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CONVERGENCE OF AN ALTERNATING $\boldsymbol{A}$-ϕ SCHEME FOR QUASI-MAGNETOSTATIC EDDY CURRENT PROBLEM *1)

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Abstract

We propose in this paper an alternating $\boldsymbol{A}$-ϕ method for the quasi-magnetostatic eddy current problem by means of finite element approximations. Bounds for continuous and discrete error in finite time are given. And it is verified that provided the time step τ is sufficiently small, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h+\tau^{1/2})$ in the L^2-norm for the magnetic field $\boldsymbol{H}(=\mu^{-1}\nabla\times\boldsymbol{A})$, where h is the mesh size, μ the magnetic permeability.

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1. Introduction

The quasi-magnetostatic eddy current model arises from Maxwell's equations as an approximation by neglecting the displacement current (see [1]-[4], [6]-[7], [10]-[11]):

$$\nabla\times\boldsymbol{E}+\mu\frac{\partial\boldsymbol{H}}{\partial t}=0, \tag{1.1}$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}=\sigma\boldsymbol{E}+\boldsymbol{J}_s, \tag{1.2}$$

where μ is the magnetic permeability of the solution domain and σ the spatially varying electrical conductivity, and $\boldsymbol{J}_s$ is source electric currents density. A constitutive equation

$$\boldsymbol{B}=\mu\boldsymbol{H}$$

relates the magnetic induction and magnetic field vectors. The divergence-free conditions

$$\nabla\cdot\boldsymbol{B}=0 \quad\text{and}\quad \nabla\cdot\boldsymbol{J}=0$$

are also imposed, indicating no point sources or sinks of electric current or magnetic induction exists inside the solution domain Ω. This is reasonable for low-frequency, high-conductivity applications like electrical machines. A number of different formulations have proposed [2, 3, 4, 6, 7, 8, 14, 15]. We consider in this paper the above eddy-current model (1.1)-(1.2) by introducing the magnetic vector potential $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{x},t)$ and the electrical scalar potential $\phi=\phi(\boldsymbol{x},t)$ as primary unknown.

As a matter of fact, the magnetic $\boldsymbol{H}$ can be expressed, in light of $\nabla\cdot\boldsymbol{B}=0$ and $\boldsymbol{B}=\mu\boldsymbol{H}$, as follows

$$\boldsymbol{H}=\frac{1}{\mu}\nabla\times\boldsymbol{A}. \tag{1.3}$$

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Combining equations (1.1) and (1.3) we have

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \tag{1.4}$$

Thus, in term of the $\boldsymbol{A}$-ϕ potentials, equation (1.2) becomes the following **curl-curl** equation:

$$\sigma\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\times(\upsilon\nabla\times\boldsymbol{A}) + \sigma\nabla\phi = \boldsymbol{J}_s. \tag{1.5}$$

Here υ is the inverse of the magnetic permeability μ (magnetic susceptibility). To maintain a divergence-free current density $\boldsymbol{J}$, the following auxiliary equation

$$\nabla\cdot(\sigma\frac{\partial \boldsymbol{A}}{\partial t} + \sigma\nabla\phi - \boldsymbol{J}_s) = 0 \tag{1.6}$$

must be solved simultaneously with equation (1.5).

In the following context we shall concentrate our attention on the finite element error analysis of the following initial-boundary value problem of equations (1.5) and (1.6):

$$\begin{cases} \sigma\dfrac{\partial \boldsymbol{A}}{\partial t} + \nabla\times(\upsilon\nabla\times\boldsymbol{A}) + \sigma\nabla\phi = \boldsymbol{J}_s, \ \Omega\times[0,T], \\ \nabla\cdot(\sigma\dfrac{\partial \boldsymbol{A}}{\partial t} + \sigma\nabla\phi - \boldsymbol{J}_s) = 0, \ \Omega\times[0,T], \\ \boldsymbol{A}\times\boldsymbol{n} = \boldsymbol{0}, \quad \partial\Omega\times[0,T], \\ \boldsymbol{A}(\cdot,0) = \boldsymbol{A}_0, \quad \Omega. \end{cases} \tag{1.7}$$

Here $\Omega\subset\mathbb{R}^3$ is a sufficiently smooth simply-connected and bounded polyhedral computational domain with boundary $\Gamma=\partial\Omega$ and $\boldsymbol{n}$ the unit normal vector to Γ. Though the equations are initially posed on the entire space $\mathbb{R}^3$, we can switch to a bounded domain by introducing an artificial boundary sufficiently removed from the region of interest. This is commonplace in engineering simulations (see [9]).

For the sake of simplicity, we confine ourselves to in this paper linear isotropic, that is, $\upsilon\in L^\infty(\Omega)$ is a bounded uniformly positive scalar function of the spatial variable $\boldsymbol{x}\in\Omega$ only. Hence, for some $\underline{\upsilon},\ \bar{\upsilon}>0$ holds $0<\underline{\upsilon}\le\upsilon\le\bar{\upsilon}$ a.e. in Ω and conductivity $\sigma\in L^\infty(\Omega)$ holds $\sigma\ge 0$ a.e. in Ω.

It is important to note that the magnetic vector potential $\boldsymbol{A}$ lacks physical meaning. The really interesting quantity is the magnetic field $\boldsymbol{H}=\upsilon\nabla\times\boldsymbol{A}$. This is reason that we can use an un-gauged formulation as in (1.7), which does not impose a constraint on $\nabla\cdot\boldsymbol{A}$ on the solution domain Ω. Obviously, this forfeits uniqueness of the solution in parts of the domain where $\sigma=0$, but the solution for $\boldsymbol{H}$ remains unique everywhere.

The $\boldsymbol{A}$-ϕ method by means of finite element approximation has been applied in the magnetostatic eddy current computation far and wide in the last two decades. The numerical results indicate that the method is a fairly valid one simulating the quasi-magnetostatic eddy current model. It is worth our while mentioning that, however, the literature on the error estimates of this method can so far not be found yet. We will in our paper devote ourselves to finite element error analysis of the proposed so-called decoupled $\boldsymbol{A}$-ϕ scheme. It is shown that provided the time step τ is small enough, the proposed algorithm yields for finite time T an error of $\mathcal{O}(h+\tau^{1/2})$ in the L^2-norm for the magnetic field $\boldsymbol{H}(=\upsilon\nabla\times\boldsymbol{A})$, where h is the mesh size.

The contents of this paper is organized as follows. In section 2, we describe the decoupled $\boldsymbol{A}$-ϕ scheme in detail. We give two forms of the method: semi-discrete finite element implicit scheme in time first and fully discrete finite element approximation. Section 3 devotes to the error estimates of the magnetic field $\boldsymbol{H}(=\upsilon\nabla\times\boldsymbol{A})$ and electrical field $\boldsymbol{E}(=-\boldsymbol{A}_t-\nabla\phi)$ with appropriate regularity assumptions.

2. Description of the Method

In this section, we describe the semidscrete and fully discrete finite element approximation of problem (1.7). First, we state some preliminary knowledge that will be frequently cited in the sequel.

2.1. Preliminaries

Throughout this paper we assume that $\Omega \subset \mathbb{R}^3$ is a sufficiently smooth bounded, simply connected polyhedral domain with connected boundary $\Gamma = \partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector to Γ. As usual, the real Sobolev space $W^{m,\,p}(\Omega)$ is defined as follows

$$W^{m,\,p}(\Omega) = \{u \in L^p(\Omega):\ D^\alpha u \in L^p(\Omega),\ \forall |\alpha| \leq m\},$$

where m, p are integers and $m > 0,\ 0 \leq p \leq \infty$, equipped with the following norm:

$$\|u\|_{m,\,p} = (\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^p dx)^{\frac{1}{p}}.$$

The space $W_0^{m,\,p}$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{m,\,p}$ norm (see [5, 13]). For $p = 2$, we denote the Hilbert spaces $W^{m,\,2}(\Omega)$ (resp., $W_0^{m,\,2}(\Omega)$) by $H^m(\Omega)$ (resp., $H_0^m(\Omega)$). The related norm is denoted by $\|\cdot\|_m$. The dual space of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$. For a fixed positive real number T, and a Banach space X, we denote by $L^p(X)$, $H^s(X)$ and $C(X)$ the space $L^p(0,T;X)$, $H^s(0,T;X)$ and $C(0,T;X)$, respectively.

For simplicity, we set

$$X = \{\boldsymbol{v} \in H^1(\Omega)^3; \boldsymbol{n} \times \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\}, \quad M = \{q \in H^1(\Omega); q|_{\partial\Omega} = 0\}.$$

Furthermore, we assume that

(H1) $\boldsymbol{A} \in L^\infty(H^2(\Omega)^3) \cap C^0(X),\ \phi \in L^\infty(H^2(\Omega)) \cap C^0(M)$;

(H2) $\boldsymbol{A}_t \in L^2(L^2(\Omega)^3)$;

(H3) $\int_0^T [\,\|\boldsymbol{A}_{tt}\|_B^2 + \|\nabla\phi_{tt}\|_B^2\,]\, dt \leq C$;

(H4) $\|\nabla\phi_t'(t,\boldsymbol{x})\|_0^2 \leq C$.

Here, and in what follows, the subscript t is employed for $\frac{\partial}{\partial t}$, and we use C as a generic constant depending of $\boldsymbol{J}_s,\ \boldsymbol{A}_0,\ \upsilon,\ \sigma$ and Ω, but not on the time step τ nor on the mesh size h; also, $B = H^\alpha(\Omega)^3, 0 \leq \alpha \leq 1$.

Based on above discusses, for a given $\boldsymbol{J}_s \in W^{2,\infty}(0,T;L^2(\Omega)^3)$ and a given initial magnetic field $\boldsymbol{A}_0 \in X \cap H^2(\Omega)^3$, the variational formulation of equations (1.7) is following:

Find $(\boldsymbol{A},\ \phi)$ in the following spaces

$$\boldsymbol{A} \in H^1(0,T;L^2(\Omega)^3) \cap L^2(0,T;X) \ \text{ and } \ \phi \in L^2(0,T;M)$$

such that it satisfies the following equations

$$\begin{cases} (\sigma\dfrac{\partial \boldsymbol{A}}{\partial t}, \boldsymbol{B}) + (\upsilon\nabla\times\boldsymbol{A},\ \nabla\times\boldsymbol{B}) \\ \qquad +(\sigma\nabla\phi,\ \boldsymbol{B}) = (\boldsymbol{J}_s,\ \boldsymbol{B}),\ \forall \boldsymbol{B} \in X, \\ (\sigma\dfrac{\partial \boldsymbol{A}}{\partial t},\ \nabla\psi) + (\sigma\nabla\phi,\ \nabla\psi) = (\boldsymbol{J}_s, \nabla\psi),\ \forall\psi \in M. \end{cases} \tag{2.1}$$

with initial-value condition $\boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0$.

In the following, we shall assume that (2.1) has a unique solution for all time and that this solution is as smooth as needed.

2.2. Semi-discrete method

In this subsection, we shall semi-discretize variational equation (2.1) in time first by the following implicit scheme: given a time step size τ, let $N = [T/\tau] - 1$; for $n \in \{1, \cdots, N\}$, let $t_n = n\tau$; given $\boldsymbol{A}^n \in X$ and $\phi^n \in M$, approximations of $A(t_n)$ and $\phi(t_n)$, respectively, find $\boldsymbol{A}^n \in X$ and $\phi^n \in M$ such that

$$\left\{ \begin{array}{l} (\sigma\partial_\tau \boldsymbol{A}^n,\ \boldsymbol{B}) + (\upsilon\nabla\times\boldsymbol{A}^n,\ \nabla\times\boldsymbol{B}) \\ \qquad +(\sigma\nabla\phi^n,\ \boldsymbol{B}) = (\boldsymbol{J}_s^n,\ \boldsymbol{B}),\ \forall \boldsymbol{B}\in X, \\ (\sigma\partial_\tau \boldsymbol{A}^n,\ \nabla\psi) + (\sigma\nabla\phi^n,\ \nabla\psi) = (\boldsymbol{J}_s^n,\ \nabla\psi),\ \forall\psi\in M, \end{array} \right. \tag{2.2}$$

where

$$\partial_\tau \boldsymbol{A}^n = \frac{\boldsymbol{A}^n - \boldsymbol{A}^{n-1}}{\tau}.$$

We are now interested in defining an alternating semidiscrete $(\boldsymbol{A},\ \phi)$ scheme for $1 \le n \le N$. We define one sequence of approximate the magnetic vector potential $\{\boldsymbol{A}^n \in X\}$ and one sequence of approximate the electrical scalar potential $\{\phi^n \in M\}$ as follows:

Algorithm 2.1. (The semidiscrete alternating $\boldsymbol{A}$-ϕ scheme)

Step 1 (*Initialization.*) *The sequences $\{\boldsymbol{A}^n \in X\}$ and $\{\phi^n \in M\}$ are initialized by $\boldsymbol{A}^0 = \boldsymbol{A}(t=0)$ and $\phi^0 = \phi(t=0)$ respectively.*

Step 2 (*Time loop.*) *For $n = 1, \cdots, N$, seek $\{\boldsymbol{A}^n \in X\}$ such that*

$$(\sigma\partial_\tau \boldsymbol{A}^n,\ \boldsymbol{B}) + (\upsilon\nabla\times\boldsymbol{A}^n,\ \nabla\times\boldsymbol{B}) = (\boldsymbol{J}_s^n,\ \boldsymbol{B}) - (\sigma\nabla\phi^{n-1},\ \boldsymbol{B}),\ \forall\boldsymbol{B}\in X \tag{2.3}$$

and find $\phi^n \in M$ such that for all $\psi \in M$,

$$(\sigma\nabla\phi^n,\ \nabla\psi) = (\boldsymbol{J}_s^n,\ \nabla\psi) - (\sigma\partial_\tau\boldsymbol{A}^n,\ \nabla\psi). \tag{2.4}$$

2.3. Fully discrete approximation

Now, we describe the fully discrete finite element approximation. Let

$$\begin{array}{c} V_h = \{\boldsymbol{v}_h \in H^1(\Omega)^3 : \boldsymbol{v}_h|_K \in \mathcal{P}_1(K)^3,\ \forall K \in \mathcal{T}_h\}, \\ Q_h = \{q_h \in H^1(\Omega) : q_h|_K \in \mathcal{P}_1(K),\ \forall K\in\mathcal{T}_h\} \end{array} \tag{2.5}$$

where $\mathcal{P}_1(K)$ is the space of linear polynomials and $\mathcal{T}_h$ is a regular partition of Ω into tetrahedrons(see [6]). We define the finite element subspaces X_h and M_h as follows:

$$X_h = \{\boldsymbol{v}_h \in V_h;\ \boldsymbol{n}\times\boldsymbol{v}_h|_{\partial\Omega} = \boldsymbol{0}\}, \quad M_h = \{q_h \in Q_h;\ q_h|_{\partial\Omega} = 0\}. \tag{2.6}$$

Noting that both the magnetic vector potential and electrical scalar potential finite element spaces X_h and M_h are referred to the same partition and both are made up with continuous functions. These finite element spaces satisfy the following approximating properties (see [5, 9, 12, 13]):

(H5) There exists $C > 0$ such that

$$\inf_{\boldsymbol{v}_h\in X_h}\{\|\boldsymbol{v}-\boldsymbol{v}_h\|_0 + h\|\nabla\times(\boldsymbol{v}-\boldsymbol{v}_h)\|_0\} \le Ch^2\|\boldsymbol{v}\|_2,\ \forall\boldsymbol{v}\in X\cap H^2(\Omega)^3.$$

(H6) There exists $C > 0$ such that for $\forall q \in M\cap H^2(\Omega)$,

$$\inf_{q_h\in M_h}\{\|q-q_h\|_0 + h\|\nabla(q-q_h)\|_0\} \le Ch^2\|q\|_2.$$

Hereafter we denote by $\hat{\boldsymbol{A}}_h^0 \in X_h$ and $\hat{\phi}_h^0 \in M_h$ an approximation to $\boldsymbol{A}_0$ and $\phi(t=0)$ such that

$$\|\boldsymbol{A}_0 - \hat{\boldsymbol{A}}_h^0\|_0 + h(\|\boldsymbol{A}_0 - \hat{\boldsymbol{A}}_h^0\|_1 + \|\phi(0) - \hat{\phi}_h^0\|_1) \le Ch^2. \tag{2.7}$$

Let us now give a fully discrete version of Algorithm 2.1: We define one sequence of approximate the magnetic vector potential $\{\boldsymbol{A}_h^n \in X_h\}$ and one sequence of approximate the electrical scalar potential $\{\phi_h^n \in M_h\}$ as follows:

Algorithm 2.2. (The fully discrete alternating $\boldsymbol{A}$-ϕ scheme)

Step 1 (*Initialization.*) *The sequences* $\{\boldsymbol{A}_h^n \in X_h\}$ *and* $\{\phi_h^n \in M_h\}$ *are initialized by* $\boldsymbol{A}_h^0 = \hat{\boldsymbol{A}}_h^0$ *and* $\phi_h^0 = \hat{\phi}_h^0$ *respectively.*

Step 2 (*Time loop.*) *For* $0 < n < N$, *seek* $\{\boldsymbol{A}_h^n \in X_h\}$ *such that*

$$\begin{aligned}&(\sigma\partial_\tau \boldsymbol{A}_h^n,\ \boldsymbol{B}_h) + (\upsilon\nabla\times\boldsymbol{A}_h^n,\ \nabla\times\boldsymbol{B}_h)\\ &= (\boldsymbol{J}_s^n,\ \boldsymbol{B}_h) - (\sigma\nabla\phi_h^{n-1},\ \boldsymbol{B}_h),\ \forall \boldsymbol{B}_h \in X_h\end{aligned} \tag{2.8}$$

and find $\phi_h^n \in M_h$ *such that for all* $\psi_h \in M_h$,

$$(\sigma\nabla\phi_h^n,\ \nabla\psi_h) = (\boldsymbol{J}_s^n,\ \nabla\psi_h) - (\sigma\partial_\tau\boldsymbol{A}_h^n,\ \nabla\psi_h). \tag{2.9}$$

Remark 2.3. (1) It is obvious that that the equation (2.8) has a unique solution whenever $\sigma(\boldsymbol{x}) \geq \underline{\sigma} > 0$. And the Poisson equation (2.9) is also well posed.

(2) Noting that Algorithm 2.2 is simple to implement: they amount to solving at each time step a parabolic problem and a Poisson problem. This technique is fast: the amount of computation is much lower than that required by coupled techniques such as those that are based on the Uzawa operator.

Hereafter we shall use repeatedly the following discrete Gronwall inequality (see [12]):

Lemma 2.4. *Let* $\delta, g_0, a_n, b_n, c_n$ *and* $\gamma_n(n = 0, 1, \cdots)$ *be a sequence of non negative numbers so that*

$$a_n + \delta\sum_{i=0}^n b_i \leq \delta\sum_{i=0}^n \gamma_i a_i + \delta\sum_{i=0}^n c_i + g_0. \tag{2.10}$$

Assume that $\gamma_i\delta < 1$ *for all* i, *and set* $\sigma_i = (1-\gamma_i\delta)^{-1}$. *Then we obtain for all* $n \geq 0$,

$$a_n + \delta\sum_{i=0}^n b_i \leq (\delta\sum_{i=0}^n c_i + g_0)\exp(\delta\sum_{i=0}^n \sigma_i\gamma_i). \tag{2.11}$$

3. Error Analysis

In this section, we present a numerical analysis of the finite element method (2.8)-(2.9). We split the error of the method into a temporal error, due to the semidiscretization (2.3)-(2.4), and a spatial error, due to fully discrete method (2.8)-(2.9).

3.1. Error estimates for semidiscrete solution

Without loss of the generality, we assume that σ and υ are constants in the sequel. (It is straightforward to extend the analysis to the non-constant or elementwise constant case by simply this coefficient inside the integrals or norm and bounding it by taking its maximum or minimum value if necessary).

Let us define the continuous errors (as for the spatial variables) as:

$$\boldsymbol{e}_c^n = \boldsymbol{A}(t_n) - \boldsymbol{A}^n,\ \ \varepsilon_c^n = \phi(t_n) - \phi^n,\ \ \theta_c^{n-1} = \phi(t_n) - \phi^{n-1} = \delta_t\phi(t_n) + \varepsilon_c^{n-1},$$

where the notation $\delta_t\phi(t_n) = \phi(t_n) - \phi(t_{n-1})$.

We now turn our attention to the (continuous) error analysis of Algorithm 2.1. Based on all above-mentioned preliminaries, the main result of this subsection is summarized in the following theorem.

Theorem 3.1. *Assume time step τ is sufficiently small and hypotheses (H1)-(H4) hold. Then the solution to the semidiscrete decoupled $\boldsymbol{A}$-ϕ scheme (2.3)-(2.4) satisfies:*

$$\upsilon\|\nabla\times\boldsymbol{e}_c^N\|_0^2+\frac{1}{2}\sigma\tau\sum_{n=1}^{N}\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2\le C\tau. \tag{3.1}$$

Proof. The whole proof is divided into the following five steps.

Step 1. It follows from (2.1) for all $(\boldsymbol{B},\ \psi)\in X\times M$,

$$\left\{\begin{array}{l}(\sigma\dfrac{\boldsymbol{A}(t_n)-\boldsymbol{A}(t_{n-1})}{\tau},\ \boldsymbol{B})+(\upsilon\nabla\times\boldsymbol{A}(t_n),\ \nabla\times\boldsymbol{B})\\ \quad+(\sigma\nabla\phi(t_n),\ \boldsymbol{B})=(\boldsymbol{J}_s^n+\sigma\boldsymbol{R}^n,\ \boldsymbol{B}),\\ (\sigma\dfrac{\boldsymbol{A}(t_n)-\boldsymbol{A}(t_{n-1})}{\tau},\ \nabla\psi)+(\sigma\nabla\phi(t_n),\ \nabla\psi)=(\boldsymbol{J}_s^n+\sigma\boldsymbol{R}^n,\ \nabla\psi).\end{array}\right. \tag{3.2}$$

where

$$\boldsymbol{R}^n\ =\ -\frac{1}{\tau}\int_{t_{n-1}}^{t_n}(t-t_{n-1})\boldsymbol{A}_{tt}(t)dt, \tag{3.3}$$

By subtracting (2.3) and (2.4) from the first equation and the second equation of (3.2), respectively, we derive the following error controlling equations:

$$(\sigma\partial_\tau\boldsymbol{e}_c^n,\ \boldsymbol{B})+(\upsilon\nabla\times\boldsymbol{e}_c^n,\ \nabla\times\boldsymbol{B})+(\sigma\nabla\theta_c^{n-1},\ \boldsymbol{B})=(\sigma\boldsymbol{R}^n,\ \boldsymbol{B}) \tag{3.4}$$

and

$$(\sigma\partial_\tau\boldsymbol{e}_c^n,\ \nabla\psi)+(\sigma\nabla\varepsilon_c^n,\ \nabla\psi)=(\sigma\boldsymbol{R}^n,\ \nabla\psi). \tag{3.5}$$

Step 2. Take $\boldsymbol{B}=2\tau\partial_\tau\boldsymbol{e}_c^n=2(\boldsymbol{e}_c^n-\boldsymbol{e}_c^{n-1})\in X$ in (3.4) and using the inequality $2(a,\ a-b)>|a|^2-|b|^2$, we get

$$\begin{array}{l}2\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n\|_0^2+2\tau(\sigma\nabla\theta_c^{n-1},\ \partial_\tau\boldsymbol{e}_c^n)\\ \quad+\upsilon(\|\nabla\times\boldsymbol{e}_c^n\|_0^2-\|\nabla\times\boldsymbol{e}_c^{n-1}\|_0^2)\le 2\tau(\sigma\boldsymbol{R}^n,\ \partial_\tau\boldsymbol{e}_c^n),\end{array} \tag{3.6}$$

On the other hand, taking $\psi=2\tau\varepsilon_c^n$ in (3.5), we obtain

$$2\sigma\tau\|\nabla\varepsilon_c^n\|_0^2+2\tau(\sigma\partial_\tau\boldsymbol{e}_c^n,\ \nabla\varepsilon_c^n)=2\tau(\sigma\boldsymbol{R}^n,\ \nabla\varepsilon_c^n). \tag{3.7}$$

Step 3. Adding up (3.6) and (3.7), we get

$$\begin{array}{l}\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\theta_c^{n-1}\|_0^2+\sigma\tau\|\nabla\varepsilon_c^n\|_0^2\\ \quad+\upsilon(\|\nabla\times\boldsymbol{e}_c^n\|_0^2-\|\nabla\times\boldsymbol{e}_c^{n-1}\|_0^2)-\sigma\tau\|\nabla\theta_c^{n-1}\|_0^2\\ \le 2\tau(\sigma\boldsymbol{R}^n,\ \partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n),\end{array} \tag{3.8}$$

Noting that

$$\begin{array}{l}2\tau(\sigma\boldsymbol{R}^n,\ \partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n)\le\dfrac{1}{2}\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+C\tau\|\boldsymbol{R}^n\|_0^2\\ \le\dfrac{1}{2}\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+C\tau^2\displaystyle\int_{t_{n-1}}^{t_n}\|\boldsymbol{A}_{tt}(t)\|_B^2dt.\end{array}$$

It then follow from (3.8) that

$$\begin{array}{l}\dfrac{1}{2}\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+\upsilon\|\nabla\times\boldsymbol{e}_c^n\|_0^2\\ \quad+\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\theta_c^{n-1}\|_0^2+\sigma\tau\|\nabla\varepsilon_c^n\|_0^2\\ \le\upsilon\|\nabla\times\boldsymbol{e}_c^{n-1}\|_0^2+\sigma\tau\|\nabla\theta_c^{n-1}\|_0^2+C\tau^2\displaystyle\int_{t_{n-1}}^{t_n}\|\boldsymbol{A}_{tt}(t)\|_B^2dt.\end{array} \tag{3.9}$$

Step 4. By the definition of θ_c^{n-1} and Hypothesis (H4), we have

$$\begin{aligned}\|\nabla\theta_c^{n-1}\|_0^2 &= \|\nabla(\phi(t_n)-\phi(t_{n-1}))+\nabla\varepsilon_c^{n-1}\|_0^2\\ &= \|\tau\nabla\phi_t'(\xi)+\nabla\varepsilon_c^{n-1}\|_0^2\\ &\le (1+\tau)\|\nabla\varepsilon_c^{n-1}\|_0^2+C\tau,\end{aligned}$$

where $\xi\in(t_{n-1},t_n)$. Hence, it is found from (3.9):

$$\begin{aligned}&\frac{1}{2}\sigma\tau\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+\upsilon\|\nabla\times\boldsymbol{e}_c^n\|_0^2+\sigma\tau\|\nabla\varepsilon_c^n\|_0^2\\ &\le\upsilon\|\nabla\times\boldsymbol{e}_c^{n-1}\|_0^2+\sigma\tau(1+\tau)\|\nabla\varepsilon_c^{n-1}\|_0^2\\ &\quad+C\tau^2+C\tau^2\int_{t_{n-1}}^{t_n}\|\boldsymbol{A}_{tt}(t)\|_B^2dt.\end{aligned}\tag{3.10}$$

Step 5. By taking the sum from $n=1$ to N in (3.10), we have

$$\begin{aligned}&\frac{1}{2}\sigma\tau\sum_{n=1}^N\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+\upsilon\|\nabla\times\boldsymbol{e}_c^N\|_0^2+\sigma\tau\|\nabla\varepsilon_c^N\|_0^2\\ &\le\upsilon\|\nabla\times\boldsymbol{e}_c^0\|_0^2+\sigma\tau\|\nabla\varepsilon_c^0\|_0^2+\sigma\tau^2\sum_{n=1}^N\|\nabla\varepsilon_c^{n-1}\|_0^2\\ &\quad+C\tau+C\tau^2\int_0^T\|\boldsymbol{A}_{tt}(t)\|_B^2dt.\end{aligned}$$

Note that we have applied the relation $\tau N\le T\le C$ in the above inequality. Using Step 1 of Algorithm 2.1 and Hypothesis (H3) and the discrete Gronwall Lemma 2.4 , we infer for sufficiently small τ that

$$\frac{1}{2}\sigma\tau\sum_{n=1}^N\|\partial_\tau\boldsymbol{e}_c^n+\nabla\varepsilon_c^n\|_0^2+\upsilon\|\nabla\times\boldsymbol{e}_c^N\|_0^2\le C\tau.\tag{3.11}$$

So far we have completed the proof of the theorem.

3.2. Error bounds for magnetic field

We now proceed to obtain error estimates for the fully discrete magnetic field $\boldsymbol{H}_h^n(=\upsilon\nabla\times\boldsymbol{A}_h^n)$ as an approximation of the semi-discrete solution $\boldsymbol{H}^n(=\upsilon\nabla\times\boldsymbol{A}^n)$ under stronger regularity assumptions on the continuous problem. We define and split the error of the method as follows:

$$\begin{aligned}\boldsymbol{e}^n &= \boldsymbol{A}(t_n)-\boldsymbol{A}_h^n=\boldsymbol{e}_c^n+\boldsymbol{e}_d^n,\\ \varepsilon^n &= \phi(t_n)-\phi_h^n=\varepsilon_c^n+\varepsilon_d^n,\end{aligned}$$

where the discrete errors are defined as:

$$\boldsymbol{e}_d^n=\boldsymbol{A}^n-\boldsymbol{A}_h^n,\quad \varepsilon_d^n=\phi^n-\phi_h^n.$$

We also introduce the following notation. Given $(B_h,\ \psi_h)\in X_h\times M_h$ arbitrary, we call:

$$\begin{aligned}E_n(h) &= \inf_{B_h\in X_h}\left(\|\nabla\times(\partial_\tau\boldsymbol{A}^n-\boldsymbol{B}_h)\|_0+\|\partial_\tau\boldsymbol{A}^n-\boldsymbol{B}_h\|_0\right)\\ &\qquad+\inf_{\psi_h\in M_h}\|\nabla\phi^n-\nabla\psi_h\|_0\\ E(h) &= \max_{1\le n\le N}E_n(h).\end{aligned}\tag{3.12}$$

Our main result of this subsection is the following:

Theorem 3.2. *Assume (H1)-(H6) hold; then there exists a constant $C > 0$ independent of τ and h such that for sufficiently small τ:*

$$\upsilon\|\nabla \times \boldsymbol{e}_d^N\|_0^2 + \frac{1}{2}\sigma\tau \sum_{n=1}^{N} \|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n\|_0^2 \le C(h^2 + E(h)^2). \tag{3.13}$$

Proof. Subtracting (2.8) and (2.9) from (2.3) and (2.4) respectively, and using their respective definition, we can see that these discrete errors satisfy the following equations, which hold for any $(\boldsymbol{B}_h,\ \psi_h) \in X_h \times M_h$:

$$(\sigma\partial_\tau \boldsymbol{e}_d^n,\ \boldsymbol{B}_h) + \upsilon(\nabla \times \boldsymbol{e}_d^n,\ \nabla \times \boldsymbol{B}_h) + (\sigma\nabla\varepsilon_d^{n-1},\ \boldsymbol{B}_h) = 0 \tag{3.14}$$

and

$$(\sigma\partial_\tau \boldsymbol{e}_d^n,\ \nabla\psi_h) + (\sigma\nabla\varepsilon_d^n,\ \nabla\psi_h) = 0. \tag{3.15}$$

Given $(\boldsymbol{B}_h,\ \psi_h) \in X_h \times M_h$, we take $2\tau(\boldsymbol{B}_h - \partial_\tau \boldsymbol{A}_h^n,\ \psi_h - \phi_h^n)$ as test functions in (3.14) and (3.15), to get

$$\begin{aligned} &2\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n\|_0^2 + 2\tau(\sigma\nabla\varepsilon_d^{n-1},\ \partial_\tau \boldsymbol{e}_d^n) + \upsilon(\|\nabla \times \boldsymbol{e}_d^n\|_0^2 + \|\nabla \times \delta_t \boldsymbol{e}_d^n\|_0^2 - \|\nabla \times \boldsymbol{e}_d^{n-1}\|_0^2) \\ &= 2\tau(\sigma(\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^{n-1}),\ \partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h) + 2\tau(\upsilon\nabla \times \boldsymbol{e}_d^n,\ \nabla \times (\partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h)) \end{aligned} \tag{3.16}$$

and

$$2\tau(\sigma\partial_\tau \boldsymbol{e}_d^n,\ \nabla\varepsilon_d^n) + 2\sigma\tau\|\nabla\varepsilon_d^n\|_0^2 = 2\tau(\sigma(\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n),\ \nabla(\phi^n - \psi_h)). \tag{3.17}$$

Adding up (3.16) and (3.17) we get

$$\begin{aligned} &\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n\|_0^2 + \sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^{n-1}\|_0^2 + \sigma\tau\|\nabla\varepsilon_d^n\|_0^2 \\ &\quad -\sigma\tau\|\nabla\varepsilon_d^{n-1}\|_0^2 + \upsilon\|\nabla \times \boldsymbol{e}_d^n\|_0^2 - \upsilon\|\nabla \times \boldsymbol{e}_d^{n-1}\|_0^2 \\ &\le 2\tau(\sigma(\partial_\tau \boldsymbol{e}_d^n + \sigma\nabla\varepsilon_d^{n-1}),\ \partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h) \\ &\quad +2\tau(\upsilon\nabla \times \boldsymbol{e}_d^n,\ \nabla \times (\partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h)) \\ &\quad +2\tau(\sigma(\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n),\ \nabla(\phi^n - \psi_h)) = \sum_{i=1}^{3}(I_i). \end{aligned} \tag{3.18}$$

Using (3.12) and the inequality $2(a,\ b) \le \gamma|a|^2 + |b|^2/\gamma,\ \gamma > 0$, we have

$$\begin{aligned} (I_1) &\le \frac{1}{2}\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \sigma\nabla\varepsilon_d^{n-1}\|_0^2 + C\tau\|\partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h\|_0^2 \\ &\le \frac{1}{2}\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \sigma\nabla\varepsilon_d^{n-1}\|_0^2 + C\tau E(h)^2, \\ (I_2) &\le C_1\tau\|\nabla \times \boldsymbol{e}_d^n\|_0^2 + C_2\tau\|\nabla \times (\partial_\tau \boldsymbol{A}^n - \boldsymbol{B}_h)\|_0^2 \\ &\le C_1\tau\|\nabla \times \boldsymbol{e}_d^n\|_0^2 + C_2\tau E(h)^2, \\ (I_3) &\le \frac{1}{2}\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \sigma\nabla\varepsilon_d^n\|_0^2 + C_3\tau\|\nabla(\phi^n - \psi_h)\|_0^2 \\ &\le \frac{1}{2}\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \sigma\nabla\varepsilon_d^n\|_0^2 + C_3\tau E(h)^2, \end{aligned}$$

Using these bounds it follows from (3.18) that,

$$\begin{aligned} &\upsilon\|\nabla \times \boldsymbol{e}_d^n\|_0^2 + \sigma\tau\|\nabla\varepsilon_d^n\|_0^2 + \frac{1}{2}\sigma\tau\|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n\|_0^2 \\ &\le \upsilon\|\nabla \times \boldsymbol{e}_d^{n-1}\|_0^2 + \sigma\tau\|\nabla\varepsilon_d^{n-1}\|_0^2 + C\tau E(h)^2 + C\tau\|\nabla \times \boldsymbol{e}_d^n\|_0^2. \end{aligned} \tag{3.19}$$

After adding up from $n = 1$ to N, we obtain

$$\begin{aligned} &\upsilon\|\nabla\times \boldsymbol{e}_d^N\|_0^2 + \sigma\tau\|\nabla\varepsilon_d^N\|_0^2 + \frac{1}{2}\sigma\tau\sum_{n=1}^{N}\|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n\|_0^2 \\ &\le \upsilon\|\nabla\times \boldsymbol{e}_d^0\|_0^2 + \sigma\tau\|\nabla\varepsilon_d^0\|_0^2 + CE(h)^2 + C\tau\sum_{n=1}^{N}\|\nabla\times \boldsymbol{e}_d^n\|_0^2. \end{aligned}$$

Note that we have applied the relation $N\tau \le T$ in above inequality. Using the discrete Gronwall Lemma 2.4 and Step 1 of Algorithm 2.2, we obtain for sufficiently small τ that

$$\upsilon\|\nabla\times \boldsymbol{e}_d^N\|_0^2 + \frac{1}{2}\sigma\tau\sum_{n=0}^{N-1}\|\partial_\tau \boldsymbol{e}_d^n + \nabla\varepsilon_d^n\|_0^2 \le C(h^2 + E(h)^2). \tag{3.20}$$

So far we complete the proof of Theorem 3.2.

Remark 3.3. For equal order interpolations of degree k, the spatial error functions $E(h)$ behaves like h^k. In general, one always has $E(h) \le Ch$.

As a consequence of the previous results, we have the following so-called global error bounds:

Corollary 3.4. *Assume that the conditions of Theorem 3.1 and Theorem 3.2 hold. Assume also that, for $n = 1, \cdots, N$, $\boldsymbol{A}^n \in H^2(\Omega)^3$ and $\phi^n \in H^2(\Omega)$, and that they are uniformly bounded in these spaces. Then there exists a constant $C > 0$ independent of τ and h such that, for small enough τ:*

$$\upsilon\|\nabla\times \boldsymbol{e}^N\|_0^2 + \frac{1}{2}\sigma\tau\sum_{n=1}^{N}\|\partial_\tau \boldsymbol{e}^n + \nabla\varepsilon_d^n\|_0^2 \le C(\tau + h^2 + E(h)^2).$$

Remark 3.5. Corollary 3.4 actually gives the error estimates on the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$. In fact, noting that $\boldsymbol{E} = -\boldsymbol{A}_t - \nabla\phi$ and $\boldsymbol{H} = \upsilon\nabla\times\boldsymbol{A}$, we can write the estimate in Corollary 3.4 as

$$\upsilon\|\boldsymbol{H}^N - \boldsymbol{H}_h^N\|_0^2 + \frac{1}{2}\sigma\tau\sum_{n=1}^{N}\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0^2 \le C(\tau + h^2 + E(h)^2), \tag{3.21}$$

where

$$\boldsymbol{E}_h^n = -\partial_\tau \boldsymbol{A}_h^n - \nabla\phi_h^n, \quad \boldsymbol{H}_h^N = \upsilon\nabla\times\boldsymbol{A}_h^N.$$

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