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A LOCKING-FREE SCHEME OF NONCONFORMING RECTANGULAR FINITE ELEMENT FOR THE PLANAR ELASTICITY *1)

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Abstract

In this paper, the authors present a locking-free scheme of the lowest order nonconforming rectangle finite element method for the planar elasticity with the pure displacement boundary condition. Optimal order error estimate, uniformly for the Lamé constant $\lambda \in (0, \infty)$ is obtained.

Mathematics subject classification: 65N30, 73V05.
Key words: Locking-free, Planar elasticity, Nonconforming finite element method.

1. Introduction

For numerical solutions of the equations of linear isotropic planar elasticity , the conforming finite element method suffers a deterioration in performance as the Lamé constant $\lambda \to \infty$, i.e., as the material becomes incompressible. It is known as the phenomenon of locking (see [1], [2], [3] and [5]). By virtue of numerical analysis, the coefficient C_λ appearing in the error estimate of the conforming finite element approximation to the planar elasticity depends on the Lamé constant λ; and $C_\lambda \to \infty$ as $\lambda \to \infty$. Thus in order to overcome the phenomenon of locking, we need to construct a finite element method such that the numerical solutions of the finite element scheme converge to the true solution of the planar elasticity, as the mesh $h \to 0$, uniformly with respect to $\lambda \in (0, \infty)$.

There are some works on locking-free finite element methods for the planar elasticity. The Crouziex-Raviart element approximations to the pure displacement boundary value problem was considered in [2] and [3] by virtue of the standard finite element analysis. The pure traction boundary value problem was considered in [5] and [10] with triangular element approximations, [11] with quadrilateral element approximations and [12] with the NRQ_1 element approximations following the argument of [11] by the mixed finite element analysis. In the previous paper [9], we have considered a higher order nonconforming rectangular finite element method for the pure displacement boundary value problem of the planar elasticity. In the present paper, we derive and analyze the locking-free scheme of the lowest order rectangular finite element for the same problem. The locking-free finite element method for the pure traction boundary value problem of the planar elasticity will be considered in our forthcoming papers.

In the following section, we present the preliminary consideration of the locking-free finite element method for the planar elasticity. Then in section 3, we derive the lowest order locking-free scheme of nonconforming rectangular finite element. In section 4, optimal order of error estimate is obtained. At last, we end this paper with some numerical experiments.

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2. Preliminary

In this section, we present the general consideration of the locking-free finite element method for the planar elasticity with the pure displacement boundary condition.

Let $\Omega \subset R^2$ be a convex domain with the boundary $\partial\Omega$,

$$\begin{cases} -\mu\Delta\vec{u} - (\mu+\lambda)grad(div\vec{u}) = \vec{f} & in \quad \Omega \\ \vec{u} = 0 & on \quad \partial\Omega. \end{cases} \tag{2.1}$$

The corresponding variational problem is as follows

$$\begin{cases} \text{to find} \quad \vec{u} \in V, \text{ such that} \\ a(\vec{u},\vec{v}) = (\vec{f},\vec{v}) \quad \forall\vec{v} \in V, \end{cases} \tag{2.2}$$

where $V = (H_0^1(\Omega))^2$,

$$\begin{aligned} a(\vec{u},\vec{v}) &= \int_\Omega \{\mu\, grad\vec{u} : grad\vec{v} + (\mu+\lambda)(div\vec{u})\,(div\vec{v})\}dx \\ &\doteq \mu \int_\Omega \{gradu_1 \cdot gradv_1 + gradu_2 \cdot gradv_2\}dx \\ &\quad +(\mu+\lambda)\int_\Omega (div\vec{u})(div\vec{v})dx, \end{aligned} \tag{2.3}$$

$$(\vec{f},\vec{v}) = \int_\Omega \vec{f}\cdot\vec{v}dx, \tag{2.4}$$

and $\lambda \in (0,\infty), \mu \in [\mu_1,\mu_2], 0 < \mu_1 < \mu_2$, are the Lamé constants. Since the bilinear form $a(\cdot,\cdot)$ (2.3) is V-elliptic, there exists an unique solution of the problem (2.2). Now we consider the conforming finite element approximation to the problem (2.2). For the sake of simplicity, we assume that Ω is a convex polygon. Let $\Im_h$ be the regular triangulation of $\Omega, V_h \subset V$ be the conforming finite element space with respect to $\Im_h$, then the finite element approximation to the problem (2.2) is as follows:

$$\begin{cases} \text{to find} \quad \vec{u}_h \in V_h, \text{ such that} \\ a(\vec{u}_h,\vec{v}_h) = (\vec{f},\vec{v}_h) \quad \forall\vec{v}_h \in V_h. \end{cases} \tag{2.5}$$

The following error estimate holds

Theorem 2.1 (see[9]). *Assume that $\vec{u} \in (H^2(\Omega))^2$ and $\vec{u}_h$ are the solutions of the problems (2.2) and (2.5) respectively, then*

$$\|\vec{u}-\vec{u}_h\|_{1,\Omega} \le C_\lambda \cdot h|\vec{u}|_{2,\Omega}, \quad C_\lambda = C\sqrt{2\mu+\lambda}, \tag{2.6}$$

where $C = Const. > 0$ is independent of h and λ.

From theorem 2.1, it can be seen that the solution $\vec{u}_h$ of the conforming finite element approximation (2.5) converges to the solution $\vec{u}$ of the problem (2.2) as $h \to 0$, for each fixed λ; but we can not say any thing about the convergence of $\vec{u}_h$ when $\lambda \to \infty$. In fact, Brenner et al.[2] proved that the solution $\vec{u}_h$ of the conforming linear finite element approximation, with respect to triangulation $\Im_h$, might not converge to the solution $\vec{u}$ of the problem (2.2) when $\lambda \to \infty$. It is known as the phenomenon of locking. By the argument of [9], it can be seen that to overcome the locking, the crucial point is to construct a finite element space V_h, and an interpolation operator $\Pi_h : (H^1(\Omega))^2 \to V_h$, such that the following commutativity property holds

$$div\Pi_h\vec{u} = \gamma_h div\vec{u}, \tag{2.7}$$

where $\gamma_h : L^2(\Omega) \to W_h$ is another operator, and W_h is a piecewise polynomial space with lower degree than those in V_h; and the following error estimate is required:

$$\|div\vec{u} - \gamma_h(div\vec{u})\|_{0,\Omega} \le Ch|div\vec{u}|_{1,\Omega}. \tag{2.8}$$

Then by the regularity of the problem (2.2)(see [2][3]), we have:

$$\|\vec{u}\|_{2,\Omega} + \lambda|div\vec{u}|_{1,\Omega} \leq C\|\vec{f}\|_{0,\Omega}; \tag{2.9}$$

and it is hopeful to obtain optimal error estimate uniform with respect to $\lambda \in (0, \infty)$:

$$\|\vec{u} - \vec{u}_h\|_{1,\Omega} \leq ch\|\vec{f}\|_{0,\Omega}, \tag{2.10}$$

where $C = Const. > 0$ independent of h and λ.

However in order to satisfy the conditions (2.7) and (2.8), the finite element space V_h must be nonconforming in general. Let $V_h \subset (L^2(\Omega))^2$ be a nonconforming finite element space with respect to the regular triangulation $\Im_h$, and

$$a_h(\vec{u}_h, \vec{v}_h) = \sum_T \int_T \{\mu\ grad\vec{u}_h : grad\vec{v}_h + (\mu + \lambda) div\vec{u}_h \cdot div\vec{v}_h\} dx. \tag{2.11}$$

Assume that $\|\vec{v}_h\|_h = a_h(\vec{v}_h, \vec{v}_h)^{\frac{1}{2}}$ is a norm on V_h. The nonconforming finite element approximation of the problem (2.2) is

$$\begin{cases} \text{to find} \quad \vec{u}_h \in V_h, \text{ such that} \\ a_h(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h) \quad \forall v_h \in V_h, \end{cases} \tag{2.12}$$

and the following abstract error estimate holds (see [4]):

Theorem 2.2. (the second Strang lemma)

Let $\vec{u}$ and $\vec{u}_h$ be the solutions of the problem (2.2) and (2.12) respectively, then

$$\|\vec{u} - \vec{u}_h\|_h \leq C\left\{ \inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_h + \sup_{0 \neq \vec{w}_h \in V_h} \frac{|a_h(\vec{u}, \vec{w}_h) - (\vec{f}, \vec{w}_h)|}{\|\vec{w}_h\|_h} \right\} \tag{2.13}$$

where $C = Const. > 0$ independent of h.

3. The Lowest Order Nonconforming Rectangular Finite Element

In this section, we derive a locking-free scheme by constructing a kind of nonconforming rectangular finite element based on the analysis in section 2.

Let Ω be a rectangle, $\Im_h$ be a regular subdivision of Ω, and $T \in \Im_h$ be a rectangle. First, we find that the simplest choice of the operator γ_h satisfying the estimate (2.8) is

$$\gamma_T(div\vec{v}) = \frac{1}{|T|} \int_T div\vec{v} dxdy \tag{3.1}$$

with $|T| = \int_T 1dxdy$, and $\gamma_h : L^2(\Omega) \to W_h$— the piecewise constant space with respect to $\Im_h$, such that

$$\gamma_h w|_T = \gamma_T w \qquad \forall\, w \in L^2(\Omega) \tag{3.2}$$

Secondly, we will find such operator Π_h(and then the finite element space V_h), that the condition (2.7) is satified. For any given $T \in \Im_h$, denote A_i, $l_i = \overline{A_i A_{i+1}}(i \bmod 4)$, $1 \leq i \leq 4$, $(x_0, y_0) = \frac{1}{4}\sum_{i=1}^4 A_i$ and $|l_1| = |l_3| = 2h_1$, $|l_2| = |l_4| = 2h_2$(see Fig 3.1) as its four vertices, four edges, center and lengths of edges respectively.

Let

$$\Pi_h\vec{v} = \vec{a}_0 + \vec{a}_1(x - x_0) + \vec{a}_2(y - y_0) + \begin{pmatrix} c_1(y - y_0)^2 \\ c_2(x - x_0)^2 \end{pmatrix}, \tag{3.3}$$

then we have the following result

Lemma 3.1. *Let*

$$\begin{cases} \vec{a}_1 = \dfrac{1}{|T|} \displaystyle\int_T \frac{\partial \vec{v}}{\partial x} dxdy, \\ \vec{a}_2 = \dfrac{1}{|T|} \displaystyle\int_T \frac{\partial \vec{v}}{\partial y} dxdy, \end{cases} \tag{3.4}$$

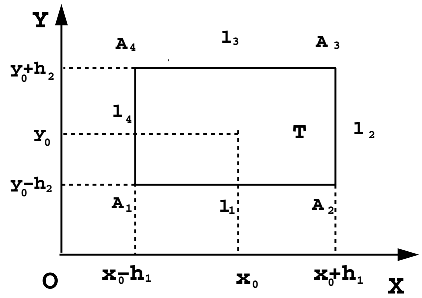

Fig 3.1

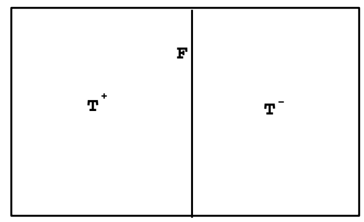

Fig 3.2

then

$$div\Pi_T\vec{v} = \gamma_T(div\vec{v}). \tag{3.5}$$

Proof. From (3.3), it can be seen that

$$div\Pi_T\vec{v} = \vec{a}_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \vec{a}_2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.6}$$

And from (3.1), by Green's formula we have

$$\begin{aligned} \gamma_T(div\vec{v}) &= \frac{1}{|T|}\int_T div\vec{v}dxdy = \frac{1}{|T|}\sum_{i=1}^{4}\int_{l_i}\vec{v}\cdot\vec{\nu}_i ds \\ &= \frac{1}{|T|}\left\{\left(\int_{l_2} - \int_{l_4}\right)\vec{v}dy\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\int_{l_3} - \int_{l_1}\right)\vec{v}dx\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \\ &= \frac{1}{|T|}\left\{\int_T \frac{\partial\vec{v}}{\partial x}dxdy\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_T \frac{\partial\vec{v}}{\partial y}dxdy\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}. \end{aligned} \tag{3.7}$$

In view of (3.6) and (3.7), the lemma is proved.

Finally, we will find the nonconforming finite element space V_h, such that for any $v_h \in V_h$, the integral of the jump of which across the common edge of any two adjacent elements T^+ and T^- vanishes. In the analysis of nonconforming finite element method (see [6], [8]), this property is very important to ensure the convegence of nonconforming finite element approximation. We have the following lemma.

Lemma 3.2. *Assume that the coefficients $\vec{a}_1$ and $\vec{a}_2$ satisfy (3.4), then the equation $\int_{l_2}\Pi_T\vec{v}dy = \int_{l_2}\vec{v}dy$ is equivalent to the equation $\int_{l_4}\Pi_T\vec{v}dy = \int_{l_4}\vec{v}dy$, and the equation $\int_{l_3}\Pi_T\vec{v}dy = \int_{l_3}\vec{v}dy$ is equivalent to the equation $\int_{l_1}\Pi_T\vec{v}dy = \int_{l_1}\vec{v}dy$.*

Proof. In fact, if $\int_{l_2}\Pi_T\vec{v}dy = \int_{l_2}\vec{v}dy$, i.e.,

$$\int_{l_2}\Pi_T\vec{v}dy = 2h_2\vec{a}_0 + \frac{2h_1h_2}{|T|}\int_T\frac{\partial\vec{v}}{\partial x}dxdy + \begin{pmatrix} c_1\cdot\frac{2}{3}h_2^3 \\ c_2\cdot 2h_1^2h_2 \end{pmatrix} = \int_{l_2}\vec{v}dy, \tag{3.8}$$

then

$$\begin{aligned} \int_{l_4}\Pi_T\vec{v}dy &= 2h_2\vec{a}_0 - \frac{2h_1h_2}{|T|}\int_T\frac{\partial\vec{v}}{\partial x}dxdy + \begin{pmatrix} c_1\cdot\frac{2}{3}h_2^3 \\ c_2\cdot 2h_1^2h_2 \end{pmatrix} \\ &= \int_{l_2}\vec{v}dy - \frac{4h_1h_2}{|T|}\int_T\frac{\partial\vec{v}}{\partial x}dxdy = \int_{l_2}\vec{v}dy - \int_T\frac{\partial\vec{v}}{\partial x}dxdy \\ &= \int_{l_4}\vec{v}dy, \end{aligned} \tag{3.9}$$

since by Green's formula, $\int_T \frac{\partial \vec{v}}{\partial x} dxdy = \int_{l_2} \vec{v}dy - \int_{l_4} \vec{v}dy$. And the second equivalence can be proved in the same way. The proof is completed.

Thus from the following equations:

$$\begin{cases} \int_{l_2} \Pi_T \vec{v} dy = 2h_2 \vec{a}_0 + \frac{2h_1h_2}{|T|} \int_T \frac{\partial \vec{v}}{\partial x} dxdy + \begin{pmatrix} \frac{2}{3}h_2^3 \cdot c_1 \\ 2h_1^2 h_2 \cdot c_2 \end{pmatrix} = \int_{l_2} \vec{v}dy, \\ \int_{l_3} \Pi_T \vec{v} dy = 2h_1 \vec{a}_0 + \frac{2h_1h_2}{|T|} \int_T \frac{\partial \vec{v}}{\partial y} dxdy + \begin{pmatrix} 2h_1h_2^2 \cdot c_2 \\ \frac{2}{3}h_1h_2^2 \cdot c_1 \end{pmatrix} = \int_{l_3} \vec{v}dx, \end{cases} \tag{3.10}$$

we can determine the coefficients $\vec{a}_0 = (a_{01}, a_{02})^T, c_1$ and c_2 as follows:

$$\begin{cases} a_{01} = \frac{1}{2h_2} \left\{ \frac{3}{4} \left(\int_{l_2} + \int_{l_4} \right) v_1 dy - \frac{h_2}{4h_1} \left(\int_{l_3} + \int_{l_1} \right) v_1 dx \right\}, \\ a_{02} = \frac{1}{2h_2} \left\{ -\frac{1}{4} \left(\int_{l_2} + \int_{l_4} \right) v_2 dy + \frac{3h_2}{4h_1} \left(\int_{l_3} + \int_{l_1} \right) v_2 dx \right\}; \end{cases} \tag{3.11}$$

and

$$\begin{cases} c_1 = \frac{3}{2|T|h_2^2} \left\{ h_2 \left(\int_{l_3} + \int_{l_1} \right) v_1 dx - h_1 \left(\int_{l_2} + \int_{l_4} \right) v_1 dy \right\}, \\ c_2 = \frac{3}{2|T|h_1^2} \left\{ -h_2 \left(\int_{l_3} + \int_{l_1} \right) v_2 dx + h_1 \left(\int_{l_2} + \int_{l_4} \right) v_2 dy \right\}. \end{cases} \tag{3.12}$$

Substituting (3.4), (3.11) and (3.12) into (3.3), after some arrangements we have $\Pi_T \vec{v} = (\Pi_{1,T} v_1, \Pi_{2,T} v_2)^t$ defined as follows: Let

$$\xi = \frac{(x - x_0)}{h_1}, \qquad \eta = \frac{(y - y_0)}{h_2}, \tag{3.13}$$

then

$$\Pi_{1,T} v_1 = \sum_{i=1}^{4} \frac{1}{|l_i|} \int_{l_i} v_1 ds \cdot p_{1i}(\xi, \eta), \tag{3.14}$$

where

$$\begin{cases} p_{11}(\xi,\eta) = \frac{1}{4}(-1 - 2\eta + 3\eta^2), \\ p_{12}(\xi,\eta) = \frac{1}{4}(3 + 2\xi - 3\eta^2), \\ p_{13}(\xi,\eta) = \frac{1}{4}(-1 + 2\eta + 3\eta^2), \\ p_{14}(\xi,\eta) = \frac{1}{4}(3 - 2\xi - 3\eta^2), \end{cases} \tag{3.15}$$

and

$$\Pi_{2,T} v_2 = \sum_{i=1}^{4} \frac{1}{|l_i|} \int_{l_i} v_2 ds \cdot p_{2i}(\xi, \eta), \tag{3.16}$$

where

$$\begin{cases} p_{21}(\xi,\eta) = \frac{1}{4}(3 - 2\eta - 3\xi^2), \\ p_{22}(\xi,\eta) = \frac{1}{4}(-1 + 2\xi + 3\xi^2), \\ p_{23}(\xi,\eta) = \frac{1}{4}(3 + 2\eta - 3\xi^2), \\ p_{24}(\xi,\eta) = \frac{1}{4}(-1 - 2\xi + 3\xi^2). \end{cases} \tag{3.17}$$

Lemma 3.3. *The interpolations (3.14) and (3.16) are preserved in $P_1(T)$, i.e.,*

$$\Pi_{i,T} v_i = v_i \qquad \forall v_i \in P_1(T), \qquad i = 1, 2. \tag{3.18}$$

Proof: The equations (3.18) can be proved for $v_i = 1$, $v_i = \xi = \frac{(x-x_0)}{h_1}$ and $v_i = \eta = \frac{(y-y_0)}{h_2}$ by straightforward calculations.

Now we can set the nonconforming finite element space V_h as follows(see Fig 3.2):

$$V_h = \Big\{ \ \vec{v}_h : \quad \vec{v}_h \in P_T, \quad \int_F \vec{v}_h^+ ds = \int_F \vec{v}_h^- ds, \ \forall F \subset \partial T^+ \cap \partial T^-, \\ \text{and} \quad \int_F \vec{v}_h ds = 0, \forall F \subset \partial\Omega \Big\} \ . \tag{3.19}$$

where $\vec{v}_h^+ = \vec{v}_h|_{T^+}, \vec{v}_h^- = \vec{v}_h|_{T^-}$ and $P_T = \begin{pmatrix} P_{1T} \\ P_{2T} \end{pmatrix}, P_{1T} = \text{span}\{p_{1i}\}_{i=1}^4$ and $P_{2T} = \text{span}\{p_{2i}\}_{i=1}^4$.
Let $\Pi_h : (H^1(\Omega))^2 \to V_h$ be defined as follows: for any given $\vec{v} \in (H^1(\Omega))^2$,

$$\Pi_h \vec{v}|_T = \Pi_T \vec{v}, \qquad \forall T \in \Im_h. \tag{3.20}$$

Lemma 3.4.

$$\| \cdot \|_h = \sqrt{a_h(\cdot,\cdot)}. \tag{3.21}$$

is the norm on the space V_h.

Proof. It is sufficient to prove that $\forall \ \vec{v}_h \in V_h$, if $\|\vec{v}_h\|_h = 0$ then $\vec{v}_h = 0$. In fact, if $\|\vec{v}_h\|_h = 0, i.e., a_h(\vec{v}_h, \vec{v}_h) = 0$, then by the definition (2.14), we have

$$|\vec{v}_h|_{1,T} = 0 \qquad \forall T \in \Im_h,$$

which means that

$$\vec{v}_h|_T = \vec{C}_T\text{— Constant vector depedent on} \quad T, \ \forall \, T \ \in \Im_h.$$

Since the mean values of integrals of $\vec{v}_h$ on the common edge F of two adjacent elements T^+ and T^- are the same, then it can be seen that

$$\vec{v}_h = \vec{C}\text{— Constant vector on} \quad \Omega,$$

from which and since the integral of $\vec{v}_h$ vanishes on each edge $F \subset \partial\Omega$, we have $\vec{v}_h = 0$. The proof is completed.

4. A Locking-free Finite Element Scheme

We consider the following finite element approximation to the problem (2.2):

$$\begin{cases} \text{Find} \quad \vec{u}_h \in V_h, \text{such that} \\ a_h(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h) \quad \forall v_h \in V_h \end{cases} \tag{4.1}$$

where V_h is defined by (3.19).

We now present the error estimate of (4.1) which is uniform with respect to $\lambda \in (0, \infty)$. First, we have the following lemmas.

Lemma 4.1.

$$\Pi_h \vec{v} \in V_h \qquad \forall \vec{v} \in (H^1(\Omega))^2. \tag{4.2}$$

Proof. From lemma 3.2 and the relations (3.10), it can be seen that $\forall F = \partial T^+ \cap \partial T^-$ (c.f.Fig 3.2)

$$\int_F \Pi_{T^+} \vec{v} ds = \int_F \Pi_{T^-} \vec{v} ds = \int_F \vec{v} ds,$$

and $\forall F \cap \partial\Omega$,

$$\int_F \Pi_T \vec{v} ds = \int_F \vec{v} ds = 0,$$

then the proof is completed.

Lemma 4.2. *The following interpolation error estimate holds:*

$$\|\vec{v} - \Pi \vec{v}\|_{1,T} \le Ch|\vec{v}|_{2,T} \qquad \forall \vec{v} \in (H^2(T))^2, \ T \in \Im_h. \tag{4.3}$$

where $C = Const. > 0$ is independent of h.

Proof. By lemma 3.3 and the arguement for the affine equivalent finite element method (see [4]), the lemma is proved.

We have the following error estimate:

Theorem 4.3. *Assume that $\vec{f} \in (L^2(\Omega))^2$, the solution of the problem (2.2) $\vec{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^2$, and $\vec{u}_h$ is the solution of the finite element approximation (4.1), then the following error estimate holds*

$$\|\vec{u} - \vec{u}_h\|_h \leq Ch\|\vec{f}\|_{0,\Omega}, \tag{4.4}$$

where $C = Const. > 0$ is independent of h and λ.

Proof. By the second Strang lemma (Theorem 2.2), it is sufficient to estimate the approximate error (the first term on the right hand side of (2.13)) and the nonconforming finite element error (the second term on the right hand side of (2.13)). First, the approximate error can be estimated as follows:

By lemma 3.1 and lemma 4.2, we have

$$\begin{aligned} \inf_{\vec{v}_h \in V_h} \|\vec{u} - \vec{v}_h\|_h^2 &\leq \|\vec{u} - \Pi_h \vec{u}\|_h^2 = a_h(\vec{u} - \Pi_h \vec{u}, \vec{u} - \Pi_h \vec{u}) \\ &= \mu \sum_T |\vec{u} - \Pi_T \vec{u}|_{1,T}^2 + (\mu + \lambda) \sum_T \|div\vec{u} - div\Pi_T \vec{u}\|_{0,T}^2 \\ &= \mu \sum_T |\vec{u} - \Pi_T \vec{u}|_{1,T}^2 + (\mu + \lambda) \sum_T \|div\vec{u} - \gamma_T(div\vec{u})\|_{0,T}^2 \\ &\leq \mu h^2 |\vec{u}|_{2,\Omega}^2 + (\mu + \lambda) h^2 |div\vec{u}|_{1,\Omega}^2 \\ &\leq Ch^2(|\vec{u}|_{2,\Omega}^2 + \lambda |div\vec{u}|_{1,\Omega}^2) \leq Ch^2 \|\vec{f}\|_{0,\Omega}^2, \end{aligned} \tag{4.5}$$

in the last inequality, we have used the regularity of $\vec{u}$ (2.9).

Next, the nonconforming finite element error can be estimated as follows: Using Green's formula and considering that $\vec{u}$ is the solution of the problem (2.2), we have

$$\begin{aligned} E_h(\vec{u}, \vec{w}_h) &= a_h(\vec{u}, \vec{w}_h) - (\vec{f}, \vec{w}_h) \\ &= \sum_T \int_T \{\mu grad\vec{u} : grad\vec{w}_h + (\mu + \lambda) div\vec{u} \cdot div\vec{w}_h\} dx - \int_\Omega \vec{f} \cdot \vec{w}_h dx \\ &= -\int_\Omega \{\mu \Delta \vec{u} + (\mu + \lambda) grad(div\vec{u})\} \vec{w}_h dx - \int_\Omega \vec{f} \cdot \vec{w}_h dx \\ &\quad + \sum_T \int_{\partial T} \{\mu \partial_\nu \vec{u} \cdot \vec{w}_h + (\mu + \lambda) div\vec{u} \cdot \vec{w}_h \cdot \vec{\nu}\} ds \\ &= \mu \sum_T \int_{\partial T} \partial_\nu \vec{u} \cdot \vec{w}_h ds + (\mu + \lambda) \sum_T \int_{\partial T} div\vec{u} \cdot \vec{w}_h \cdot \vec{\nu} ds. \end{aligned} \tag{4.6}$$

Denote

$$P_0^T(\vec{w}) = \frac{1}{|T|} \int_T \vec{w} dx. \tag{4.7}$$

Since on the common edge $F = \partial T^+ \cap \partial T^-$ of adjacent elements T^+ and T^-, the following relation holds: $\forall \vec{w}_h \in V_h$,

$$\int_F \vec{w}_h^+ ds = \int_F \vec{w}_h^- ds, \tag{4.8}$$

where

$$\vec{w}_h^+ = \vec{w}_h|_{T^+}, \qquad \vec{w}_h^- = \vec{w}_h|_{T^-}, \tag{4.9}$$

and on the edge $F \subset \partial\Omega$,

$$\int_F \vec{w}_h ds = 0, \tag{4.10}$$

by the error estimate of nonconforming finite element (see [8],[6]), we have

$$\begin{aligned} |\sum_T \int_{\partial T} \partial_\nu \vec{u} \cdot \vec{w}_h ds| &\leq \sum_T \sum_{i=1}^{2} \|\partial_i \vec{u} - p_0^T(\partial_i \vec{u})\|_{0,\partial T} \cdot \|\vec{w}_h - p_0^T(\vec{w}_h)\|_{0,\partial T} \\ &\leq Ch|\vec{u}|_{2.\Omega} \cdot \|\vec{w}_h\|_h, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} |\sum_T \int_{\partial T} div\vec{u} \cdot \vec{w}_h \cdot \vec{\nu} ds| &\leq \sum_T \|div\vec{u} - p_0^T(div\vec{u})\|_{0,\partial T} \cdot \|\vec{w}_h - p_0^T(\vec{w}_h)\|_{0,\partial T} \\ &\leq Ch|div\vec{u}|_{1,\Omega} \cdot \|\vec{w}_h\|_h. \end{aligned} \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.6), we have

$$|E_h(\vec{u}, \vec{w}_h)| \leq Ch\{|\vec{u}|_{2,\Omega} + \lambda|div\vec{u}|_{1,\Omega}\} \cdot \|\vec{w}_h\|_h. \tag{4.13}$$

By (4.13) and the regularity of the solution $\vec{u}$ (2.9), it can be seen that

$$\sup_{0 \neq \vec{w}_h \in V_h} \frac{|E_h(\vec{u}, \vec{w}_h)|}{\|\vec{w}_h\|_h} \leq Ch\|\vec{f}\|_{0,\Omega}, \tag{4.14}$$

where $C = Const. > 0$ is independent of h and λ. The proof is completed.

5. Numerical Experiments

In order to check the convergence of the finite element scheme described in this paper as $\lambda \to \infty$, we give two numerical examples in the rest of this section.

5.1. A cantilever beam with a parabolic end load

This example appears in R. Kouhia and R. Stenberg's paper[10]. It is a standard test problem of a cantilever beam subjected to a parabolically end shear. For the sake of convenience in notation, we use the Poisson ratio ν. It relates to μ and λ by the following identities:

$$\mu = \frac{1}{2(1+\nu)}, \qquad \lambda = \frac{2\mu\nu}{1-2\nu}. \tag{5.1}$$

Let $\Omega = [0, L] \times [-c, c]$ and $\Gamma_0 = L \times [-c, c]$. A true solution of (2.1) is

$$u_1(x,y) = -\tfrac{P(1-\nu^2)y}{4c^3E}\left\{3x(2L-x) + \tfrac{(2-\nu)(y^2-c^2)}{1-\nu}\right\}. \tag{5.2}$$

$$u_2(x,y) = \tfrac{P(1-\nu^2)}{4c^3E}\left\{(L-x)^3 - L^3 + \tfrac{(4+\nu)c^2x}{1-\nu} + 3L^2x + \tfrac{3\nu(L-x)y^2}{1-\nu}\right\}, \tag{5.3}$$

with $P = -1$, $L = 16$, $c = 2$, and $E = 1$. On a part of the boundary, $\vec{u}$ satisfies the following normal boundary traction condition ($\mathbf{I}$ is the identity tensor):

$$\{\mu[\nabla\vec{u} + (\nabla\vec{u})^t] + \lambda \text{div}\vec{u}\,\mathbf{I}\} \cdot \vec{n} = \vec{g} := (0, 0.75P(c^2 - y^2)/c^3)^t, \quad (x,y) \in \Gamma_0. \tag{5.4}$$

We obtain the nonhomogeneous displacement boundary conditions for (2.1) by virtue of (5.2) and (5.3). Denote h_x and h_y as the mesh length along $x-$ and $y-$direction respectively. [10] solves the problem by the traction condition (5.4), then obtains the discrete displacement on Γ_0. We use the displacement boundary conditions to obtain the discrete boundary traction $\vec{g}_h$ and compare it with $\vec{g}$. In Table 5.1 and 5.2, we present error information in the energy norm and L^2 norm on different meshes, for different ν.

Table 5.1. $\nu = 0.3$

h_x	h_y	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$	$\frac{\|\vec{u}-\vec{u}_h\|_h}{\|\vec{u}\|_h}$	$\frac{\|\vec{g}-\vec{g}_h\|_{0,\Gamma_0}}{\|\vec{g}\|_{0,\Gamma_0}}$
4.0	2.0	0.008949	0.097070	1.354426
2.0	1.0	0.002241	0.034419	0.654926
1.0	1/2	0.000560	0.024350	0.341142
1/2	1/4	0.000140	0.012176	0.178884
1/4	1/8	0.000035	0.006088	0.092761

Table 5.2. $\nu = 0.49999$

h_x	h_y	$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}}$	$\frac{\|\vec{u}-\vec{u}_h\|_h}{\|\vec{u}\|_h}$	$\frac{\|\vec{g}-\vec{g}_h\|_{0,\Gamma_0}}{\|\vec{g}\|_{0,\Gamma_0}}$
4.0	2.0	0.009743	0.096717	1.201420
2.0	1.0	0.002433	0.048420	0.622112
1.0	1/2	0.000608	0.024205	0.327500
1/2	1/4	0.000152	0.012100	0.173906
1/4	1/8	0.000038	0.006049	0.091157

Error analyse. From Table 5.1 and 5.2 we can see that, each element is refined iteratively by bisecting its four edges. The edge lengths, relative errors in L^2-norm and in the energy norm, and relative boundary traction errors are reduced respectively as follows:

$$h_x = 4 \times 2^{-n}, \qquad h_y = 2 \times 2^{-n},$$

$$\frac{\|\vec{u}-\vec{u}_h\|_{0,\Omega}}{\|\vec{u}\|_{0,\Omega}} \approx 0.008949 \times 2^{-2n}, \qquad \frac{\|\vec{u}-\vec{u}_h\|_h}{\|\vec{u}\|_h} \approx 0.09707 \times 2^{-n},$$

$$\frac{\|\vec{g}-\vec{g}_h\|_{0,\Gamma_0}}{\|\vec{g}\|_{0,\Gamma_0}} \approx 1.20142 \times 2^{-n}, \quad n = 0, 1, \cdots, 4.$$

Thus we can conclude that, for this example, error reductions behave uniformly with respect to $\lambda \to +\infty$ as follows:

$$\|\vec{u}-\vec{u}_h\|_{0,\Omega} \approx O(h^2), \quad \|\vec{u}-\vec{u}_h\|_h \approx O(h), \quad \|\vec{g}-\vec{g}_h\|_{0,\Gamma_0} \approx O(h).$$

Thus these results coincide with our convergence result (inequality (4.4)) very well.

5.2. Fixed boundary problem

The second is mathematical example. We give a more complicated righthand side

$$\vec{f} = (e^{4(x+y)}, e^{4(x-y)}), \tag{5.5}$$

and solve (2.1) with homogeneous boundary conditions.

The domain $\Omega = [0,1] \times [0,1]$ is divided into the combination of uniform squares. The length of any edge of each element is h. We list error information in Table 5.3, 5.4, and 5.5 for different λ.

Table 5.3. $\lambda = 10$

h_1	h_2	$\|\vec{u}_{h_1}\|_{0,\Omega}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{0,\Omega}}{\|\vec{u}_{h_1}\|_{0,\Omega}}$	$\|\vec{u}_{h_1}\|_{h_1}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{h_1}}{\|\vec{u}_{h_1}\|_{h_1}}$
0.1	0.2	1.4333304	0.13081827	14.8139829	0.36546064
0.05	0.1	1.4137843	0.0375355	14.8315507	0.19148415
0.01	0.02	1.4071949	0.00164618	14.8323388	0.0403195

Table 5.4. $\lambda = 10^5$

h_1	h_2	$\|\vec{u}_{h_1}\|_{0,\Omega}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{0,\Omega}}{\|\vec{u}_{h_1}\|_{0,\Omega}}$	$\|\vec{u}_{h_1}\|_{h_1}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{h_1}}{\|\vec{u}_{h_1}\|_{h_1}}$
0.1	0.2	1.1153696	0.15905983	9.7854903	0.32077705
0.05	0.1	1.0799019	0.045995382	9.5561538	0.15702522
0.01	0.02	1.0655821	0.00194632	9.4691434	0.03115786

Table 5.5. $\lambda = 10^{10}$

h_1	h_2	$\|\vec{u}_{h_1}\|_{0,\Omega}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{0,\Omega}}{\|\vec{u}_{h_1}\|_{0,\Omega}}$	$\|\vec{u}_{h_1}\|_{h_1}$	$\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{h_1}}{\|\vec{u}_{h_1}\|_{h_1}}$
0.1	0.2	1.1153765	0.15906064	9.7846438	0.32076529
0.05	0.1	1.0799063	0.04599705	9.55526	0.15701656
0.01	0.02	1.0655821	0.00410217	9.4469238	0.03134559

It should be noted that since there is no explicit solution of the problem with given $\vec{f}$ in (5.5), we compare the numerical solutions over two different meshes.

From Table 5.3-5.5 (for $\lambda = 10, 10^5$, and 10^{10}), we can see that, all relative L^2-errors $\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{0,\Omega}}{\|\vec{u}_{h_1}\|_{0,\Omega}}$ and energy-errors $\frac{\|\vec{u}_{h_1}-\vec{u}_{h_2}\|_{h_1}}{\|\vec{u}_{h_1}\|_{h_1}}$ tend to zero, as the mesh size $h \to 0$, **uniformly** with respect to $\lambda \in (0, \infty)$. This means that our scheme is locking-free, provided its convergence which is proved in Theorem 4.3.

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