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AD GALERKIN ANALYSIS FOR NONLINEAR PSEUDO-HYPERBOLIC EQUATIONS*[1)]

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Abstract

AD (Alternating direction) Galerkin schemes for d-dimensional nonlinear pseudo-hyperbolic equations are studied. By using patch approximation technique, AD procedure is realized, and calculation work is simplified. By using Galerkin approach, highly computational accuracy is kept. By using various priori estimate techniques for differential equations, difficulty coming from non-linearity is treated, and optimal H^1 and L^2 convergence properties are demonstrated. Moreover, although all the existed AD Galerkin schemes using patch approximation are limited to have only one order accuracy in time increment, yet the schemes formulated in this paper have second order accuracy in it. This implies an essential advancement in AD Galerkin analysis.

Key words: nonlinear, pseudo-hyperbolic equation, alternating direction, numerical analysis

1. Introduction

Consider the nonlinear pseudo-hyperbolic equation with memory term given by

$$\begin{aligned} q(u)u_{tt} &= \nabla\cdot(a(u)\nabla u_t + b(u)\nabla u \\ &+ \textstyle\int_0^t c(u(\tau))\nabla u(\tau)d\tau) + p(u)\nabla u_t + r(u)\nabla u + f(u), && x\in\Omega, t\in J. \\ u(x,t) &= 0, && x\in\partial\Omega, t\in J. \\ u(x,0) &= u_0(x), u_t(x,0) = u_{t0}(x), && x\in\Omega. \end{aligned} \tag{1.1}$$

where $\Omega\subset R^d$ ($d\ge 2$ is the dimension of the space) is an open bounded domain with piecewise smooth boundary $\partial\Omega$. $x=(x_1,\cdots,x_d)$. $J=[0,T]$. $\phi(u)=\phi(x,t,u)$ for $\phi=q,a,b,p,r,f$, $c(u(\tau))=c(t,\tau,x,u(x,\tau))$, and $u_0(x)$, $u_{t0}(x)$ are known functions.

We assume that:

1) there exist positive constants q^*, q_*, a^* and a_* such that $q^*\ge q(x,t,\psi)\ge q_*, a^*\ge a(x,t,\psi)\ge a_*$, for all $x\in\Omega, t\in J, \psi\in R$.

2) the function q is Lipschitz continuous with respect to t and u.

3) the functions b,c are bounded, a,b,c and the derivatives c_u, c_τ, $c_{\tau\tau}$ are Lipschitz continuous with respect to t, the derivatives a_u, b_u are Lipschitz continuous with respect to t and u.

4) the functions p,r are bounded, p,r and f are Lipschitz continuous with respect to u.

Equation (1.1) is also called pseudo-hyperbolic integro-differential equation, it is widely used in the fields of visco-elasticity, nuclear physics and biological mechanics. There is some work on its qualitative analysis and numerical solution [3,4,11]. When the memory term $c(u(\tau))\equiv 0$, (1.1) is called pseudo-hyperbolic equation in a usual meaning, which often appears in visco-elasticity

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theory, for example, in the propagation of sound in viscous media and other phenomena of similar nature [1]. There are also some numerical methods for it [9]. But the existed numerical approaches for these two equations are limited to Galerkin schemes, which have highly accuracy, but need fairly complicated calculation. In this paper, we first consider their AD (alternating direction) Galerkin solutions. AD Galerkin method was first propounded by Douglas and Dupont [6,7], and was verified very efficient in numerical approach of parabolic and hyperbolic equations, it can keep highly accuracy of Galerkin method, and can solve large multi-dimensional problems as a series of smaller one-dimensional problems by AD technique, hence can simplify computational work. Here, we will use patch approximation [7] to treat $q(u)$, to realize AD procedure, and use various priori estimate techniques for differential equations to treat difficulty coming from non-linearity, and to obtain optimal H^1 and L^2 convergence of our schemes. Something deserving of mention is that all the existed AD Galerkin schemes using patch approximation are limited to have only one order accuracy in time increment, while the schemes established here have second order accuracy in it. This means an essential improvement in AD Galerkin analysis.

Since pseudo-hyperbolic equation can be regarded as a special case of pseudo-hyperbolic integro-differential equation, we may just study the numerical analysis for the latter, and let the approximation terms derived from $c(u(\tau))$ equal zero to obtain corresponding results for the former. Besides, before studying AD Galerkin scheme, we consider a Galerkin analysis first for convenience.

An outline of the paper is as follows. In Section 2, a Galerkin scheme and its convergence analysis are described. In Section 3, an AD Galerkin procedure and its analysis are given. In Section 4, start-up procedures for the preceding schemes are discussed and generalization is made.

In this paper, the letter K will be a generic constant, and may be different each time it is used. ϵ will be an arbitrarily small constant. Let $(\phi,\psi)=\int_\Omega \phi\psi dx$, and let the norms in the Banach space follow those in [7] and [8].

Divide $[0,T]$ into L small intervals with equal step length $\Delta t=\frac{T}{L}$, denote $t_l=l\Delta t$, $t_{l+\frac{1}{2}}=(l+\frac{1}{2})\Delta t$, $\phi^l=\phi(t_l)$, $\phi^{l+\frac{1}{2}}=\frac{1}{2}(\phi^{l+1}+\phi^l)$, $E\phi^l=2\phi^{l-1}-\phi^{l-2}$, $d_t\phi^l=\frac{1}{\Delta t}(\phi^{l+1}-\phi^l)$, $\partial_t\phi^l=\frac{1}{2\Delta t}(\phi^{l+1}-\phi^{l-1})$, and $\partial_{tt}\phi^l=\frac{1}{(\Delta t)^2}(\phi^{l+1}-2\phi^l+\phi^{l-1})$.

2. Analysis for Galerkin scheme

The weak form of (1.1) can be writen as

$$\begin{aligned}&w=u_t,\\&(q(u)w_t,v)+(a(u)\nabla w+b(u)\nabla u+\int_0^t c(u(\tau))\nabla u(\tau)d\tau,\nabla v)\\&-(p(u)\nabla w+r(u)\nabla u,v)=(f(u),v),\qquad \forall v\in H_0^1(\Omega),t\in J.\end{aligned}\tag{2.1}$$

Let μ =span$(N_1,\cdots,N_m)\subset H_0^1(\Omega)$ be the finite element space associated with a quasi-regular polygonalization of Ω such that the elements have diameters bounded by h, let the index of μ be the integer k. Let $\lambda>\frac{1}{4}(a^*/q_*)$ be a constant, $c_{nl}(U)=c(t_n,t_{l+\frac{1}{2}},x,U^{l+\frac{1}{2}})$, and $\phi^n(U)=\phi(x,t_n,U^n)$ for $\phi=q,a,b,p,r,f$. A Galerkin scheme is obtained by finding $U^{n+1},W^{n+1}\in\mu$ such that

$$\begin{aligned}&(q^n\partial_tW^n,v)+\lambda(\Delta t)^2(q^n\nabla\partial_{tt}W^n,\nabla v)\\&+(a^n(U)\nabla W^n+b^n(U)\nabla U^n+\Delta t\sum_{l=0}^{n-1}c_{nl}(U)\nabla U^{l+\frac{1}{2}},\nabla v)\\&-(p^n(U)\nabla W^n+r^n(U)\nabla U^n,v)\\&=(f^n(U),v)+(\tilde{q}^nE\partial_tW^n,v),\qquad \forall v\in\mu,\\&d_tU^n=W^{n+\frac{1}{2}},\qquad n=3,4,5,\cdots.\end{aligned}\tag{2.2}$$

where the initial valuation of $U^i, W^i (i = 0, 1, 2, 3)$ will be addressed in Section 4, $\tilde{q}^n = q^n - q^n(U)$, and q^n is an approximation to $q^n(U)$ which is chosen as follows: let $\Omega_i = supp(N_i)$, and $\Omega_{ij} = \Omega_i \cap \Omega_j$, then on the patch of elements Ω_{ij}, let $q^n_{ij} = \sqrt{q^n(x^i)}\sqrt{q^n(x^j)}$, where $x^i \in \Omega_i, q^n(x^i) = q(x^i, t_n, U^n(x^i))$. The patch approximation q^n may be multi-valued, for practical computation, x^i is usually chosen to be the ith node in Ω.

One can prove that for h sufficiently small,

$$\sup|\tilde{q}^n| = \sup_{\substack{x\in\Omega_{ij}\\ 1\le i,j\le m}} |q^n_{ij} - q(x, t_n, U^n(x))| \le K_0 h(1 + \|\nabla U^n\|_\infty), \quad for \quad 0 \le n \le L,$$
$$|q^n - q^{n-1}| \le K_0 \Delta t(1 + \|d_t U^{n-1}\|_\infty), \qquad for \quad 1 \le n \le L.$$

For κ sufficiently large, we introduce the following Sobolev-Volterra projection: finding $\tilde{u} : [0, T] \to \mu$ such that

$$(a(u)\nabla(u_t - \tilde{u}_t) + b(u)\nabla(u - \tilde{u}) + \int_0^t c(u(\tau))\nabla(u - \tilde{u})(\tau)d\tau, \nabla v)$$
$$-(p(u)\nabla(u_t - \tilde{u}_t) + r(u)\nabla(u - \tilde{u}), v) + \kappa(u_t - \tilde{u}_t, v) = 0, \quad \forall v \in \mu. \tag{2.3}$$

Let $\eta = u - \tilde{u}$, with an analogous reasoning to that in [4, 5], we derive

Lemma 2.1. *If $u \in H^2(J, H^{k+1}(\Omega))$, then*

$$\|\eta\|_{L^\infty(L^2)} + \|\eta_t\|_{L^2(L^2)} + \|\eta_{tt}\|_{L^2(L^2)}$$
$$+h[\|\nabla\eta\|_{L^\infty(L^2)} + \|\nabla\eta_t\|_{L^2(L^2)} + \|\nabla\eta_{tt}\|_{L^2(L^2)}] \le Kh^{k+1}.$$

moreover, if $u \in H^4(J, H^1(\Omega))$ and $k \ge \frac{d}{2}$, then

$$\sum_{i=0}^{2} \|\nabla \tfrac{\partial^i \tilde{u}}{\partial t^i}\|_{L^\infty(L^\infty)} + \|\tilde{u}_t\|_{L^\infty(L^\infty)} + \|\tilde{u}_{ttt}\|_{L^\infty(H^1)} + \|\tilde{u}_{tttt}\|_{L^2(H^1)} \le K.$$

Denote $\tilde{w} = \tilde{u}_t$, $\rho = w - \tilde{w}$, $\xi^n = U^n - \tilde{u}^n$, and $\theta^n = W^n - \tilde{w}^n$, then $U^n - u^n = \xi^n - \eta^n$, $W^n - w^n = \theta^n - \rho^n$.

Theorem 2.1. *Assume that $k \ge \frac{d}{2}$, if*

$$\|\theta^2\|_1 + \|\theta^3\|_1 + \|\xi^2\|_1 + (\Delta t)^{\frac{1}{2}} \sum_{l=0}^{1} \|\xi^l\|_1$$
$$+(\Delta t)^{\frac{1}{2}}(\|\partial_t\theta^1\| + \|\partial_t\theta^2\|) = O(h^{k+1} + (\Delta t)^2), \tag{2.4}$$

then for Galerkin scheme (2.2),

$$\|\partial_t(W - w)\|_{L^2(L^2)} + \|W - w\|_{L^\infty(L^2)} + h\|W - w\|_{L^\infty(H^1)}$$
$$+\|U - u\|_{L^\infty(L^2)} + h\|U - u\|_{L^\infty(H^1)} = O(h^{k+1} + (\Delta t)^2).$$

Proof. Subtracting (2.1) from (2.2), using relation (2.3), and denoting $\hat{c}_{nl}(u) = c(t_n, t_{l+\frac{1}{2}}, x, u(x, t_{l+\frac{1}{2}}))$, $c_{nl}(u) = c(t_n, t_{l+\frac{1}{2}}, x, u^{l+\frac{1}{2}})$, $\varepsilon_{nl}(c, \nabla\tilde{u}) = \int_{t_l}^{t_{l+1}} c(t_n, \tau, x, u(x, \tau))\nabla\tilde{u}(x, \tau)d\tau - \Delta t\hat{c}_{nl}(u)\nabla\tilde{u}^{l+\frac{1}{2}}$, we get for $\forall v \in \mu$, $n \ge 3$,

$$(q^n\partial_t\theta^n-\tilde{q}^nE\partial_t\theta^n),v)+(\lambda(\Delta t)^2q^n\nabla\partial_{tt}\theta^n+a^n(U)\nabla\theta^n,\nabla v)=:\sum_{i=1}^{2}P_i^n$$
$$=\sum_{i=1}^{8}Q_i^n=:(q^n(U)[(w_t^n-\partial_tw^n)+\partial_t\rho^n]+[q^n(u)-q^n(U)]w_t^n$$
$$+\tilde{q}^n(E\partial_t\tilde{w}^n-\partial_t\tilde{w}^n)-\kappa\rho^n+[f^n(U)-f^n(u)]+[p^n(U)-p^n(u)]\nabla\tilde{w}^n$$
$$+[r^n(U)-r^n(u)]\nabla\tilde{u}^n+p^n(U)\nabla\theta^n+r^n(U)\nabla\xi^n,v)$$
$$-(b^n(U)\nabla\xi^n,\nabla v)-\lambda(\Delta t)^2(q^n\nabla\partial_{tt}\tilde{u}^n,\nabla v)+([a^n(u)-a^n(U)]\nabla\tilde{w}^n,\nabla v)$$
$$+([b^n(u)-b^n(U)]\nabla\tilde{u}^n,\nabla v)-(\Delta t\sum_{l=0}^{n-1}c_{nl}(U)\nabla\xi^{l+\frac{1}{2}},\nabla v)+(\sum_{l=0}^{n-1}\varepsilon_{nl}(c,\nabla\tilde{u}),\nabla v)$$
$$+(\Delta t\sum_{l=0}^{n-1}\{[c_{nl}(u)-c_{nl}(U)]+[\hat{c}_{nl}(u)-c_{nl}(u)]\}\nabla\tilde{u}^{l+\frac{1}{2}},\nabla v). \tag{2.5}$$

Choose $v=\partial_t\theta^n$, multiply (2.5) by $2\Delta t$ and sum for $n=3,4,\cdots,N(3\leq N\leq L)$, by hölder's inequality, we have

$$2\Delta t\sum_{n=3}^{N}P_1^n\geq 2(q_*-\epsilon)\Delta t\sum_{n=3}^{N}\|\partial_t\theta^n\|^2$$
$$-\bar{K}\Delta th^2\sum_{n=3}^{N-1}(1+\|\nabla\xi^n\|_\infty)^2(\|\partial_t\theta^{n-1}\|^2+\|\partial_t\theta^{n-2}\|^2). \tag{2.6}$$
$$2\Delta t\sum_{n=3}^{N}P_2^n=\lambda[(q^N(U)\nabla\theta^{N+1},\nabla\theta^{N+1})+(q^N(U)\nabla\theta^N,\nabla\theta^N)$$
$$+([\tfrac{a^N(U)}{\lambda q^N(U)}-2]q^N(U)\nabla\theta^N,\nabla\theta^{N+1})]$$
$$+\lambda[(\tilde{q}^N\nabla\theta^{N+1},\nabla\theta^{N+1})+(\tilde{q}^N\nabla\theta^N,\nabla\theta^N)-2(\tilde{q}^N\nabla\theta^N,\nabla\theta^{N+1})$$
$$+((q^{N-1}-q^N)\nabla\theta^N,\nabla\theta^N)-(q^2\nabla\theta^3,\nabla\theta^3)-(q^1\nabla\theta^2,\nabla\theta^2)+2(q^2\nabla\theta^2,\nabla\theta^3)$$
$$+\sum_{n=3}^{N}((q^{n-2}-q^n)\nabla\theta^{n-1},\nabla\theta^{n-1})-2\sum_{n=3}^{N}((q^{n-1}-q^n)\nabla\theta^{n-1},\nabla\theta^n)]$$
$$-(a^2(U)\nabla\theta^2,\nabla\theta^3)-\sum_{n=3}^{N}([a^n(U)-a^{n-1}(U)]\nabla\theta^{n-1},\nabla\theta^n),$$

rewrite the first term on the right side of this equality as P_{21}^N, and recall $\lambda>\frac{1}{4}a^*/q_*$, then [7] there exists a constant $\beta_0>0$ such that $P_{21}^N\geq\beta_0(\|\nabla\theta^{N+1}\|^2+\|\nabla\theta^N\|^2)$, thus by assumptions 1)-3), we obtain

$$2\Delta t\sum_{n=3}^{N}P_2^n\geq[\beta_0-\bar{K}h(1+\|\nabla\xi^n\|_\infty)](\|\nabla\theta^{N+1}\|^2+\|\nabla\theta^N\|^2)$$
$$-K(\|\nabla\theta^3\|^2+\|\nabla\theta^2\|^2)-K\Delta t\sum_{n=3}^{N}(1+\|d_t\xi^{n-1}\|_\infty+\|d_t\xi^{n-2}\|_\infty)\|\nabla\theta^{n-1}\|^2$$
$$-K\Delta t\sum_{n=3}^{N}(1+\|d_t\xi^{n-1}\|_\infty)(\|\nabla\theta^{n-1}\|^2+\|\nabla\theta^n\|^2). \tag{2.7}$$

Now turn to the right hand of (2.5), we see

$$
\begin{aligned}
2\Delta t\sum_{n=3}^{N}Q_1^n &\le \epsilon\Delta t\sum_{n=3}^{N}\|\partial_t\theta^n\|^2+K[(\Delta t)^4+\|\eta\|^2_{L^2(L^2)}+\|\rho\|^2_{L^2(L^2)}+\|\rho_t\|^2_{L^2(L^2)}]\\
&\quad +\bar{K}\Delta t h^2\sum_{n=3}^{N}(1+\|\nabla\xi^n\|_\infty)^2(\Delta t)^4+K\Delta t\sum_{n=3}^{N}(\|\xi^n\|_1^2+\|\nabla\theta^n\|^2),
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
2\Delta t\sum_{n=3}^{N}\sum_{i=2}^{8}Q_i^n &\le K[(\Delta t)^4+\|\eta\|^2_{L^\infty(L^2)}+\|\eta_t\|^2_{L^2(L^2)}+\|\xi^2\|_1^2+\|\nabla\theta^{\frac{5}{2}}\|^2]\\
&\quad +K\|\xi^N\|_1^2+\epsilon\|\nabla\theta^{N+\frac12}\|^2+K\Delta t\sum_{n=2}^{N}\|\xi^n\|^2\\
&\quad +K\Delta t\sum_{n=0}^{N-1}\|\xi^{n+\frac12}\|_1^2+K\Delta t\sum_{n=3}^{N}(\|d_t\xi^{n-1}\|_1^2+\|\nabla\xi^{n-1}\|^2)\\
&\quad +K\Delta t\sum_{n=3}^{N}(1+\|d_t\xi^{n-1}\|_\infty)^2\|\nabla\theta^{n-\frac12}\|^2+K\Delta t\sum_{n=3}^{N}\|\nabla\theta^{n-\frac12}\|^2,
\end{aligned}
\tag{2.9}
$$

where assumptions 1), 2), 4) have been used to show (2.8), relation

$$
\sum_{n=3}^{N}(\phi^n,\psi^{n+1}-\psi^{n-1})=(\phi^N,\psi^{N+\frac12})-(\phi^2,\psi^{\frac52})-\sum_{n=3}^{N}(\phi^n-\psi^{n-1},\psi^{n-\frac12}),
\tag{2.10}
$$

conditions 1)-3), and the fact $\|\varepsilon_{nl}(c,\nabla\tilde{u})\|^2\le K(\Delta t)^5\int_{t_l}^{t_{l+1}}\sum_{i=0}^{2}\|\nabla\frac{\partial^i\tilde{u}}{\partial t^i}\|^2dt+K(\Delta t)^6\|\nabla\tilde{u}^{l+\frac12}\|^2$ have been adopted to get (2.9).

Noticing that $d_t\xi^l=\theta^{l+\frac12}+(\tilde{w}^{l+\frac12}-d_t\tilde{u}^l)$ leads to

$$
\|d_t\xi^l\|_\infty\le 1+h^{-\frac{d}{2}}\|\theta^{l+\frac12}\|,\quad \|\xi^n\|_1^2\le K^*[(\Delta t)^4+\|\xi^2\|_1^2+\Delta t\sum_{l=3}^{n}\|\theta^{l-\frac12}\|_1^2],
$$

where $K^*\ge 1$, applying these inequalities and

$$
\|\theta^{N+1}\|^2+\|\theta^N\|^2\le K(\|\theta^2\|^2+\|\theta^3\|^2)+K\Delta t\sum_{n=3}^{N}\|\theta^{n-1}\|^2+\epsilon\Delta t\sum_{n=3}^{N}\|\partial_t\theta^n\|^2
$$

to the combination of relation (2.5)-(2.9) implies that

$$
\begin{aligned}
&\Delta t\sum_{n=3}^{N}\|\partial_t\theta^n\|^2+\|\theta^{N+1}\|_1^2+\|\theta^N\|_1^2\\
&\le\sum_{i=1}^{10}B_i^N=:K_*[(\Delta t)^4+\|\eta\|^2_{L^\infty(L^2)}+\|\eta_t\|^2_{L^2(L^2)}+\|\rho_t\|^2_{L^2(L^2)}+\|\theta^2\|_1^2+\|\theta^3\|_1^2\\
&\quad +\|\xi^2\|_1^2+\Delta t\sum_{l=0}^{1}\|\xi^l\|_1^2+\Delta t(\|\partial_t\theta^1\|^2+\|\partial_t\theta^2\|^2)]+\bar{K}_1\Delta t h^{2-d}\sum_{n=3}^{N}\|\xi^n\|_1^2(\Delta t)^4\\
&\quad +\bar{K}_1Th^2(\Delta t)^4+\bar{K}_2\Delta t h^{2-d}\sum_{n=3}^{N}\|\xi^n\|_1^2(\|\partial_t\theta^{n-1}\|^2+\|\partial_t\theta^{n-2}\|^2)\\
&\quad +\bar{K}_2\Delta t h^2\sum_{n=3}^{N}(\|\partial_t\theta^{n-1}\|^2+\|\partial_t\theta^{n-2}\|^2)+\bar{K}_3h^{1-\frac{d}{2}}\|\xi^n\|_1(\|\nabla\theta^{N+1}\|^2+\|\nabla\theta^N\|^2)\\
&\quad +\bar{K}_3h(\|\nabla\theta^{N+1}\|^2+\|\nabla\theta^N\|^2)+\frac{K_2}{2}\Delta t\sum_{n=3}^{N}\|\theta^n\|_1^2\\
&\quad +\frac{K_3}{2}\Delta t\sum_{n=3}^{N}h^{-\frac{d}{2}}(\|\theta^{n-\frac12}\|+\|\theta^{n-\frac32}\|)(\|\theta^{n-1}\|_1^2+\|\theta^n\|_1^2)\\
&\quad +\frac{K_4}{2}\Delta t\sum_{n=3}^{N}(h^{-\frac{d}{2}}\|\theta^{n-\frac12}\|)^2(\|\theta^{n-1}\|_1^2+\|\theta^n\|_1^2).
\end{aligned}
\tag{2.11}
$$

From Lemma 2.1 and assumption (2.4), $B_1^N \le K_1[h^{2k+2}+(\Delta t)^4]$. Since the conclusion expected in Theorem 2.1 involves $h^{k+1}+(\Delta t)^2$, we may consider that there exist two positive constants θ_1, θ_2 such that $\theta_1 h^{k+1} \le (\Delta t)^2 \le \theta_2 h^{k+1}$. Denote constants $\delta = k - \frac{d}{2} + 1 \ge 1$, $S \ge max\{K_2, K_3, K_4\}$ such that $\frac{2\sqrt{K^*(1+T)}}{ST} \le 1$. And let $G = \frac{e^{-(ST+1)^2/(ST)}}{2K_1 ST}$, $\theta_3 = \max\{1, \theta_2\}$, and $h_0 = min\{(\frac{G}{\theta_3})^{\frac{1}{\delta}}, 1, \frac{1}{2}\sqrt{\frac{K_*}{K_1 T}}, \frac{1}{2\sqrt{2\bar{K}_2}}, \frac{1}{4\bar{K}_3}\}$, then for $0 < h \le h_0$, we have $h^{-\frac{d}{2}}(h^{k+1} + (\Delta t)^2) \le 2G$. Set $\tilde{K} = K_1 e^{(ST+1)^2/(ST)} (\ge 1)$, it's natural that $\|\xi^2\|_1 \le \tilde{K}(h^{k+1} + (\Delta t)^2)$. Now make inductive hypothesis that

$$\|\theta^{l-\frac{3}{2}}\|_1 \le \tilde{K}(h^{k+1} + (\Delta t)^2), \quad \|\theta^{l-\frac{1}{2}}\|_1 \le \tilde{K}(h^{k+1} + (\Delta t)^2) \tag{2.12}$$

has been shown valid for $l = 3, 4, \cdots, N$, then a further deduction shows that for $n = 3, 4, \cdots, N$, $h^{-\frac{d}{2}}\|\xi^n\|_1 \le 2\tilde{K}G\sqrt{K^*(1+T)} \le 1$, and hence $B_2^N \le B_3^N \le \frac{K_*}{4}(\Delta t)^4$, $B_4^N \le B_5^N \le \frac{1}{4}\Delta t \sum_{n=1}^{N-1} \|\partial_t \theta^n\|^2$, $B_6^N \le B_7^N \le \frac{1}{4}(\|\nabla\theta^{N+1}\|^2 + \|\nabla\theta^N\|^2)$, then apply Gronwall's inequality to the simplification of (2.11), we confirm

$$\Delta t \sum_{n=3}^{N} \|\partial_t\theta^n\|^2 + \|\theta^{N+1}\|_1^2 + \|\theta^N\|_1^2 \le \tilde{K}[h^{2k+2} + (\Delta t)^4],$$

$$\|\xi^{N+1}\|_1^2 + \|\xi^N\|_1^2 \le K[(\Delta t)^4 + \|\xi^2\|_1^2 + \Delta t \sum_{n=2}^{N+1} \|\theta^n\|_1^2] \le K[h^{2k+2} + (\Delta t)^4],$$

and inequality (2.12) is also valid for $l = N+1$, which completes the proof.

Using $v = \partial_t W^n$ in (2.2), we derive

Theorem 2.2. *Assume that the conditions of Theorem 2.1 hold, then for Galerkin scheme (2.2),*

$$\begin{aligned} &\Delta t \sum_{n=3}^{N} \|\partial_t W^n\|^2 + \|\nabla W^{N+1}\|^2 + \|\nabla W^N\|^2 + \|\nabla U^{N+1}\|^2 + \|\nabla U^N\|^2 \\ \le\ & K[\|\nabla W^2\|^2 + \|\nabla W^3\|^2 + \|\nabla U^2\|^2 + \Delta t \sum_{l=0}^{1} \|\nabla U^l\|^2 \\ &+\Delta t(\|\partial_t W^1\|^2 + \|\partial_t W^2\|^2) + \Delta t \sum_{n=3}^{N} \|f^n(U)\|^2]. \end{aligned}$$

3. Analysis for AD Galerkin scheme

In this part, let $\Omega = (L_1, R_1) \times (L_2, R_2) \times \cdots \times (L_d, R_d)$ be a rectangular solid in R^d. For $j = 1, 2, \cdots, d$, let $\mu_j = span(\gamma_1^j(x_j), \gamma_2^j(x_j), \cdots, \gamma_{M_j}^j(x_j)) \subset H_0^1([L_j, R_j])$, $\alpha_j \in \{0, 1\}$; denote $|\alpha| = \sum_{j=1}^{d} \alpha_j$, $D^\alpha\phi = \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_d^{\alpha_d}}$, then $D^0\phi = \phi$. Denote

$$\begin{aligned} H = \{\, \phi \mid\ & \phi, D^\alpha\phi \in L^2(\Omega) \ for\ |\alpha| = 1, 2, \cdots, d; \\ & \|D^\alpha\phi\| \le K h^{j-|\alpha|}\|\phi\|_j,\ for\ j = 0, 1, 2\cdots, |\alpha|\ and\ |\alpha| = 2, \cdots, d\,\}. \end{aligned}$$

Let $\otimes$ represent the tensor product operator, $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_d = span(N_1, \cdots, N_m) \subset$

$H_0^1(\Omega)\cap H$ be a k degree finite dimensional space. Consider the following AD Galerkin scheme

$$\begin{aligned}
&(q^n\partial_t W^n, v) + \tfrac{1}{2}\Delta t \sum_{|\alpha|=1}^{d}(2\lambda\Delta t)^{|\alpha|}(q^n D^\alpha \partial_{tt}W^n, D^\alpha v)\\
&+(a^n(U)\nabla W^n + b^n(U)\nabla U^n + \Delta t\sum_{l=0}^{n-1} c_{nl}(U)\nabla U^{l+\frac{1}{2}}, \nabla v)\\
&\quad -(p^n(U)\nabla W^n + r^n(U)\nabla U^n, v)\\
&\quad = (f^n(U), v) + (\tilde{q}^n E\partial_t W^n, v), \qquad \forall v\in \mu,\\
&\qquad d_t U^n = W^{n+\frac{1}{2}}, \qquad n = 3,4,5,\cdots.
\end{aligned}\tag{3.1}$$

where the symbols q^n, λ, $a^n(U)$, $b^n(U)$, $c_{nl}(U)$, $p^n(U)$, $r^n(U)$, $f^n(U)$ and $\tilde{q}^n$ have the same meaning as in scheme (2.2).

If we set $W^n = \sum_{j=1}^{m}\vartheta_j^n N_j$ and $U^n = \sum_{j=1}^{m}\Lambda_j^n N_j$, and let

$$G^n = \begin{bmatrix} q^n(x^1) & 0 & 0 & \cdots & 0\\ 0 & q^n(x^2) & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & q^n(x^m)\end{bmatrix}$$

and $M = (\sum_{|\alpha|=0}^{d}(2\lambda\Delta t)^{|\alpha|}(D^\alpha N_i, D^\alpha N_s))$ be $m\times m$ matrices, then the matrix problem associated with (3.1) can be expressed by

$$(G^n)^{\frac{1}{2}} M (G^n)^{\frac{1}{2}}(\vartheta^{n+1} - 2\vartheta^n + \vartheta^{n-1}) = 2\Delta t\Psi^n, \quad \Lambda^{n+1} = \Lambda^n + \Delta t\vartheta^{n+\frac{1}{2}},$$

where $(\Psi^n)_j = (p^n(U)\nabla W^n + r^n(U)\nabla U^n - q^n d_t W^{n-1} + f^n(U) + \tilde{q}^n E\partial_t W^n, v) - (a^n(U)\nabla W^n + b^n(U)\nabla U^n + \Delta t\sum_{l=0}^{n-1} c_{nl}(U)\nabla U^{l+\frac{1}{2}}, \nabla v)$, $v = N_j$, and $U^i, W^i (i = 0,1,\cdots,n)$ are known.

Thus the calculation work of (3.1) can be carried out as below:

$$\begin{aligned}
&(G^n)^{\frac{1}{2}}\Theta^n = 2\Delta t\Psi^n, \ \ M\Phi^n = \Theta^n, \ \ (G^n)^{\frac{1}{2}}\Upsilon^n = \Phi^n,\\
&\vartheta^{n+1} = 2\vartheta^n - \vartheta^{n-1} + \Upsilon^n, \ \ \Lambda^{n+1} = \Lambda^n + \Delta t\vartheta^{n+\frac{1}{2}}.
\end{aligned}$$

The diagonal matrix G^n involves only point evaluations and hence can be formed quickly and inverted easily. Now the main work focuses on resolving $M\Phi^n = \Theta^n$. Denote $\langle\phi,\psi\rangle_j = \int_{L_j}^{R_j}\phi\psi dx_j$, let I_j be $M_j\times M_j$ unit matrix, $C_j = (\langle\gamma_i^j(x_j), \gamma_s^j(x_j)\rangle_j)$ and $A_j = (\langle(\gamma_i^j(x_j))', (\gamma_s^j(x_j))'\rangle_j)$ be $M_j\times M_j$ matrices $(j = 1,\cdots,d)$. If the nodes are numbered in x_1 direction first, then in x_2, and so on, and finally in x_d direction, then $M\Phi^n = \Theta^n$ is equivalent to a simpler AD procedure

$$I_1\otimes\cdots\otimes I_{j-1}\otimes(C_j + 2\lambda\Delta t A_j)\otimes I_{j+1}\otimes\cdots\otimes I_d\Phi_j^n = \Phi_{j-1}^n, \quad j = 1,2\cdots,d,$$

where $\Phi_0^n = \Theta^n$ and $\Phi^n = \Phi_d^n$. Since M is independent of time, it only need to be decomposed once, and this decomposition can be used at each time step.

Theorem 3.1. *Assume that* $k\geq\max\{3,\frac{d}{2}\}$, *if*

$$\begin{aligned}
&\|\theta^2\|_1 + \|\theta^3\|_1 + \|\xi^2\|_1 + (\Delta t)^{\frac{1}{2}}\sum_{l=0}^{1}\|\xi^l\|_1 + (\Delta t)^{\frac{1}{2}}(\|\partial_t\theta^1\| + \|\partial_t\theta^2\|)\\
&\quad + \sum_{|\alpha|=2}^{d}(\Delta t)^{\frac{|\alpha|+1}{2}}\|D^\alpha d_t\theta^2\| = O(h^{k+1} + (\Delta t)^2),
\end{aligned}\tag{3.2}$$

then for AD Galerkin scheme (3.1),

$$\|\partial_t(W-w)\|_{L^2(L^2)} + \|W-w\|_{L^\infty(L^2)} + h\|W-w\|_{L^\infty(H^1)} + \|U-u\|_{L^\infty(L^2)}$$
$$+h\|U-u\|_{L^\infty(H^1)} + \sum_{|\alpha|=2}^{d}(\Delta t)^{\frac{|\alpha|+1}{2}}\|D^\alpha d_t(W-w)\|_{L^\infty(L^2)} = O(h^{k+1}+(\Delta t)^2).$$

Proof. The error equation got from (2.1), (3.1) and (2.3) is $\sum_{i=1}^{3} P_i^n = \sum_{i=1}^{9} Q_i^n$, where $P_i^n (i = 1, 2)$ and $Q_i^n (i = 1, 2, \cdots, 8)$ follow their forms in (2.5), and

$$P_3^n = \tfrac{1}{2}\Delta t \sum_{|\alpha|=2}^{d} (2\lambda\Delta t)^{|\alpha|}(q^n D^\alpha \partial_{tt}\theta^n, D^\alpha v),$$
$$Q_9^n = -\tfrac{1}{2}\Delta t \sum_{|\alpha|=2}^{d} (2\lambda\Delta t)^{|\alpha|}(q^n D^\alpha \partial_{tt}\tilde{w}^n, D^\alpha v).$$

Taking $v = \partial_t\theta^n$ in the error equation shows

$$\begin{aligned} 2\Delta t \sum_{n=3}^{N} P_3^n \geq\ & \tfrac{1}{2}\Delta t \sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}[q_*\|D^\alpha d_t\theta^N\|^2 - K\|D^\alpha d_t\theta^2\|^2 \\ & -K\Delta t \sum_{n=3}^{N}(1+\|d_t\xi^{n-1}\|_\infty)\|D^\alpha d_t\theta^{n-1}\|^2], \\ 2\Delta t \sum_{n=3}^{N} Q_9^n \leq\ & K[(\Delta t)^4 + \|\theta^{\frac{5}{2}}\|_1^2] + \epsilon\|\theta^{N+\frac{1}{2}}\|_1^2 \\ & +K\Delta t \sum_{n=3}^{N}(1+\|d_t\xi^{n-1}\|_\infty)^2\|\theta^{n-\frac{1}{2}}\|_1^2, \end{aligned}$$

where (2.10) and $k \geq \max\{3, \frac{d}{2}\}$ have been employed for the second inequality. Combine these relations with the existed results for other terms in Section 2, use a similar inductive hypothesis reasoning, we accomplish the proof of Theorem 3.1.

Taking $v = \partial_t W^n$ in (3.1), we have

Theorem 3.2. *Assume that the conditions of Theorem 3.1 hold, then for AD Galerkin scheme (3.1),*

$$\begin{aligned} & \Delta t \sum_{n=3}^{N}\|\partial_t W^n\|^2 + \|\nabla W^{N+1}\|^2 + \|\nabla W^N\|^2 \\ & +\|\nabla U^{N+1}\|^2 + \|\nabla U^N\|^2 + \sum_{|\alpha|=2}^{d}(\Delta t)^{|\alpha|+1}\|D^\alpha d_t W^N\|^2 \\ \leq\ & K[\|\nabla W^2\|^2 + \|\nabla W^3\|^2 + \|\nabla U^2\|^2 + \Delta t\sum_{l=0}^{1}\|\nabla U^l\|^2 \\ & +\Delta t(\|\partial_t W^1\|^2 + \|\partial_t W^2\|^2) + \sum_{|\alpha|=2}^{d}(\Delta t)^{|\alpha|+1}\|D^\alpha d_t W^2\|^2 + \Delta t\sum_{n=3}^{N}\|f^n(U)\|^2]. \end{aligned}$$

4. Initialization procedures and generalization

To start procedures (2.2) and (3.1), we need to define perfect initial value that satisfies (2.4) and (3.2) respectively. One way to do this is to define

$$U^0 = \tilde{u}^0, W^0 = \tilde{u}_t^0.$$

$$(a(\bar{u}^n)\nabla W^n + b(\bar{u}^n)\nabla U^n + \Delta t \sum_{l=0}^{n-1} c_{nl}(\bar{u}^l)\nabla U^{l+\frac{1}{2}}, \nabla v)$$

$$-(p(\bar{u}^n)\nabla W^n + r(\bar{u}^n)\nabla U^n, v) + \kappa(W^n, v)$$

$$= (a(\bar{u}^n)\nabla \bar{u}_t^n + b(\bar{u}^n)\nabla \bar{u}^n + \Delta t \sum_{l=0}^{n-1} c_{nl}(\bar{u}^l)\nabla \tfrac{\bar{u}^{l+1}+\bar{u}^l}{2}, \nabla v)$$

$$-(p(\bar{u}^n)\nabla \bar{u}_t^n + r(\bar{u}^n)\nabla \bar{u}^n, v) + \kappa(\bar{u}_t^n, v), \quad \forall v \in \mu, n = 1, 2, 3. \tag{4.1}$$

where $\tilde{u}^0, \tilde{u}_t^0$ and κ are provided by (2.3), $\phi(\bar{u}^n) = \phi(t_n, x, \bar{u}^n)$ for $\phi = a, b, p, r$; $c_{nl}(\bar{u}^l) = c(t_n, t_{l+\frac{1}{2}}, x, \frac{\bar{u}^{l+1}+\bar{u}^l}{2})$, $\bar{u}^n = u_0 + n\Delta t u_{t0} + \frac{(n\Delta t)^2}{2} u_{tt}^0$, $\bar{u}_t^n = u_{t0} + n\Delta t u_{tt}^0 + \frac{(n\Delta t)^2}{2} u_{ttt}^0$, for $n = 1, 2, 3$. u_{tt}^0, u_{ttt}^0 can be calculated by (1.1) directly.

Denote $\bar{c}_{nl}(U) = \frac{1}{2}[c(t_n, t_{l+\frac{1}{2}}, x, U^{l+\frac{1}{2}}) + c(t_{n+1}, t_{l+\frac{1}{2}}, x, U^{l+\frac{1}{2}})]$, $l = 0, 1, \cdots, n-1$; $\bar{c}_{nn}(U) = \frac{1}{2} c(t_{n+1}, t_{n+\frac{1}{2}}, x, U^{n+\frac{1}{2}})$. Another initialization is given by

$$U^0 = \tilde{u}^0, W^0 = \tilde{u}_t^0.$$

$$(q^{n+\frac{1}{2}}(U) d_t W^n, v) + (a^{n+\frac{1}{2}}(U)\nabla W^{n+\frac{1}{2}}$$

$$+ b^{n+\frac{1}{2}}(U)\nabla U^{n+\frac{1}{2}} + \Delta t \sum_{l=0}^{n} \bar{c}_{nl}(U)\nabla U^{l+\frac{1}{2}}, \nabla v)$$

$$-(p^{n+\frac{1}{2}}(U)\nabla W^{n+\frac{1}{2}} + r^{n+\frac{1}{2}}(U)\nabla U^{n+\frac{1}{2}}, v) = (f^{n+\frac{1}{2}}(U), v), \quad \forall v \in \mu,$$

$$d_t U^n = W^{n+\frac{1}{2}}, \qquad n = 0, 1, 2. \tag{4.2}$$

We can prove that relation (2.4) stands for $k \geq \frac{d}{2}$ with definition (4.1) or (4.2), while (3.2) stands for $k \geq \frac{d}{2}$ with (4.1), and for $k \geq \max\{3, \frac{d}{2}\}$ with (4.2).

Obviously schemes (2.2) and (3.2) are both linear algebraic equation systems about U^{n+1} and W^{n+1}, from theorems in this paper, we see, they are all uniquely solvable, and all have optimal H^1 and L^2 convergence properties.

Remark. The idea in this paper can be extended to generalized nonlinear systems of pseudo-hyperbolic equations. Consider

$$q_s u_{stt} = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{sij}\frac{\partial u_{st}}{\partial x_j} + b_{sij}\frac{\partial u_s}{\partial x_j}$$

$$+ \int_0^t c_{sij}\frac{\partial u_s(\tau)}{\partial x_j} d\tau) + \sum_{i=1}^{d}(p_{si}\frac{\partial u_{st}}{\partial x_i} + r_{si}\frac{\partial u_s}{\partial x_i}) + f_s, \quad x \in \Omega, t \in J, \ s = 1, 2, \cdots, S$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, t \in J.$$

$$u(x,0) = u_0(x), u_t(x,0) = u_{t0}(x), \qquad x \in \Omega. \tag{4.3}$$

where $u = (u_1, u_2, \cdots, u_S)$ is unknown vector variable. $a_{sij} = a_{sij}(x,t,u)$, $\bar{A}_s = (a_{sij})$ is a $d \times d$ symmetric and positive-defined matrix, $q_s = q_s(x,t,u)$,$b_{sij} = b_{sij}(x,t,u)$, $c_{sij} = c_{sij}(t, \tau, x, u(x,\tau))$, etc.

We can present similar schemes and obtain similar conclusion as we do here.

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