta+m+j-1}{k} k! t^{k-\eta-m+1-j} = \binom{\eta}{k} k! t^{k-\eta}, \quad k = 0,1,2,...,m-1,$$

that is to say

$$\sum_{j=0}^{m-1} C_{jt}\binom{\eta+m+j-1}{k} t^{1-m-j} = \binom{\eta}{k}, \quad k = 0,1,2,...,m-1. \tag{3.16}$$

The system of equations with unknowns $C_{jt}, \quad j = 1,2,...,m-1$, must have a unique solution since the system of function $\{(x-a)^{\eta+m+j-1}\}_{j=0}^{m-1}$ is an extended Chebyshev systems [2, pp. 9], with application solving (3.16) with unknowns $C_{jt}t^{1-m-j}$, $j = 1,2,...,m-1$, we get the value of $C_{jt}t^{1-m-j}$ which is independent of t, thus $|C_{jt}t^{1-m-j}| = O(1)$.

That is to say

$$|C_{jt}| = O(t^{m-1+j}), \quad j = 1, 2, ..., m-1. \tag{3.17}$$

Now we notice that on $[-1, a]$ and $[a + \frac{1}{t}, 1]$, $g(x) - g_t(x) = 0$ and on $[a, a + 1/t)$

$$\begin{aligned} |g(x) - g_t(x)| &= \left| \sum_{j=0}^{m-1} C_{jt}(x-a)^{\eta+m+j-1} - (x-a)^{\eta} \right| \\ &\le \sum_{j=0}^{m-1} \left| C_{jt} t^{-\eta-m-j+1} + t^{-\eta} \right| = O(t^{-\eta}), \end{aligned}$$

so

$$\lim_{n\to\infty} \|g_t - g\| = 0. \tag{3.18}$$

It is easy to see that the convergence is uniform on $[-1, a]$.

By Lemma 4 for fixed t it has

$$\lim_{n\to\infty} \|H^*_{nm}(g_t) - g_t\| = 0. \tag{3.19}$$

It is to see that the convergence is also uniform on $[-1, a]$. Now we choose t so large that

$$|g_t(x) - g(x)| < \frac{\varepsilon}{3} \frac{1}{2C_1 + 1}, \quad x \in [-1, 1]. \tag{3.20}$$

$$|\tbinom{\eta}{j} t^{-\eta}| \le \frac{\rho}{6m} \frac{1}{C(C_1 + C_2\rho + 1)} \varepsilon. \tag{3.21}$$

For this t, choosing $n_1 \ge n_0$ so large that while $n \ge n_1$ we have

$$|H^*_{nm}(g_t, a)| = |H^*_{nm}(g_t, a) - g_t(a)| < \frac{\varepsilon}{3}.$$

In this case it has

$$\begin{aligned} &|H^*_{nm}(g, a)| = |H^*_{nm}(g, a) - g(a)| \\ &= |H^*_{nm}(g - g_t, a) + H^*_{nm}(g_t, a)| \le |H^*_{nm}(g - g_t, a)| + \frac{\varepsilon}{3}, \end{aligned}$$

for term $|H^*_{nm}(g - g_t, a)|$ we have

$$\begin{aligned} |H^*_{nm}(g - g_t, a)| &= \left| \sum_{k=1}^{n} \sum_{j=0}^{m_k-1} \left[g^{(j)}(x_k) - g_t^{(j)}(x_k) \right] \frac{1}{j!} B_{jk}(a)(a - x_k)^j L_k(a) \right| \\ &\le \left| \sum_{k=1}^{n} \left[g(x_k) - g_t(x_k) \right] B_{0k}(a) L_k(a) \right| \\ &+ \left| \sum_{k=1}^{n} \sum_{j=1}^{m_k-1} \left[g^{(j)}(x_k) - g_t^{(j)}(x_k) \right] \frac{1}{j!} B_{jk}(a)(a - x_k)^j L_k(a) \right| \\ &\le \frac{\varepsilon}{3} + \left| \sum_{k=1}^{n} \sum_{j=1}^{m_k-1} \left[g^{(j)}(x_k) - g_t^{(j)}(x_k) \right] \frac{1}{j!} B_{jk}(a)(a - x_k)^j L_k(a) \right|, \end{aligned}$$

Let us estimate $S_2 =: \left|\sum_{k=1}^{n}\sum_{j=1}^{m_k-1}\left[g^{(j)}(x_k)-g_t^{(j)}(x_k)\right]\frac{1}{j!}B_{jk}(a)(a-x_k)^jL_k(a)\right|$.

First by the definition of $g(x)$ and $g_t(x)$, (3.16) we obtain

$$\begin{aligned}
S_2 &= \left|\sum_{a<x_k<a+\frac{1}{t}}\sum_{j=1}^{m_k-1}\left[g^{(j)}(x_k)-g_t^{(j)}(x_k)\right]\frac{1}{j!}B_{jk}(a)(a-x_k)^jL_k(a)\right| \\
&\le \sum_{a<x_k<a+\frac{1}{t}}\sum_{j=1}^{m_k-1}\left|\left[\binom{\eta}{j}-\sum_{l=0}^{m-1}C_{lt}\binom{\eta+m+l-1}{j}(x_k-a)^{m-1+l}\right]B_{jk}(a)L_k(a)(a-x_k)^{\eta}\right| \\
&\le C\sum_{\substack{a<x_k<a+\frac{1}{t}\\ k\in I_{1n}}}\sum_{j=1}^{m_k-1}\left|2\binom{\eta}{j}t^{-\eta}B_{jk}(a)L_k(a)\right| \\
&+ C\sum_{\substack{a<x_k<a+\frac{1}{t}\\ k\in I_{2n}}}\sum_{j=1}^{m_k-1}\left|2\binom{\eta}{j}t^{-\eta}B_{jk}(a)L_k(a)\right| := C\{S_{21}+S_{22}\},
\end{aligned}$$

For S_{21} and S_{22} by Theorem A and (3.5)

$$\begin{aligned}
S_{21} &\le 2C\sum_{\substack{a<x_k<a+\frac{1}{t}\\ k\in I_{1n}}}\sum_{j=1}^{m_k-1}|\binom{\eta}{j}t^{-\eta}|\frac{|B_{0k}(a)L_k(a)|}{\rho} \\
&\le 2Cm\sum_{\substack{a<x_k<a+\frac{1}{t}\\ k\in I_{1n}}}|\binom{\eta}{j}t^{-\eta}|\frac{|A_{0k}(a)|}{\rho} \le \frac{2mC(C_1+1)}{\rho}\left|\binom{\eta}{j}t^{-\eta}\right|, \\
S_{22} &= 2C\sum_{\substack{a<x_k<a+\frac{1}{t}\\ k\in I_{2n}}}\sum_{j=1}^{m_k-1}\left|\binom{\eta}{j}t^{-\eta}\right||B_{jk}(a)L_k(a)| \le 2mCC_2\left|\binom{\eta}{j}t^{-\eta}\right|,
\end{aligned}$$

If t is properly chosen we have $S_{21}+S_{22}<\frac{\varepsilon}{3}$.

By means of these estimats we have

$$|H_{nm}^*(g-g_t,a)| \le \frac{2}{3}\varepsilon,$$

then it has

$$|H_{nm}^*(g,a)| < \varepsilon. \tag{3.22}$$

So by Statement 1 and definition of $g(x)$ we have

$$\left|\sum_{x_k>a}V_k(\eta;a)(x_k-a)^{\eta}L_k(a)\right| < \varepsilon, \tag{3.23}$$

where we suppose that $x_k \ne a$, $\quad 1\le k\le n$. But if $x_k=a$, then the corresponding terms in the sum (3.23) vanish. So we have $|\sum_{x_k\ge a}...| < \varepsilon$ uniformly in a, $-1\le a\le 0$. Using the function

$$g^*(x) = \begin{cases}(a-x)^{\eta}, & x\in[-1,a],\\ 0, & x\in[a,1].\end{cases}$$

with $0 \le a \le 1$, we similarly get relation

$$\left|\sum_{x_k \le a} V_k(\eta;a)(a-x_k)^\eta L_k(a)\right| < \varepsilon$$

uniformly in a, $0 \le a \le 1$, a.e., summarizing we have

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n} V_k(\eta;x)|x-x_k|^\eta L_k(x)\right\| = 0. \tag{3.24}$$

Put

$$S := \left|\sum_{k=1}^{n} V_k(\eta;x)|x-x_k|^\eta L_k(x)\right| \le \left|\sum_{k\in I_{1n}} ...\right| + \left|\sum_{k\in I_{2n}} ...\right| := S_3 + S_4,$$

where S_4 can be writen as

$$S_4 \le \sum_{k\in I_{2n}} |x-x_k|^\eta |B_{0k}(x)| L_k(x) +$$
$$\sum_{j=1}^{m-1} |(\eta)_j| \sum_{k\in I_{2n}} |x-x_k|^\eta \left|\frac{1}{j!}B_{jk}(x)\right| L_k(x) := S_5 + S_6.$$

For S_5 by (3.3) , (3.2) and Hölder inequality we have

$$\begin{aligned}
&\sum_{k\in I_2} |x-x_k|^\eta |B_{0k}(x)| L_k(x)\\
&= \sum_{k\in I_2} \{|x-x_k||B_{0k}(x)|L_k(x)\}^\eta \{|B_{0k}(x)|L_k(x)\}^{1-\eta}\\
&\le \left\{\sum_{k\in I_2} |x-x_k||A_{0k}(x)|\right\}^\eta \left\{\sum_{k\in I_2} |B_{0k}(x)|L_k(x)\right\}^{1-\eta} = o(1),
\end{aligned}$$

Now let us estimate S_6. For arbitrary fixed j, $1 \le j \le m-1$, here $(\eta)_j = \eta(\eta-1)...(\eta-j+1)$, let $p = j/\eta$, $q = j/(j-\eta)$, using Theorem A, (3.4), (3.5) and Hölder inequality

$$\begin{aligned}
&\sum_{k\in I_{2n}} |(\eta)_j||x-x_k|^\eta \left|\frac{1}{j!}B_{jk}(x)L_k(x)\right|\\
&= \sum_{k\in I_{2n}} |(\eta)_j| \left\{\left|\frac{1}{j!}B_{jk}(x)L_k(x)\right|^{1/p} |x-x_k|^\eta\right\} \left\{\left|\frac{1}{j!}B_{jk}(x)L_k(x)\right|^{1/q}\right\}\\
&\le c|(\eta)_j| \left\{\sum_{k\in I_{2n}} |A_{1k}(x)|\right\}^{1/p} \left\{\sum_{k\in I_{2n}} |B_{1k}(x)L_k(x)|\right\}^{1/q} = o(1).
\end{aligned}$$

So applying (3.24), we get $S_3 = o(1)$ uniformly in x.

$$\lim_{n\to\infty}\left\|\sum_{k\in I_{1n}} V_k(\eta;x)|x-x_k|^\eta L_k(x)\right\| = o(1). \tag{3.25}$$

By $|x-x_i| = 2|\frac{x-x_i}{2}| \le 2\left|\frac{x-x_i}{2}\right|^\eta = 2^{1-\eta}|x-x_i|^\eta$ we obtain

$$\lim_{n\to\infty}\left\|\sum_{k\in I_{1n}} V_k(\eta;x)|x-x_k| L_k(x)\right\| = 0. \tag{3.26}$$

So by Statement 1 it has

$$\lim_{n\to\infty}\left\|\sum_{k\in I_{1n}}|A_{jk}|\right\|=0,\quad j=1,2,...,m_k-1. \tag{3.27}$$

Then by (3.4) and Theorem A we have

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{n}|A_{jk}|\right\|=0,\quad j=1,2,...,m-1.$$

By Lemma 5 and Statement 3 shows that (2.5) holds for all polynomials and also for every $f\in C[-1,1]$ by Weierstrass theorem. □

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