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A HYBRID SMOOTHING-NONSMOOTH NEWTON-TYPE ALGORITHM YIELDING AN EXACT SOLUTION OF THE P_0-LCP [*1)]

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Abstract

We propose a hybrid smoothing-nonsmooth Newton-type algorithm for solving the P_0 linear complementarity problem (P_0-LCP) based on the techniques used in the non-smooth Newton method and smoothing Newton method. Under some assumptions, the proposed algorithm can find an exact solution of P_0-LCP in finite steps. Preliminary numerical results indicate that the proposed algorithm is promising.

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1. Introduction

It is well-known that many mathematical programming problems can be reformulated as a non-smooth equation. By using general Jacobian in the sense of Clarke [4], one can treat directly the non-smooth equation and design a few iterative Newton-type algorithms to solve the problem. This is known as the *non-smooth Newton method* [6, 15]. Moreover, to overcome the difficulties arising from non-differentiability of the non-smooth equation, one can smooth the non-smooth equation by using some smoothing functions. Instead of the non-smooth equation, one investigates a system of the parameterized smoothing equations. Furthermore, one can design a few iterative Newton-type algorithms to solve the problem. This is just the *non-interior continuation method / smoothing Newton method*, which has been used extensively to solve a few mathematical programming problems [1, 2, 8, 16].

It is also known that the iterative method only generates generally an approximation solution of the problem concerned. In order to obtain an exact solution of the problem, many algorithms with the finite termination property have been proposed to solve some linear optimization problems including the linear programming [14, 18], the linear complementarity problem [6, 17], the box constrained variational inequality problem [3, 13], and the vertical linear complementarity problem [5]. It is shown in each above-mentioned algorithm that an exact solution of the problem can be found in one step when an iterate is sufficiently close to this solution.

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In this paper, we consider the P_0 linear complementarity problem (P_0-LCP) of finding a vector $(x, y) \in R^n \times R^n$ such that

$$x \geq 0, \quad y \geq 0, \quad x^T y = 0, \quad Mx + q - y = 0, \tag{1.1}$$

where the matrix $M \in R^{n\times n}$ is a P_0-matrix and the vector $q \in R^n$. By exploiting the techniques of non-smooth Newton methods [6] and smoothing Newton methods [9, 16], we give a hybrid smoothing-nonsmooth Newton-type algorithm for the P_0-LCP (1.1) where we use a smoothing function introduced by Huang-Han-Chen [9] in the smoothing Newton step. It is shown that our algorithm can find an exact solution of the P_0-LCP (1.1) in finite steps under some assumptions. We implement the proposed algorithm for several standard test problems by a MATLAB code. The preliminary numerical results indicate that the algorithm is promising.

The rest of this paper is organized as follows. We give some properties of the smoothing function introduced in [9] and some basic concepts in the next section. Then we propose a hybrid smoothing-nonsmooth Newton-type algorithm for the P_0-LCP (1.1). In section 3, we show the finite termination property of the proposed algorithm. Some numerical results are given in section 4.

The following notions will be used throughout this paper. All vector are column vectors, the subscript T denotes transpose, R^n (respectively, R) denotes the space of n-dimensional real column vectors (respectively, real numbers), R^n_+ and R^n_{++} denote the nonnegative and positive orthants of R^n, R_+ (respectively, R_{++}) denotes the nonnegative (respectively, positive) orthant in R. We define $N := \{1, 2, \cdots, n\}$. For any vector $u \in R^n$, we denote by $diag\{u_i : i \in N\}$ the diagonal matrix whose ith diagonal element is u_i and $vec\{u_i : i \in N\}$ the vector u. For simplicity, we use (u, v) for the column vector $(u^T, v^T)^T$. The matrix I represents the identity matrix of arbitrary dimension. The symbol $\|\cdot\|$ stands for the 2-norm. We denote by S the solution set of the P_0-LCP (1.1).

2. Algorithm Description

It is easy to see that the problem (1.1) is equivalent to the following non-smooth equations

$$F(w) := F(x, y) := \begin{pmatrix} y - Mx - q \\ \min\{x, y\} \end{pmatrix} = 0, \tag{2.1}$$

that is, $(x, y) \in S$ if and only if $F(w) = 0$. Since $F(w)$ is a locally Lipschitz-continuous operator, we can define its generalized Jacobian (see [6])

$$\partial F(w) = \{V_a | a_i = 1 \text{ if } x_i < y_i, \quad a_i = 0 \text{ if } x_i > y_i, \quad a_i \in [0, 1] \text{ if } x_i = y_i, \quad i \in N\},$$

where

$$V_a := \begin{pmatrix} -M & I \\ D_a & I - D_a \end{pmatrix}, \quad D_a := diag(a_1, \cdots, a_n).$$

Moreover, the non-smooth function $\min\{x, y\}$ in (2.1) can be smoothed by using the smoothing function $\phi : R^3 \to R$ defined by

$$\phi(\mu, a, b) = (1 + \mu)(a + b) - \sqrt{(1 - \mu)^2(a - b)^2 + 4\mu^2}, \tag{2.2}$$

which was introduced by Huang-Han-Chen [9]. We need the following properties of smoothing function (2.2) which can be found in [9].

Lemma 2.1. *Let $(\mu, a, b) \in R^3$ and $\phi(\mu, a, b)$ be defined by (2.2). Then $\phi(\mu, a, b)$ is continuously differentiable in $R_{++} \times R^2$. Moreover, $\phi(0, a, b) = 0$ if and only if $a, b \geq 0, ab = 0$.*

Let $z := (\mu, w) := (\mu, x, y) \in R^{2n+1}$ and

$$H(z) := \begin{pmatrix} \mu \\ y - Mx - q \\ \Phi(z) \end{pmatrix}, \quad \text{where} \quad \Phi(z) := \begin{pmatrix} \phi(\mu, x_1, y_1) \\ \vdots \\ \phi(\mu, x_n, y_n) \end{pmatrix}. \tag{2.3}$$

By Lemma 2.1, we know that the P_0-LCP (1.1) is equivalent to the equation $H(z) = 0$ in the sense that their solution sets are coincident.

The following lemma will be used in our analysis later. We omit its proof (for details, see Lemma 2.5 and Theorem 3.1 in [9]).

Lemma 2.2. *Let $z = (\mu, x, y) \in R_{++} \times R^{2n}$ and $H(z)$ be defined by (2.3). Then*

(i) *$H(z)$ is continuously differentiable with its Jacobian*

$$H'(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -M & I \\ x + y - v(z) & (1+\mu)I - D_1(z) & (1+\mu)I - D_2(z) \end{pmatrix},$$

where

$$\begin{aligned} v(z) &:= vec\left\{ \frac{-(1-\mu)(x_i - y_i)^2 + 4\mu}{\sqrt{(1-\mu)^2(x_i - y_i)^2 + 4\mu^2}} : \quad i \in N \right\}, \\ D_1(z) &:= diag\left\{ \frac{(1-\mu)^2(x_i - y_i)}{\sqrt{(1-\mu)^2(x_i - y_i)^2 + 4\mu^2}} : \quad i \in N \right\}, \\ D_2(z) &:= diag\left\{ \frac{-(1-\mu)^2(x_i - y_i)}{\sqrt{(1-\mu)^2(x_i - y_i)^2 + 4\mu^2}} : \quad i \in N \right\}. \end{aligned}$$

If M is a P_0-matrix, then the matrix $H'(z)$ is nonsingular on $R_{++} \times R^{2n}$.

(ii) *$H(z)$ is coercive on $R_{++} \times R^{2n}$, i.e., $\lim_{\|w\| \to \infty} \|H(z)\| = \infty$ holds for each $\mu > 0$.*

We are now giving a formal description of our algorithm.

Algorithm 2.1. (*A hybrid smoothing-nonsmooth Newton-type method*)

Step 0. (Initalization)

Choose $\delta, \sigma, \eta, \eta_1 \in (0,1)$ and $\mu_0 \in R_{++}$. Let $\bar{u} := (\mu_0, 0) \in R_{++} \times R^{2n}$ and $x^0 \in R^n$ be an arbitrary point. Let $y^0 := Mx^0 + q$, $w^0 := (x^0, y^0)$ and $z^0 := (\mu_0, w^0)$. Choose $\gamma \in (0,1)$ such that $\gamma \|H(z^0)\| < 1$. Take $\alpha_0 := \|F(w^0)\|$. Set $k := 0$.

Step 1. (One of the Termination Criteria)

If $H(z^k) = 0$, stop. Otherwise, define a function $\rho : R^{2n+1} \to R_+$ by

$$\rho(z^k) = \rho(\mu_k, w^k) := \gamma \|H(z^k)\| \min\{1, \|H(z^k)\|\}.$$

Step 2. (Non-smooth Newton Step)

Choose $V_k \in \partial F(w^k)$. If V_k is singular, then go to Step 3; otherwise, compute $\Delta \hat{w}^k$ from

$$V_k \Delta \hat{w}^k = -F(w^k). \tag{2.4}$$

If

$$F(w^k + \Delta \hat{w}^k) = 0, \tag{2.5}$$

stop. If the following three inequalities are satisfied

$$\|F(w^k + \Delta \hat{w}^k)\| \le \eta \alpha_k, \tag{2.6}$$

$$\mu_0\rho(\mu_k, w^k + \Delta\hat{w}^k) \le \mu_k, \tag{2.7}$$

$$\|H(\mu_k, w^k + \Delta\hat{w}^k)\| \le \eta_1\|H(z^k)\|, \tag{2.8}$$

then set

$$\mu_{k+1} := \mu_k, w^{k+1} := w^k + \Delta\hat{w}^k, \alpha_{k+1} := \|F(w^k + \Delta\hat{w}^k)\|, k := k+1$$

and go to Step 2; otherwise, go to Step 3.

Step 3. (Smoothing Newton Step)

Compute $\Delta z^k := (\Delta\mu_k, \Delta w^k) \in R^{2n+1}$ by

$$H(z^k) + H'(z^k)\Delta z^k = \rho(z^k)\bar{u}. \tag{2.9}$$

Let m_k be the smallest nonnegative integer such that

$$\|H(z^k + \delta^{m_k}\Delta z^k)\| \le [1 - \sigma(1-\gamma\mu_0)\delta^{m_k}]\|H(z^k)\|. \tag{2.10}$$

Let $\lambda_k := \delta^{m_k}$. Set

$$z^{k+1} := z^k + \lambda_k\Delta z^k, \quad \alpha_{k+1} := \|F(w^k + \lambda_k\Delta w^k)\|,$$

and $k := k+1$. Go to Step 1.

Remark 2.1. (i) To solve the general Newton equation (2.4) is a key step in the nonsmooth Newton method; whereas to solve the smoothing Newton equation (such as, (2.9)) is a main step in the smoothing Newton method. Thus, Algorithm 2.1 combines the techniques used in the non-smooth Newton method and the smoothing Newton method. We call Step 2 as the non-smooth Newton step and Step 3 as the smoothing Newton step.

(ii) Generally, the algorithm needs to solve two linear system of equations at each iteration. However, if V_k is singular or V_k is nonsingular but either the condition (2.5) or condition (2.6)–(2.8) is accepted, then the algorithm only needs to solve one linear system of equations at kth iteration.

(iii) By Lemma 2.2 (i), it is not difficult to see that Algorithm 2.1 is well-defined.

(iv) If $F(w^k) \neq 0$ for all $k \ge 0$, then Algorithm 2.1 will not stop at Step 2 for all $k \ge 0$.

(v) It is not difficult to see from (2.1), (2.3), and Lemma 2.1 that if $H(z^k) = 0$ for some $k \ge 0$, then $F(w^k) = 0$ for the same $k \ge 0$; and that if $F(w^k) \neq 0$ for all $k \ge 0$, then $H(z^k) \neq 0$ for all $k \ge 0$.

If a point $w \in R^{2n}$ such that $F(w) = 0$, then w is an exact solution of (1.1). Thus, from the condition (2.5) in Algorithm 2.1 it is not difficult to see the following result holds.

Theorem 2.3. *If Algorithm 2.1 terminates at Step 2 for some $k \ge 0$, then $w^k + \Delta\hat{w}^k$ is an exact solution of (1.1).*

3. Finite Termination Analysis

In this section, we show that the stopping criteria given in either Step 1 or Step 2 of Algorithm 2.1 must be met when k is sufficiently large under some assumptions, that is, $F(w^k) = 0$ holds for some $k \ge 0$, and hence an exact solution is obtained in a finite number of the iteration.

We use the following assumption, which is weaker than many conditions used in previous literatures to ensure the boundedness of iteration sequences (see [10, 7]).

Assumption 3.1. *The solution set S of (1.1) is nonempty and bounded.*

To show our main result, we need to use the following two theorems.

Theorem 3.2. *Suppose that M is a P_0 matrix. Let the sequence $\{z^k = (\mu_k, w^k)\}$ be generated by Algorithm 2.1. If $F(w^k) \neq 0$ for all $k \geq 0$, then $\{z^k\}$ is an infinite sequence where $\mu_k \in R_{++}$ is monotonically decreasing and satisfies $\mu_k \geq \rho(z^k)\mu_0$ for all $k \geq 0$.*

Proof. Similar to the proofs given in [16, 9], by the inequalities (2.6)–(2.8), it is not difficult to show the theorem is true. We omit the proof.

In order to show the boundedness of the sequence $\{z^k = (\mu_k, w^k)\}$ generated by Algorithm 2.1, we need the following lemma.

Lemma 3.3. *Let $\{t_k\}$ and $\{s_k\}$ be two infinite sequences and satisfy $t_k > 0$, $s_k \geq 0$ for all $k \geq 0$, and $\lim_{k\to\infty} t_k = 0$, $\lim_{k\to\infty} s_k = 0$. Let H be defined by (2.3) and $\{(x^k, y^k)\} \subset R^{2n}$ be the sequence with $\|H(t_k, x^k, y^k)\| \leq s_k$. If M is a P_0-matrix and Assumption 3.1 holds, then $\{(x^k, y^k)\}$ is bounded.*

Proof. Similar to the proof of Theorem 2.1 in [10], it is not difficult to show the lemma is true.

Theorem 3.4. *Suppose that M is a P_0 matrix. Let $\{z^k = (\mu_k, w^k)\}$ be the iteration sequence generated by Algorithm 2.1. If $F(w^k) \neq 0$ for all $k \geq 0$. Then*

(i) there has at least one of $\{\|F(w^k)\|\}$ and $\{\|H(z^k)\|\}$ converges to zero as k tends to $+\infty$;

(ii) if Assumption 3.1 is satisfied, $\{z^k\}$ is bounded and each accumulation point of the sequence $\{w^k\}$ is a solution of (1.1).

Proof. If $F(w^k) \neq 0$ for all $k \geq 0$, then $\|H(z^k)\| \neq 0$ for all $k \geq 0$. Therefore, Algorithm 2.1 generates an infinite sequence $\{z^k\} = \{z^{\hat{k}}\} + \{z^{\bar{k}}\}$, where $z^{\hat{k}}$ is generated by Step 2 and $z^{\bar{k}}$ is generated by Step 3. Then it is not difficult to see that one of the following two cases must occur.

Case (A): the sequence $\{z^{\bar{k}}\}$ is finite. Case (B): the sequence $\{z^{\bar{k}}\}$ is infinite.

We first prove (i). If Case (A) occurs, then the sequence $\{z^{\hat{k}}\}$ is infinite. Hence, there must exist a sufficiently large integer $k_0 > 0$ such that $z^k = z^{\hat{k}}$ for all $k > k_0$. Thus, it follows from the inequality (2.6) that

$$\|F(w^k)\| \leq \eta^{k-k_0}\alpha_{k_0}, \quad \forall k > k_0,$$

which, together with $\eta \in (0, 1)$, implies that $\{\|F(w^k)\|\}$ tends to zero as k tends to $+\infty$. If Case (B) occurs, then (2.10) and (2.8) imply that the entire sequence $\{\|H(z^k)\|\}$ is monotonically decreasing, which together with $\|H(z^k)\| \geq 0$ implies that the limit of $\{\|H(z^k)\|\}$ exists, denoted by θ_*. Suppose that $\{\|H(z^k)\|\}$ does not converge to zero, i.e., $\theta_* > 0$. Then $\{\|H(z^{\bar{k}})\|\}$ also converges to $\theta_* > 0$ as $\bar{k} \to \infty$ and from Theorem 3.2 and the definition of $\rho(\cdot)$ we have

$$0 < \rho(z^{\bar{k}+1})\mu_0 \leq \mu_{\bar{k}+1} \leq \mu_{\bar{k}},$$

which, together with Lemma 2.2(ii), implies that $\{z^{\bar{k}}\}$ is bounded. Let $z^* = (\mu_*, w^*)$ be an accumulation point of $\{z^{\bar{k}} = (\mu_{\bar{k}}, w^{\bar{k}})\}$. Without loss of generality, we assume that $\{z^{\bar{k}}\}$ converges to z^*. Then, it follows from the continuity of H and the definition of $\rho(\cdot)$ that $\{\mu_{\bar{k}}\}$ and $\{\rho(z^{\bar{k}})\}$ converge to μ_* and ρ_*, respectively; and that

$$\theta_* = \|H(z^*)\| > 0, \quad \rho_* = \gamma\theta_* \min\{1, \theta_*\} > 0, \quad 0 < \rho_*\mu_0 \leq \mu_* \leq \mu_0. \tag{3.1}$$

Hence, by (2.10), we have $\lim_{\bar{k}\to\infty} \lambda_{\bar{k}} = 0$. On one hand, from the fact that $\lambda_{\bar{k}}/\delta = \delta^{m_{\bar{k}}-1}$ can not satisfy the condition (2.10), we obtain that

$$\|H(z^{\bar{k}} + \delta^{m_{\bar{k}}-1}\Delta z^{\bar{k}})\| > [1 - \sigma(1 - \gamma\mu_0)\delta^{m_{\bar{k}}-1}]\|H(z^{\bar{k}})\|,$$

which implies

$$[\|H(z^{\bar{k}}+\delta^{m_{\bar{k}}-1}\Delta z^{\bar{k}})\|-\|H(z^{\bar{k}})\|]/\delta^{m_{\bar{k}}-1}>-\sigma(1-\gamma\mu_0)\|H(z^{\bar{k}})\|.$$

Furthermore, by Theorem 3.1 in [15], we have $\Delta z^{\bar{k}}$ is bounded and then

$$[H(z^*)/\|H(z^*)\|]^T H'(z^*)\Delta z^* \geq -\sigma(1-\gamma\mu_0)\|H(z^*)\|. \tag{3.2}$$

On the other hand, by (2.9), we have

$$H(z^*)^T H'(z^*)\Delta z^* = -\|H(z^*)\|^2+\rho_* H(z^*)^T\bar{u}. \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\rho_*\mu_0\|H(z^*)\| \geq \rho_* H(z^*)^T\bar{u} \geq [1-\sigma(1-\gamma\mu_0)]\|H(z^*)\|^2. \tag{3.4}$$

(3.1) and (3.4) imply that $\gamma\mu_0 \geq 1-\sigma(1-\gamma\mu_0)$, i.e.,

$$(1-\sigma)(1-\gamma\mu_0)\leq 0,$$

which contradicts with the fact $\sigma<1,\ \gamma\mu_0\leq\gamma\|H(z^0)\|<1$. Therefore, we have $\theta_*=0$. Hence the above argument proved (i).

Next we prove (ii). If Case (A) occurs, then the sequence $\{z^{\hat{k}}\}$ is infinite. Hence, there must exist a sufficiently large integer $k_0>0$ such that z^k is generated by Step 2 in Algorithm 2.1 for all $k>k_0$. Then it is not difficult to see that $\lim_{k\to\infty}\mu_k=\mu_{k_0}>0$. Thus, it follows from the inequality (2.7) that $\{\|H(z^k)\|\}$ is bounded. Furthermore, by Lemma 2.2(ii) we obtain that $\{z^k\}$ is bounded. If Case (B) occurs, then from the proof of (i) that $\|H(z^k)\|\to 0$ as $k\to\infty$. Hence, by the definition of H (2.3), we have

$$\lim_{k\to\infty}\mu_k=0. \tag{3.5}$$

It follows from (2.8) and (2.3) that there exists $c_k\geq 0$ with

$$c_k=\max\{\eta_1^k\|H(z^0)\|,\mu_k+\|F(w^k)\|\},$$

which, together with the proof of (i), implies that

$$\lim_{k\to\infty}c_k=0,\quad \|H(\mu_k,w^k)\|\leq c_k. \tag{3.6}$$

Since M is a P_0 matrix and Assumption 3.1 holds, Lemma 3.3 and (3.5)–(3.6) imply that $\{w^k\}$ is bounded. Hence, by (3.5) we obtain that $\{z^k\}$ is bounded. This completes the proof of the boundedness of the iteration sequence. In addition, since the iteration sequence is bounded, $\{w^k\}$ has at least one accumulation point w^*. Furthermore, by (i) we have $F(w^*)=0$, and hence $w^*\in S$. This completes the proof of (ii).

In the analysis of our main result, we need the following lemma which is proposed in [6].

Lemma 3.5. *Let $w^*\in R^{2n}$ denote a solution of the problem (1.1) and $B_\varepsilon:=\{w\in R^{2n}\mid \|w-w^*\|<\varepsilon\}$ ($\varepsilon>0$). Then there exists a positive number $\varepsilon(w^*)$ such that*

$$F(w)-V_a(w-w^*)=0 \tag{3.7}$$

for all $w\in B_{\varepsilon(w^*)}$ and all matrices $V_a\in\partial F(w)$.

Now we present the main result of this section.

Theorem 3.6. *Suppose that M is a P_0 matrix. For any solution $w^*\in R^{2n}$ of the problem (1.1), assume that every matrix in $\partial F(w^*)$ is nonsingular and Assumption 3.1 holds. Then Algorithm 2.1 terminates at an exact solution of (1.1) in finite steps.*

Proof. Suppose that Algorithm 2.1 does not stop in finite steps. Then we know $F(w^k) \neq 0$ for all $k \geq 0$ and the stopping criteria in Step 2 are inactive all the time. By Theorem 3.2, an infinite sequence $\{z^k = (\mu_k, w^k)\}$ is generated by Algorithm 2.1. Since Assumption 3.1 is satisfied, it follows from Theorem 3.4 that $\{z^k\}$ is bounded and each accumulation point of the sequence $\{w^k\}$ is a solution of (1.1). Without loss of generality, we assume that $\{w^k\}$ converges to w^*. Thus w^* is a solution of (1.1) and every matrix in $\partial F(w^*)$ is nonsingular. It follows from Proposition 3.1 in [15] that all matrices $V_k \in \partial F(w^k)$ are nonsingular for all sufficient large k. By the fact $\lim_{k\to\infty} w^k = w^*$ and Lemma 3.5, there exists $\bar{k} \geq 0$ such that the equation (2.4) is solvable and

$$(w^{\bar{k}} + \Delta w^{\bar{k}}) - w^* = w^{\bar{k}} - V_{\bar{k}}^{-1} F(w^{\bar{k}}) - w^* = -V_{\bar{k}}^{-1}(F(w^{\bar{k}}) - V_{\bar{k}}(w^{\bar{k}} - w^*) = 0.$$

That is,

$$w^{\bar{k}} + \Delta w^{\bar{k}} = w^*.$$

Since $F(w^*) = 0$, we have $F(w^{\bar{k}} + \Delta w^{\bar{k}}) = 0$. Step 2 in Algorithm 2.1 and (2.1) imply that $w^{\bar{k}} + \Delta w^{\bar{k}}$ is a solution of (1.1) and Algorithm 2.1 terminates at the $\bar{k}$th iteration. This is a contradiction. Hence, $F(w^{\bar{k}}) = 0$ for some $\bar{k} \geq 0$. That is, Algorithm 2.1 can find an exact solution of (1.1) in finite steps.

4. Numerical Results

In this section we present some numerical experiments of Algorithm 2.1 by using a MATLAB code. Throughout the computational experiments, the parameters used in the algorithm were $\sigma = 0.25$, $\delta = 0.75$, $\eta = 0.8$, $\eta_1 = 0.99$, $\mu_0 = 0.75$. Take $\gamma = \min\{1/\|H(z^0)\|, 0.99\}$. The starting point $(x^0, y^0) \in R^{2n}$ has been chosen as follows: let $x^0 \in R^n$ as in the examples and set $y^0 := Mx^0 + q$. In Step 1, we used $\|H(z^k)\| \leq 10^{-8}$ as the stopping rule; and in Step 2, we used $\|F(w^k)\| \leq 10^{-8}$ as the stopping rule. The numerical results are summarized in Tables 1 and 2 for different problems. In Tables 1 and 2, EXAM denotes the number of test examples, DIM denotes the number of the variables in the problems, x^0 denotes the starting point, TIT denotes the total number of iterations, NSLSE2 denotes the number of solving linear system of equations in Step 2, NSLSE3 denotes the number of solving linear system of equations in Step 3, and TERM denotes that the algorithm will terminate at which step. In the following, we give a brief description of the tested problems.

Example 4.1. This is the first example of Kanzow [11] in Section 5 with five variables, which has several solutions including two degenerate ones, namely, $x^{*1} = (0.2, 0.4, 0, 0.5, 0)^T$, $y^{*1} = (0, 0, 0, 0, 0.6)^T$ and $x^{*2} = (0, 1, 0, 0.5, 0)^T$, $y^{*2} = (0, 0, 1, 0, 2)^T$. We used the same starting points as in [11]. The tested results are listed in Table 1.

Example 4.2. This is the second example of Kanzow [11] in Section 5 with four variables, in which the matrix M is degenerate. The solution is $x^* = (4/3, 7/9, 4/9, 2/9)^T$, $y^* = (0, 0, 0, 0)^T$ and $x^{*2} = (0, 1, 0, 0.5, 0)^T$, $y^{*2} = (0, 0, 1, 0, 2)^T$. We used the same starting points as in [11]. The tested results are listed in Table 1.

Example 4.3. This is the third example of Kanzow [11] in Section 5 with seven variables, in which the matrix M is degenerate. $x^* = (0.27, 2.09, 0, 0.54, 0.45, 0, 0)^T$, $y^{*1} = (0, 0, 1.72, 0, 0, 1.63, 0.59)^T$ is the nondegenerate solution of the problem. We used the same starting points as in [11]. The tested results are listed in Table 1.

Example 4.4. This is the fourth example of Kanzow [11] in Section 5 with five variables, which has a nondegenerate solution, namely, $x^* = (1, 0, 1, 1.2, 0.4)^T$, $y^* = (0, 1, 0, 0, 0)^T$. We used the same starting points as in [11]. The tested results are listed in Table 1.

Example 4.5. This is the fifth example of Kanzow [11] in Section 5 with n variables. The solution is $x^* = (0, \ldots, 0, 1)^T$, $y^* = (1, \ldots, 1, 0)^T$. For this example, Lemke's complementarity pivot algorithm and Cottle and Danzig's principal pivoting method are known to run in exponential time. As in [11], we used $(0, \ldots, 0)$ as a starting point. This example was also tested by

Table 1: The numerical results of Examples 4.1-4.4

EXAM	DIM	x^0	$\|F(x^0, y^0)\|$	TIT	NSLSE2	NSLSE3	TERM
4.1	5	$(0,\dots,0)$	12.2361	2	1	1	Step 2
		$(1,\dots,1)$	2.2361	2	2	1	Step 2
		$(10,\dots,10)$	22.3607	3	2	1	Step 2
		$(100,\dots,100)$	223.6068	3	2	1	Step 2
		$(1000,\dots,1000)$	2236.1	3	2	1	Step 2
4.2	4	$(0,\dots,0)$	10.7703	2	2	0	Step 2
		$(1,\dots,1)$	2	2	2	1	Step 2
		$(10,\dots,10)$	40.8534	2	2	1	Step 2
		$(100,\dots,1000)$	433.1385	2	2	1	Step 2
		$(1000,\dots,1000)$	4356.1	2	2	1	Step 2
4.3	7	$(0,\dots,0)$	3.6401	4	4	2	Step 2
		$(1,\dots,1)$	2.2361	2	2	1	Step 2
		$(10,\dots,10)$	68.1249	3	2	1	Step 2
		$(100,\dots,100)$	735.5549	3	2	1	Step 2
		$(1000,\dots,1000)$	7410.1	3	2	1	Step 2
4.4	5	$(0,\dots,0)$	1	2	2	0	Step 2
		$(1,\dots,1)$	8.1854	1	1	0	Step 2
		$(10,\dots,10)$	76.6877	1	1	0	Step 2
		$(100,\dots,100)$	762.103	1	1	0	Step 2
		$(1000,\dots,1000)$	7616.3	1	1	0	Step 2

Table 2: The numerical results of Examples 4.5 and 4.6

EXAM	x^0	DIM	$\|F(x^0, y^0)\|$	TIT	NSLSE3	NSLSE3	TERM
4.5	$(0,\dots,0)$	8	2.8284	3	3	1	Step 2
		16	4	3	3	1	Step 2
		32	5.6569	3	3	1	Step 2
		64	8	3	3	1	Step 2
		128	11.3137	3	3	1	Step 2
		256	16	3	3	1	Step2
	$(1,\dots,1)$	8	2.6458	1	1	0	Step 2
		16	3.873	1	1	0	Step 2
		32	5.5678	1	1	0	Step 2
		64	7.9373	1	1	0	Step 2
		128	11.2649	1	1	0	Step 2
		256	15.9687	1	1	0	Step 2
4.6	$(0,\dots,0)$	8	2.8284	2	2	1	Step 2
		16	4	2	2	1	Step 2
		32	5.6569	2	2	1	Step 2
		64	8	2	2	1	Step 2
		128	11.3137	2	2	1	Step 2
		256	16	2	2	1	Step 2
	$(1,\dots,1)$	8	2.8284	2	2	1	Step 2
		16	4	2	2	1	Step 2
		32	5.6569	3	3	2	Step 2
		64	8	3	3	2	Step 2
		128	11.3137	3	3	2	Step 2
		256	16	3	3	2	Step 2

Kanzow [12] and Burker and Xu [1] by using the starting point $x^0 = (1, \ldots, 1), y^0 := Mx^0 + q$. We also tested this problem by using this starting point. The tested results are listed in Table 2.

Example 4.6. This is the sixth example of Kanzow [11] in Section 5 with n variables. The solution is $x^* = (1, 0, \ldots, 0)^T$, $y^* = (0, 1, \ldots, 1)^T$. As in Example 4.5, both Lemke's complementarity pivot algorithm and Cottle and Danzig's principal pivoting method run in exponential time. As in [11], we used $(0, \ldots, 0)$ as a starting point. This example was also tested by Kanzow [12] and Burker and Xu [1] by using the starting point $x^0 = (1, \ldots, 1), y^0 := Mx^0 + q$. We also tested this problem by using this starting point. The tested results are listed in Table 2.

From Tables 1 and 2, we can obtain the following observations:

- All problems tested have been solved using only a small number of iterations, which is significantly better than those appearing in [1, Section 5], [11, Section 7], and [12, Section 5].
- For each problems tested, our algorithm terminated at Step 2, and hence an exact solution of the problem was obtained by Theorem 2.3.
- For each problems tested, we tested it by using different starting point or different dimensions of the problem. However, vary of the number of iterations is very small.

Our computational results indicate that the proposed hybrid smoothing-nonsmooth Newton-type algorithm works very well for all tested problems in this paper. We expect that the method can be used to solve practical large-scale problems efficiently.

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