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HAT AVERAGE MULTIRESOLUTION WITH ERROR CONTROL IN 2-D [*1)]

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Abstract

Multiresolution representations of data are a powerful tool in data compression. For a proper adaptation to the singularities, it is crucial to develop nonlinear methods which are not based on tensor product. The hat average framework permets develop adapted schemes for all types of singularities. In contrast with the wavelet framework these representations cannot be considered as a change of basis, and the stability theory requires different considerations. In this paper, non separable two-dimensional hat average multiresolution processing algorithms that ensure stability are introduced. Explicit error bounds are presented.

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1. Introduction

Multiresolution representations are one of the most efficient tools for data compression. The multi-scale representation of a signal is well adapted to *quantization* or simple *thresholding*.

A discrete sequence f^L is encoded to produce a multi-scale representation of its information contents, $(f^0, \bar{e}^1, \bar{e}^2, \dots, \bar{e}^L)$; this representation is then processed and the end result of this step is a modified multi-scale representation $(\hat{f}^0, \hat{e}^1, \hat{e}^2, \dots, \hat{e}^L)$ which is *close* to the original one, i.e. such that (in some norm)

$$||\hat{f}^0 - f^0|| \leq \epsilon_0 \qquad ||\hat{e}^k - \bar{e}^k|| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \dots, \epsilon_L$ are chosen according to some criteria specified by the user. After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set f^L. Thus, some form of stability is needed, i.e. we must require that

$$||\hat{f}^L - f^L|| \leq \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L)$$

where $\sigma(\cdot, \dots, \cdot)$ satisfies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L) = 0.$$

The stability analysis for linear prediction processes can be carried out using tools coming from wavelet theory, subdivision schemes and functional analysis (see [11]), however none of these techniques is applicable in general when the prediction process is nonlinear.

The discrete multiresolution framework of Harten [11] was developed to use nonlinear reconstruction processes. In signal and image examples [6], [4], [7], [9], we can see the nonlinear process allows a better adapted treatment of singularities. In these cases, stability can be

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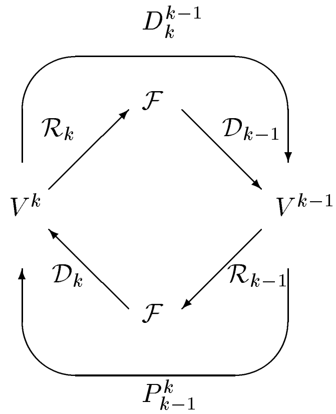

Figure 1: Definition of the operators

ensured by modifying the encoding algorithm. The idea of a modified-encoding to deal with nonlinear multiresolution schemes is due to Harten. One dimensional algorithms in several settings can be found in [10], [7]. The goal of a modified-encoding procedure is to keep track of the accumulation error in processing the values in the multi-scale representation. A synchronization of the encoding and decoding algorithms is obtained [5].

In this paper, we consider the hat average multiresolution setting [8]. In this framework, we can develop nonlinear schemes adapted to the presence of different types of discontinuities as δ's, jumps and corners [9]. The space of such functions is used, for instance, in vortex methods for the numerical solution of fluid dynamics problems.

In the framework of point values and cell averages we developed the stability for tensor product in [3]-[1] and in the non separable case in [2]. For a good adaptation to the singularities we have to consider the non separable approach.

The aim of this paper is to present non separable two-dimensional hat average multiresolution algorithms that ensure stability in the case of nonlinear prediction processes. We introduce a modified encoding for any reconstruction type. The multivariate context of tensor product emerges only as a particular case.

The paper is organized as follows: We recall the basic ingredients of the Harten's multiresolution framework in next section, focussing in the hat average setting 2.1. The error-control algorithms are discussed in 3. Finally, we give stability results in 4.

2. Harten's Framework

Harten's framework is based on two fundamental tools: discretization $\mathcal{D}_k$ and reconstruction $\mathcal{R}_k$. The discretization operator obtains discrete information from a (non-discrete) signal ($f \in \mathcal{F}$) at a particular resolution level k. The reconstruction operator, on the other hand, produces an approximation to a signal from its discrete values. This reconstruction can be nonlinear, and then better adapted to the considered problem.

Using these two operators we can connect linear vectors spaces (see figure 1), V^k, that represent in some way the different resolution levels (k increasing implies more resolution), i.e.,

$$\begin{aligned} D_k^{k-1} &: V^k \to V^{k-1}, \quad \textit{decimation}, \\ P_{k-1}^k &: V^{k-1} \to V^k, \quad \textit{prediction}. \end{aligned}$$

We focus on the specific case corresponding to the hat average discretization.

2.1. Hat average multiresolution analysis

Let us consider the unit interval $[0,1]$ and the sequence of nested dyadic grids $X^k = \{x_i^k\}$, $x_i^k = ih_k$, $h_k = 2^{-k}h_0$, $J_k = 2^k J_0$.

The discretization operator is based on integrating scaled translates of the hat function:

$$w(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{1}$$

that is:

$$(D_k f)_i = f_i^k =< f, w_i^k >, \quad w_i^k = \frac{1}{h_k} w(\frac{x}{h_k} - i). \tag{2}$$

$$D_k : \begin{cases} \mathcal{F} & \to V^k \\ f & \mapsto f^k = (f_i^k)_{i=0}^{J_k - 1} \end{cases} \tag{3}$$

where V^k is the space of real sequences of length $J_k - 1$ (these averages contain information of f over the whole interval $[0,1]$) and $\mathcal{F}$ is the space of piecewise smooth functions in $[0,1]$ with a finite number of δ-type singularities in $(0,1)$ (because the hat function is continuous).

The hat function satisfies the following dilation equation:

$$w(x) = \frac{1}{2}[w(2x-1) + 2w(x) + w(2x+1)]. \tag{4}$$

This implies that

$$f_i^{k-1} = \frac{1}{4} f_{2i-1}^k + \frac{1}{2} f_{2i}^k + \frac{1}{4} f_{2i+1}^k, \tag{5}$$

and that the prediction errors satisfy

$$\bar{e}_{2i}^k = -\frac{1}{2}\bar{e}_{2i-1}^k - \frac{1}{2}\bar{e}_{2i+1}^k. \tag{6}$$

In particular, we only need to keep $d_i^k := \bar{e}_{2i-1}^k$, for $i = 1, \dots, J_{k-1}$.

To complete the construction we need to define the prediction operators.

Let $f = f_p + \sum_l h_l \delta(x - a_l)$, $0 < a_l < 1$, be represented as the sum of a piecewise smooth function in $[0,1]$, f_p, with a finite number of δ-jumps in $(0,1)$. Define the "second primitive" as

$$H(x) = \int_0^x \int_0^y f_p(z) dz dy + \sum_l h_l (x - a_l)_+ - qx, \tag{7}$$

where

$$(x - a_l)_+ \begin{cases} x - a & if \quad x > a \\ 0 & otherwise, \end{cases} \tag{8}$$

and

$$q = \int_0^1 \int_0^y f_p(z) dz dy + \sum_l h_l. \tag{9}$$

$H(x)$ is a continuous piecewise function with satisfies the following relation:

$$f_i^k = \frac{1}{h_k^2}(H_{i+1}^k - 2H_i^k + H_{i-1}^k), \quad 1 \leq i \leq J_k - 1, \tag{10}$$

where $H_i^k = H(x_i^k)$.

We can interpolate the point values of the "second primitive" by any interpolation procedure (note that it is not required linear) $\mathcal{I}_k(x; H^k)$ and define

$$(\mathcal{R}_k f^k)(x) := \frac{d^2}{dx^2}\mathcal{I}_k(x; H^k). \tag{11}$$

The prediction operator is now computed from $\mathcal{R}_k$, and this leads to

$$(P_{k-1}^k f^{k-1})_i = \frac{1}{h_k^2}(\mathcal{I}_{k-1}(x_{i-1}; H^{k-1}) - 2\mathcal{I}_{k-1}(x_i; H^{k-1}) + \mathcal{I}_{k-1}(x_{i+1}; H^{k-1})). \tag{12}$$

Thus, following [8] one can define the so called direct (13) and inverse (14) multiresolution transforms as

$$f^L \to Mf^L \begin{cases} \text{Do} \quad k = L, \ldots, 1 \\ \quad f_i^{k-1} = \frac{1}{4}(f_{2i-1}^k + 2f_{2i}^k + f_{2i+1}^k) & 1 \leq i \leq J_{k-1} - 1, \\ d_i^k \quad = f_{2i-1}^k - (P_{k-1}^k f^{k-1})_{2i-1} & 1 \leq i \leq J_{k-1}, \end{cases} \tag{13}$$

and

$$Mf^L \to M^{-1}Mf^L \begin{cases} \text{Do} \quad k = 1, \ldots, L \\ f_{2i-1}^k = (P_{k-1}^k f^{k-1})_{2i-1} + d_i^k & 1 \leq i \leq J_{k-1}, \\ \quad f_{2i}^k = 2f_i^{k-1} - \frac{1}{2}(f_{2i-1}^{k-1} + f_{2i+1}^{k-1}) & 1 \leq j \leq J_{k-1} - 1. \end{cases} \tag{14}$$

For more details in this reconstruction we refer [8] and for nonlinear adapted reconstructions we refer [9].

3. Multiresolution Schemes with Error-control

In this section, we describe a modification of the encoding technique within the hat-average framework. It is designed to monitor the cumulative compression error and compress accordingly. The simplest data compression procedure is truncation. This type of data compression is used primarily to reduce the "dimensionality" of the data. A different strategy, which is used to reduce the digital representation of the data is "quantization". Observe that in both cases [3], denoting by $pr(e_{i,j}^k, \epsilon_k)$ the processed value of $\bar{e}_{i,j}^k$, we obtain

$$|\bar{e}_{i,j}^k - \hat{e}_{i,j}^k| \leq \epsilon_k. \tag{15}$$

The algorithmic description of the modified encoding is as follows:

$$
\begin{aligned}
&for \quad k = L, \dots, 1\\
&\quad for \quad i = 1, \dots, J_{k-1} - 1\\
&\quad for \quad j = 1, \dots, J_{k-1} - 1\\
&\qquad \bar{f}^{k-1}_{i,j} = \tfrac{1}{16}(\bar{f}^k_{2i-1,2j-1} + 2\bar{f}^k_{2i-1,2j} + \bar{f}^k_{2i-1,2j+1}\\
&\qquad\qquad +2\bar{f}^k_{2i,2j-1} + 4\bar{f}^k_{2i,2j} + 2\bar{f}^k_{2i,2j+1}\\
&\qquad\qquad +\bar{f}^k_{2i+1,2j-1} + 2\bar{f}^k_{2i+1,2j} + \bar{f}^k_{2i+1,2j+1})\\
&\quad end\\
&\quad end\\
&end\\
&\text{Set } \hat{f}^0 = \bar{f}^0\\
&for \quad k = 1, \dots, L\\
&\quad for \quad i = 1, \dots, J_{k-1} - 1\\
&\qquad f^P_{2i-1,J_{k-1}-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,J_{k-1}-1}\\
&\qquad \hat{e}^k_{2i-1,J_{k-1}-1} = \mathrm{pr}([\bar{f}^k_{2i-1,J_{k-1}-1} - f^P_{2i-1,J_{k-1}-1}], \epsilon_k)\\
&\qquad \hat{f}^k_{2i-1,J_{k-1}-1} = f^P_{2i-1,J_{k-1}-1} + \hat{e}^k_{2i-1,J_{k-1}-1}\\
&\quad end\\
&\quad for \quad j = 1, \dots, J_{k-1} - 1\\
&\qquad f^P_{J_{k-1}-1,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{J_{k-1}-1,2j-1}\\
&\qquad \hat{e}^k_{J_{k-1}-1,2j-1} = \mathrm{pr}([\bar{f}^k_{J_{k-1}-1,2j-1} - f^P_{J_{k-1}-1,2j-1}], \epsilon_k)\\
&\qquad \hat{f}^k_{J_{k-1}-1,2j-1} = f^P_{J_{k-1}-1,2j-1} + \hat{e}^k_{J_{k-1}-1,2j-1}\\
&\quad end\\
&\quad for \quad i, j = 1, \dots, J_{k-1} - 1\\
&\qquad f^P_{2i-1,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j-1}\\
&\qquad \hat{e}^k_{2i-1,2j-1} = \mathrm{pr}([\bar{f}^k_{2i-1,2j-1} - f^P_{2i-1,2j-1}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k)\\
&\qquad \hat{f}^k_{2i-1,2j-1} = f^P_{2i-1,2j-1} + \hat{e}^k_{2i-1,2j-1}\\
&\qquad f^P_{2i-1,2j} = (P^k_{k-1}\hat{f}^{k-1})_{2i-1,2j}\\
&\qquad \hat{e}^k_{2i-1,2j} = \mathrm{pr}([\bar{f}^k_{2i-1,2j} - f^P_{2i-1,2j}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k)\\
&\qquad \hat{f}^k_{2i-1,2j} = f^P_{2i-1,2j} + \hat{e}^k_{2i-1,2j}\\
&\qquad f^P_{2i,2j-1} = (P^k_{k-1}\hat{f}^{k-1})_{2i,2j-1}\\
&\qquad \hat{e}^k_{2i,2j-1} = \mathrm{pr}([\bar{f}^k_{2i,2j-1} - f^P_{2i,2j-1}] - [\bar{f}^{k-1}_{i,j} - \hat{f}^{k-1}_{i,j}], \epsilon_k)\\
&\qquad \hat{f}^k_{2i,2j-1} = f^P_{2i,2j-1} + \hat{e}^k_{2i,2j-1}\\
&\qquad \hat{f}^k_{2i,2j} = \tfrac{1}{4}(16\hat{f}^{k-1}_{i,j} - \hat{f}^k_{2i-1,2j-1} - 2\hat{f}^k_{2i-1,2j} - \hat{f}^k_{2i-1,2j+1}\\
&\qquad\qquad -2\hat{f}^k_{2i,2j-1} - 2\hat{f}^k_{2i,2j+1}\\
&\qquad\qquad -\hat{f}^k_{2i+1,2j-1} - 2\hat{f}^k_{2i+1,2j} - \hat{f}^k_{2i+1,2j+1})\\
&\quad end\\
&end
\end{aligned}
\tag{16}
$$

4. Stability Analysis

We use the following matrix norms:

$$\| A \|_1 = max_{\|x\|_1} \| Ax \|_1, \tag{17}$$

$$\| A \|_\infty = max_{\|x\|_\infty} \| Ax \|_\infty, \tag{18}$$

$$\| A \|_2 = max_{\|x\|_2} \| Ax \|_2 . \tag{19}$$

The proofs are similar with norm 1 and ∞. Moreover, to derive the l^2 bound, we simply note that for a vector, x, one has $\| x \|_2 \leq \| x \|_1$. Thus we are proving the propositions with $p = \infty$.

Proposition 1. *Given a discrete sequence $\bar{f}^L$ and a tolerance level ϵ, if the truncation parameters ϵ_k in the modified encoding algorithm (16) are chosen so that*

$$\epsilon_k = \frac{\epsilon}{2} \cdot q^{L-k}, \qquad 0 < q < 1,$$

then the sequence $\hat{f}^L = M^{-1}\{\bar{f}^0, \hat{e}^1, \ldots, \hat{e}^L\}$ satisfies

$$\| \bar{f}^L - \hat{f}^L \|_p \leq (1-q)^{-1}\epsilon \tag{20}$$

for $p = \infty, 1$ and 2.

Proof. Let us define the cumulative compression error at the k-th level by $\mathcal{E}^k_{i,j}$,

$$\mathcal{E}^k_{i,j} = \bar{f}^k_{i,j} - \hat{f}^k_{i,j},$$

and the modified prediction error at the k-th level by $e^k_{i,j}$,

$$e^k_{i,j} = \bar{f}^k_{i,j} - (P^k_{k-1}\hat{f}^{k-1})_{i,j}.$$

With this notation we get that

$$\begin{aligned} \| \mathcal{E}^k \|_\infty = \max_i(\max(&\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}|) + |\mathcal{E}^k_{2i-1,J_k-1}|), \\ &\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{2i,2j-1}| + |\mathcal{E}^k_{2i,2j}|) + |\mathcal{E}^k_{2i,J_k-1}|), \\ &\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{J_k-1,2j-1}| + |\mathcal{E}^k_{J_k-1,2j}|) + |\mathcal{E}^k_{J_k-1,J_k-1}|))). \end{aligned}$$

We start with the first factor of the maximum. We have that

$$\mathcal{E}^k_{2i-1,2j-1} = \begin{cases} e^k_{2i-1,2j-1} & if \quad |e^k_{2i-1,2j-1} - \mathcal{E}^{k-1}_{i,j}| \leq \epsilon_k \\ \mathcal{E}^{k-1}_{i,j} & otherwise \end{cases}$$

$$\mathcal{E}^k_{2i-1,2j} = \begin{cases} e^k_{2i-1,2j} & if \quad (|e^k_{2i-1,2j} - \mathcal{E}^{k-1}_{i,j}| \leq \epsilon_k) \\ \mathcal{E}^{k-1}_{i,j} & otherwise \end{cases}$$

Let us now examine the possibilities:

1)

$$\mathcal{E}^k_{2i-1,2j-1} = \mathcal{E}^k_{2i-1,2j} = \mathcal{E}^{k-1}_{i,j}$$

then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq 2|\mathcal{E}^{k-1}_{i,j}| \ . \tag{21}$$

2)

$$\begin{aligned} \mathcal{E}^k_{2i-1,2j-1} &= e^k_{2i-1,2j-1} \\ \mathcal{E}^k_{2i-1,2j} &= e^k_{2i-1,2j} \end{aligned}$$

then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq 2(|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k) \ . \tag{22}$$

3)

$$\begin{aligned} \mathcal{E}^k_{2i-1,2j-1} &= \mathcal{E}^{k-1}_{i,j} \\ \mathcal{E}^k_{2i-1,2j} &= e^k_{2i-1,2j} \end{aligned}$$

then

$$|\mathcal{E}^k_{2i-1,2j}| + |\mathcal{E}^k_{2i-1,2j}| \leq 2(|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k) \ . \tag{23}$$

4)

$$\begin{aligned} \mathcal{E}^k_{2i-1,2j-1} &= e^k_{2i-1,2j-1} \\ \mathcal{E}^k_{2i-1,2j} &= \mathcal{E}^{k-1}_{i,j} \end{aligned}$$

then

$$|\mathcal{E}^k_{2i-1,2j}| + |\mathcal{E}^k_{2i-1,2j}| \leq 2(|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k) \ . \tag{24}$$

So, for the first factor, we have:

$$\frac{1}{J_k - 1}\Big(\sum_{j=1}^{\frac{J_k}{2}-1} (|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}|) + |\mathcal{E}^k_{2i-1,J_k-1}|\Big)$$

$$\leq \frac{1}{J_k - 1}\Big(\sum_{j=1}^{\frac{J_k}{2}-1} (2(\epsilon_k + |\mathcal{E}^{k-1}_{i,j}|)) + \epsilon_k\Big)$$

$$= \frac{1}{J_k - 1}(\sum_{j=1}^{\frac{J_k}{2}-1}(2|\mathcal{E}_{i,j}^{k-1}|) + (J_k - 1)\epsilon_k)$$

$$\leq \| \mathcal{E}^{k-1} \|_\infty + \epsilon_k.$$

On the other hand, for the last factor we have directly:

$$\frac{1}{J_k - 1}\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}_{J_k-1,2j-1}^k| + |\mathcal{E}_{J_k-1,2j}^k|) + |\mathcal{E}_{J_k-1,J_k-1}^k|)$$

$$\leq \frac{1}{J_k - 1}((\frac{J_k}{2} - 1)2\epsilon_k + \epsilon_k) = \epsilon_k.$$

Finally, from (26) - (62) (see appendix 4.1) we obtain for the second factor:

$$\frac{1}{J_k - 1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}_{2i,2j-1}^k| + |\mathcal{E}_{2i,2j}^k|) + |\mathcal{E}_{2i,J_k-1}^k|)$$

$$\leq \frac{1}{J_k - 1}(\sum_{j=1}^{\frac{J_k}{2}-1}(4\epsilon_k + 2|\mathcal{E}_{i,j}^{k-1}|) + \epsilon_k)$$

$$= \| \mathcal{E}^{k-1} \|_\infty + (\frac{2J_k - 3}{J_k - 1})\epsilon_k$$

$$= \| \mathcal{E}^{k-1} \|_\infty + 2\epsilon_k.$$

Then:

$$\| \mathcal{E}^L \|_\infty \leq \| \mathcal{E}^{L-1} \|_\infty + 2\epsilon_L \leq \ldots \leq 2\sum_{l=1}^{L}\epsilon_l, \tag{25}$$

taking $\epsilon_k = \frac{\epsilon}{2}q^{L-k}$ where $0 < q < 1$ we obtain $\sum_{l=1}^{L}\epsilon_l \leq \epsilon\frac{1}{1-q}$ and hence (20) for $p = \infty$.

If the reconstruction operators $\mathcal{R}_k$ are linear functionals, the error-control technique we have described allows us to control the quality of the decoded data instead of the compression rate. If the reconstruction operators are non linear (data dependent) this algorithm guarantees the stability of the data compression procedure.

The advantages of linear models are obvious. Not only can they be analyzed in a mathematically tractable manner, as it is the case of stability; but they also yield relatively efficient and fast algorithms. However, these advantages are bought at a price: the visual quality of the resulting images are quite often unsatisfactory. In general, the adaptive algorithms yield perceptually more pleasing results than linear ones.

Throughout this paper we have introduced some error-control theorems for nonlinear hat average multiresolutions in 2-d. Nonlinearity is stressed in the sense that they are not invariant by translations and dilations, though these classes of multiresolution keep some nice properties that other well known multiresolution do not have. Our algorithms are general, and not only for the tensor product case.

Some question, however, remain to be studied. The choice of the particular nonlinear multiresolution, the compression strategies, the norms, ... , among some others, important aspects of this topic, which are still under discussion.

4.1. Appendix: Proof for the second factor

We had to consider the second factor of

$$\begin{aligned}\| \mathcal{E}^k \|_\infty = \max_i(\max(&\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{2i-1,2j-1}|+|\mathcal{E}^k_{2i-1,2j}|)+|\mathcal{E}^k_{2i-1,J_k-1}|),\\ &\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{2i,2j-1}|+|\mathcal{E}^k_{2i,2j}|)+|\mathcal{E}^k_{2i,J_k-1}|),\\ &\frac{1}{J_k-1}(\sum_{j=1}^{\frac{J_k}{2}-1}(|\mathcal{E}^k_{J_k-1,2j-1}|+|\mathcal{E}^k_{J_k-1,2j}|)+|\mathcal{E}^k_{J_k-1,J_k-1}|))).\end{aligned}$$

We have:

$$|\mathcal{E}^k_{2i,2j-1}|+|\mathcal{E}^k_{2i,2j}| = \max(|\mathcal{E}^k_{2i,2j-1}+\mathcal{E}^k_{2i,2j}|, |\mathcal{E}^k_{2i,2j-1}-\mathcal{E}^k_{2i,2j}|) \ . \tag{26}$$

Note that:

$$\begin{aligned}\mathcal{E}^k_{2i,2j-1}+\mathcal{E}^k_{2i,2j} &= 4\mathcal{E}^{k-1}_{i,j}-\frac{1}{4}\mathcal{E}^k_{2i-1,2j-1}-\frac{1}{2}\mathcal{E}^k_{2i-1,2j}\\ &- \frac{1}{4}\mathcal{E}^k_{2i-1,2j+1}+\frac{1}{4}\mathcal{E}^k_{2i,2j-1}-\frac{1}{4}\mathcal{E}^k_{2i,2j+1}\\ &- \frac{1}{4}\mathcal{E}^k_{2i+1,2j-1}-\frac{1}{2}\mathcal{E}^k_{2i+1,2j}-\frac{1}{4}\mathcal{E}^k_{2i+1,2j+1},\end{aligned} \tag{27}$$

$$\begin{aligned}\mathcal{E}^k_{2i,2j}-\mathcal{E}^k_{2i,2j-1} &= 4\mathcal{E}^{k-1}_{i,j}-\frac{1}{4}\mathcal{E}^k_{2i-1,2j-1}-\frac{1}{2}\mathcal{E}^k_{2i-1,2j}\\ &- \frac{1}{4}\mathcal{E}^k_{2i-1,2j+1}-\frac{3}{2}\mathcal{E}^k_{2i,2j-1}-\frac{1}{2}\mathcal{E}^k_{2i,2j+1}\\ &- \frac{1}{4}\mathcal{E}^k_{2i+1,2j-1}-\frac{1}{2}\mathcal{E}^k_{2i+1,2j}-\frac{1}{4}\mathcal{E}^k_{2i+1,2j+1},\end{aligned} \tag{28}$$

and

$$\mathcal{E}^k_{2i,2j-1} = \begin{cases} e^k_{2i,2j-1} & if \quad |e^k_{2i,2j-1}-\mathcal{E}^{k-1}_{i,j}| \le \epsilon_k \\ \mathcal{E}^{k-1}_{i,j} & otherwise \end{cases}$$

$$
\begin{aligned}
\mathcal{E}^k_{2i,2j} &= 4\mathcal{E}^{k-1}_{i,j} - \frac{1}{4}\mathcal{E}^k_{2i-1,2j-1} - \frac{1}{2}\mathcal{E}^k_{2i-1,2j} - \frac{1}{4}\mathcal{E}^k_{2i-1,2j+1} \\
&- \frac{1}{2}\mathcal{E}^k_{2i,2j-1} - \frac{1}{2}\mathcal{E}^k_{2i,2j+1} \\
&- \frac{1}{4}\mathcal{E}^k_{2i+1,2j-1} - \frac{1}{2}\mathcal{E}^k_{2i+1,2j} - \frac{1}{4}\mathcal{E}^k_{2i+1,2j+1}.
\end{aligned}
$$

Let us now examine the possibilities in (27) - (28):

1)

$$
\begin{aligned}
\mathcal{E}^k_{2i-1,2j-1} &= \mathcal{E}^k_{2i-1,2j} = \mathcal{E}^k_{2i-1,2j+1} \\
&= \mathcal{E}^k_{2i,2j-1} = \mathcal{E}^k_{2i,2j+1} \\
&= \mathcal{E}^k_{2i-1,2j-1} = \mathcal{E}^k_{2i-1,2j} \\
&= \mathcal{E}^k_{2i-1,2j+1} = \mathcal{E}^{k-1}_{i,j}
\end{aligned}
$$

then

$$
|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \tfrac{5}{2}|\mathcal{E}^{k-1}_{i,j}| \ . \tag{29}
$$

2)

$$
\begin{array}{l}
\mathcal{E}^k_{2i-1,2j-1} = \mathcal{E}^{k-1}_{i,j} \\
or \\
\mathcal{E}^k_{2i-1,2j+1} = \mathcal{E}^{k-1}_{i,j} \\
or \\
\mathcal{E}^k_{2i+1,2j-1} = \mathcal{E}^{k-1}_{i,j} \\
or \\
\mathcal{E}^k_{2i+1,2j+1} = \mathcal{E}^{k-1}_{i,j} \\
and\ the\ others\ e^k_{i,j}
\end{array}
$$

then

$$
|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{7}{4}\epsilon_k, \tfrac{15}{4}\epsilon_k). \tag{30}
$$

3)

$$
\begin{array}{l}
\mathcal{E}^k_{2i-1,2j} = \mathcal{E}^{k-1}_{i,j} \\
or \\
\mathcal{E}^k_{2i,2j+1} = \mathcal{E}^{k-1}_{i,j} \\
or \\
\mathcal{E}^k_{2i+1,2j} = \mathcal{E}^{k-1}_{i,j} \\
and\ the\ others\ e^k_{i,j}
\end{array}
$$

then

$$
|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{2}\epsilon_k, \tfrac{7}{2}\epsilon_k). \tag{31}
$$

4)

$$\mathcal{E}^k_{2i,2j-1} = \mathcal{E}^{k-1}_{i,j}$$
$$\textit{and the others } e^k_{i,j}$$

then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{5}{2}\epsilon_k, \tfrac{5}{2}\epsilon_k) \ . \tag{32}$$

5) One element with factor $-\frac{1}{4}$ and one with factor $\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{5}{4}\epsilon_k, \tfrac{13}{4}\epsilon_k). \tag{33}$$

6) One element with factor $-\frac{1}{4}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{9}{4}\epsilon_k, \tfrac{9}{4}\epsilon_k). \tag{34}$$

7) One element with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq 2\epsilon_k. \tag{35}$$

8) Each element equal to its $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq 4\epsilon_k. \tag{36}$$

9) Three with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, \tfrac{5}{2}\epsilon_k). \tag{37}$$

10) Three with factor $-\frac{1}{4}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{5}{4}\epsilon_k, \tfrac{13}{4}\epsilon_k). \tag{38}$$

11) Two with factor $-\frac{1}{2}$ and one with factor $\frac{1}{4}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{4}\epsilon_k, \tfrac{11}{4}\epsilon_k). \tag{39}$$

12) Two with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{2}\epsilon_k, \tfrac{3}{2}\epsilon_k). \tag{40}$$

13) Two with factor $-\frac{1}{4}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2(|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k), 2\epsilon_k). \tag{41}$$

14) Two with factor $-\frac{1}{4}$ and one with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k, 3\epsilon_k). \tag{42}$$

15) One with factor $-\frac{1}{2}$, one with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{7}{4}\epsilon_k, \tfrac{7}{4}\epsilon_k). \tag{43}$$

16) One with factor $-\frac{1}{2}$ and three with factor $-\frac{1}{4}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{11}{4}\epsilon_k, \tfrac{11}{4}\epsilon_k). \tag{44}$$

17) Three with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{7}{4}\epsilon_k, \tfrac{7}{4}\epsilon_k). \tag{45}$$

18) Two with factor $-\frac{1}{2}$ and two with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| - \tfrac{1}{2}\epsilon_k, \tfrac{5}{2}\epsilon_k). \tag{46}$$

19) Two with factor $-\frac{1}{4}$, one with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{2}\epsilon_k, \tfrac{3}{2}\epsilon_k). \tag{47}$$

20) One with factor $-\frac{1}{4}$ and three with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{4}\epsilon_k, \tfrac{9}{4}\epsilon_k). \tag{48}$$

21) Two with factor $-\frac{1}{2}$, one with factor $-\frac{1}{4}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{5}{4}\epsilon_k, \tfrac{5}{4}\epsilon_k). \tag{49}$$

22) Three with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k, \epsilon_k). \tag{50}$$

23) Four with factor $-\frac{1}{4}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2(|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k), \tfrac{5}{2}\epsilon_k). \tag{51}$$

24) Two with factor $-\frac{1}{2}$ and three with factor $-\frac{1}{4}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{4}\epsilon_k, \tfrac{9}{4}\epsilon_k). \tag{52}$$

25) Two with factor $-\frac{1}{4}$ and three with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(\tfrac{3}{2}|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, 2\epsilon_k). \tag{53}$$

26) Two with factor $-\frac{1}{4}$, two with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k, \epsilon_k). \tag{54}$$

27) Three with factor $-\frac{1}{2}$, one with factor $-\frac{1}{4}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{4}\epsilon_k, \tfrac{3}{4}\epsilon_k). \tag{55}$$

28) Three with factor $-\frac{1}{4}$ and three with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(\tfrac{9}{4}|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, \tfrac{7}{4}\epsilon_k). \tag{56}$$

29) Three with factor $-\frac{1}{4}$, two with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{3}{4}\epsilon_k, \tfrac{3}{4}\epsilon_k). \tag{57}$$

30) Three with factor $-\frac{1}{2}$, two with factor $-\frac{1}{4}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, \tfrac{1}{2}\epsilon_k). \tag{58}$$

31) Four with factor $-\frac{1}{4}$, one with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \epsilon_k, \epsilon_k). \tag{59}$$

32) Four with factor $-\frac{1}{4}$ and three with factor $-\frac{1}{2}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, \epsilon_k). \tag{60}$$

33) Four with factor $-\frac{1}{4}$, two with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{2}\epsilon_k, \tfrac{1}{2}\epsilon_k). \tag{61}$$

34) Three with factor $-\frac{1}{4}$, three with factor $-\frac{1}{2}$ and $\mathcal{E}^k_{2i,2j-1}$ equals to $\mathcal{E}^{k-1}_{i,j}$ and the others $e^k_{i,j}$ then

$$|\mathcal{E}^k_{2i-1,2j-1}| + |\mathcal{E}^k_{2i-1,2j}| \leq \max(2|\mathcal{E}^{k-1}_{i,j}| + \tfrac{1}{4}\epsilon_k, \tfrac{1}{4}\epsilon_k). \tag{62}$$

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