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A TRULY GLOBALLY CONVERGENT FEASIBLE NEWTON-TYPE METHOD FOR MIXED COMPLEMENTARITY PROBLEMS [*1)]

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Abstract

Typical solution methods for solving mixed complementarity problems either generate feasible iterates but have to solve relatively complicated subproblems such as quadratic programs or linear complementarity problems, or (those methods) have relatively simple subproblems such as system of linear equations but possibly generate infeasible iterates. In this paper, we propose a new Newton-type method for solving monotone mixed complementarity problems, which ensures to generate feasible iterates, and only has to solve a system of well-conditioned linear equations with reduced dimension per iteration. Without any regularity assumption, we prove that the whole sequence of iterates converges to a solution of the problem (truly globally convergent). Furthermore, under suitable conditions, the local superlinear rate of convergence is also established.

Mathematics subject classification: 65H10.
Key words: Mixed complementarity problems, Newton-type methods, Global convergence, Superlinear convergence.

1. Introduction

We consider the mixed complementarity problem, MCP for simplicity: Find a vector $x^* \in [l,u]$, such that

$$\begin{aligned} x_i^* = l_i \quad &\Rightarrow \quad F_i(x^*) \geq 0, \\ x_i^* \in (l_i, u_i) \quad &\Rightarrow \quad F_i(x^*) = 0, \\ x_i^* = u_i \quad &\Rightarrow \quad F_i(x^*) \leq 0, \end{aligned} \tag{1}$$

where $l_i \in R \cup \{-\infty\}$ and $u_i \in R \cup \{+\infty\}$ are given lower and upper bounds with $l_i < u_i$, $i = 1, \cdots, n$, F is a continuously differentiable mapping from the rectangle $[l,u]$ to R^n. The MCP (1) can also be written as the closed form of the variational inequality problem of finding a vector $x^* \in [l,u]$, such that

$$(x - x^*)^\top F(x^*) \geq 0, \quad \forall x \in [l,u]. \tag{2}$$

When $l_i = 0$ and $u_i = +\infty$ for all $i = 1, \cdots, n$, MCP reduces to the nonlinear complementarity problem of finding a vector $x^* \in R^n$, such that

$$x^* \geq 0, \; F(x^*) \geq 0, \quad {x^*}^\top F(x^*) = 0,$$

and if $l_i = -\infty$ and $u_i = +\infty$ for all $i = 1, \cdots, n$, MCP reduces to the nonlinear system of equations

$$F(x) = 0.$$

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The mixed complementarity problem and the nonlinear complementarity problem have a number of important applications in operations research, engineering problems and economics equilibrium problems, see the survey papers [6, 8, 10] for detailed examples and the references therein.

There are many iterative methods for solving the mixed complementarity problem [2, 7, 11, 12, 13, 14, 15, 19, 30]. While the projection-type methods [11, 12, 13, 14, 15] are attractive for its simplicity and global convergence, the most successful and widely used are Newton-type methods. A class of Newton-type methods are Josephy-Newton methods based on solving a series of linear complementarity problems [3, 10, 17, 30]. Given a point x^k, the Josephy-Newton methods generate the next iterate x^{k+1} by solving the following linear mixed complementarity problem

$$\text{Find } x \in [l,u], \text{ such that } \quad F^k(x)^\top(z-x) \geq 0, \ \forall z \in [l,u], \tag{3}$$

where $F^k(\cdot)$ is the first order approximation of F at x^k,

$$F^k(x) = F(x^k) + F^{'}(x^k)(x-x^k) \tag{4}$$

assuming $F(\cdot)$ is differentiable.

Another class of Newton-type method are equation-based reformulation for MCP [18, 22, 32, 33, 36]. At each iteration, this type of methods solves a linear system of equations

$$H_k d^k = -\Phi(x^k) \tag{5}$$

to find the search direction d^k, where Φ is a semismooth function with the property that

$$x \text{ solves MCP (1)} \Longleftrightarrow \Phi(x) = 0,$$

and $H_k \in \partial\Phi(x^k)$ is an element of the generalized Jacobian of Φ at x^k in the Clarke's sense [4]. Note that at each iteration, the equation-based Newton-type methods solve a linear system of equations (5), which is structurally easier to solve than linear mixed complementarity problem (3). Because of the extreme efficiency in practice, this class of Newton-type methods, especially those methods based on Fisher-Burmeister function are recently studied extensively [7, 18, 21, 22, 36, 38]. However, the generated sequence $\{x^k\}$ is not necessarily contained in the feasible set $[l,u]$. At the same time, the feasibility issue for MCP (1) is always important because some real-life applications such as in engineering design and economics [16, 29] require the data only defined in the feasible region. Hence, as mentioned in [21], "it would be extremely nice to have an algorithm that, on the one hand, generates only feasible iterates and, on the other hand, has to solve only simple subproblems". Nevertheless, there are currently only a few methods with these two properties available [21, 24, 38, 39].

A common difficulty with using the Newton-type method by solving (3) or (5) is that, while possessing fast local convergence property, there are serious problems with ensuring global convergence. To enlarge the domain of convergence of the Newton method, many globalization strategy for (3), (5) are proposed. The most natural globalization strategy is a line search procedure in the obtained Newton direction aimed at decreasing the value of some valid merit functions. However, these strategies can only ensure that the generated sequence converges to a stationary point of the merit function, which is a solution of MCP under some restrictive assumptions. These assumptions imply the boundedness of level sets of the merit function and possible uniqueness of the solution. Moreover, some of the merit functions, such as those based on the natural residual [27] and the normal map [34] are nondifferentiable, which make the line search difficult to implement. The differentiable merit functions, such as the gap function [23], the regularized gap function [9] and the D-gap function [30], are designed for special variational inequality problems and/or complementarity problems, and thus each of these globalizations has certain drawbacks. Recently, Solodov and Svaiter [35] proposed a truly globally convergent

Newton-type method for solving monotone nonlinear complementarity problems. They do not use any merit function, but use an additional projection step to globalize the domain of convergence. The method is truly convergent in the sense that the subproblems are always solvable, and the whole sequence of iterates converges to a solution of the problem without any regularity assumptions. Under natural assumptions, the method is locally superlinearly convergent.

In this paper, by combining the method of Solodov and Svaiter [35] and the equation-based feasible Newton-type method of Kanzow [21], we propose a new Newton-type method for the monotone mixed complementarity problem. Our method thus, has all the favorable properties of these methods. Specially,

a). It is well defined for an arbitrary monotone mixed complementarity problem;

b). It is truly globally convergent to a solution of the problem without any regularity assumption;

c). It has to solve just one linear system of equations at each iteration. This system is actually of reduced dimension;

d). Under some natural conditions, it is locally superlinearly convergent;

e). All iterates of the method remain feasible.

We note that most recently some efforts have been made on those methods based on Fisher-Burmeister NCP function in weakening the conditions guaranteeing convergence. Some methods, for example, the methods in [37, 31], only require the solution set is bounded and F is P_0 function. The later is weaker than the property of monotonicity, required in the proposed method. But even for monotone MCPs, these methods cannot ensure that the whole generated sequence converges to a solution.

Some words about our notation are in order. The Jacobian of a continuous differentiable function $G : R^p \to R^p$ at a point x is denoted by $G^{'}(x)$, whereas $\nabla G(x)$ is the transposed Jacobian. If $M \in R^{p\times p}$, $M = (m_{ij})$, is any given matrix and $I, J \subseteq \{1, 2, \cdots, p\}$ are two subsets of indices, then M_{IJ} denote the $|I| \times |J|$ submatrix with elements m_{ij}, $i \in I$, $j \in J$.

2. Preliminaries

In this section, we summarize some related definitions and properties that will be used in the following discussion.

Let $[l, u]$ be a rectangle in R^n. For a vector $x \in R^n$, let $[x]^+$ denote the orthogonal projection of x to $[l, u]$. Then,

$$[x]_i^+ = \begin{cases} l_i, & \text{if } x_i \le l_i, \\ x_i, & \text{if } x_i \in (l_i, u_i), \\ u_i, & \text{if } x_i \ge u_i. \end{cases}$$

Since the solution set of MCP (1) is invariant under multiplication F by some positive scalar β, it follows from ([1], p. 267) that

Lemma 2.1. *x^* is a solution of the MCP (1) if and only if*

$$x^* = [x^* - \beta F(x^*)]^+, \quad \forall \beta > 0.$$

Let $e(x, \beta)$ denote the residue function associated with mapping F, i.e.,

$$e(x, \beta) = x - [x - \beta F(x)]^+. \tag{6}$$

Then, solving MCP (1) is thus, equivalent to finding a vector $\bar{x} \in [l, u]$, such that

$$e(\bar{x}, \beta) = 0$$

for any positive constant β.

A basic property of the projection mapping $[\cdot]^+$ is

$$(x - [x]^+)^\top (y - [x]^+) \leq 0, \ \forall x \in R^n, \ \forall y \in [l, u], \tag{7}$$

which, together with the Cauchy-Schwarz inequality, means that $[\cdot]^+$ is nonexpansive, i.e.,

$$\|[x]^+ - [y]^+\| \leq \|x - y\|, \ \forall x, y \in R^n.$$

We need the following definitions concerning the functions.

Definition 2.1. *Let Ω be a nonempty closed convex subset of R^n. A mapping $F : \Omega \to R^n$ is said to be*

a). monotone on Ω, if

$$(F(x) - F(y))^\top (x - y) \geq 0, \ \forall x, y \in \Omega;$$

b). strictly monotone, if the above inequality holds strictly whenever $x \neq y$;

c). strongly monotone, if there is a constant $\alpha > 0$, such that

$$(F(x) - F(y))^\top (x - y) \geq \alpha \|x - y\|^2, \ \forall x, y \in \Omega. \tag{8}$$

Here, $\|\cdot\|$ denotes the Euclidean norm in R^n. It can be shown [25] that, when F is continuously differentiable, an necessary and sufficient condition for (8) is

$$d^\top \nabla F(x) d \geq \alpha \|d\|^2, \ \forall x \in \Omega, \quad \forall d \in R^n.$$

The following *error bound* [28] result will play a crucial role in establishing the superlinear rate of convergence of our algorithm.

Lemma 2.2. *Suppose that $\bar{x}$ is a solution of MCP (1) where $\nabla F(\bar{x})$ is positive definite, and $\rho > 0$ is a constant, then there exist constants $\theta_1 > 0$ and $\theta_2 > 0$, and a neighborhood B of $\bar{x}$, such that*

$$\theta_1 \|e(x, \rho^{-1})\| \leq \|x - \bar{x}\| \leq \theta_2 \|e(x, \rho^{-1})\|,$$

for all $x \in B$.

3. The Algorithm

As we described in Introduction, our method can be viewed as a combination of the Solodov and Svaiter's method [35] and some equation-based Newton-type methods. That is, we solve a system of linear equations (with reduced dimension) to obtain a Newton-type direction, and use a projection step to force the global convergence. However, our computation of the steplength is different from that of Solodov and Svaiter.

We now formally state our method as follows.

Algorithm 3.1. *(A truly globally convergent Newton-type method.)*

Step 0 *(Initialization)*

Choose $x^0 \in [l, u]$, $\sigma \in (0, 1)$, $\beta \in (0, 1)$, $c > 0$, $\delta > 0$, $\epsilon > 0$, $\rho > 0$ and set $k := 0$.

Step 1 *(Termination criterion)*

If $\|e(x^k, \rho^{-1})\| \leq \epsilon$, STOP.

Step 2 *(Active set strategy)*

Let

$$\delta_k := \min\left\{\delta, c\sqrt{\|e(x^k, \rho^{-1})\|}\right\},$$

and define

$$\begin{aligned} \mathcal{A}_k : &= \{i \mid x_i^k - l_i \le \delta_k \text{ or } u_i - x_i^k \le \delta_k\}, \\ \mathcal{I}_k : &= \{1, 2, \cdots, n\} \backslash \mathcal{A}_k. \end{aligned}$$

Step 3 *(Subproblem solution)*

Choose a positive semidefinite matrix $G^k \in R^{n\times n}$ and a number $\mu_k > 0$, and compute a vector $d^k \in R^n$ in the following way: For $i \in \mathcal{A}_k$, set

$$d_i^k = \begin{cases} l_i - x_i^k & \text{if} \quad l_i - x_i^k \le \delta_k, \\ u_i - x_i^k & \text{if} \quad u_i - x_i^k \le \delta_k, \end{cases}$$

then solve the linear system

$$(G^k_{\mathcal{I}_k\mathcal{I}_k} + \mu_k I)d = -F(x^k)_{\mathcal{I}_k} - G^k_{\mathcal{I}_k\mathcal{A}_k} d^k_{\mathcal{A}_k} \tag{9}$$

in order to get the components d_i^k for $i \in \mathcal{I}_k$.

Step 4 *(Acceptance criterion for Newton-type search direction)*

Set $z^k = x^k + d^k$. If $z^k \in [l, u]$ and satisfies

$$\begin{aligned} z_i^k = l_i \quad &\Rightarrow \quad F_i^k(z^k) \ge 0, \\ z_i^k \in (l_i, u_i) \quad &\Rightarrow \quad F_i^k(z^k) = 0, \\ z_i^k = u_i \quad &\Rightarrow \quad F_i^k(z^k) \le 0, \end{aligned} \tag{10}$$

and

$$\|F(z^k) - F(x^k) - G^k d^k\| \le \sigma\mu_k \|d^k\|, \tag{11}$$

then set

$$v_k := F(z^k) - F(x^k) - G^k d^k - \mu_k d^k, \tag{12}$$

$$\alpha_k := \max\{1.0/\mu_k, v_k^\top (x^k - z^k)/\|v_k\|^2\}, \tag{13}$$

and go to Step 6, else go to Step 5.

Step 5 *(Extragradient search direction)*

Set $\bar{x}^k := [x^k - \rho^{-1}F(x^k)]^+$, and $S^k := \bar{x}^k - x^k$, and find $t_k = \beta^{m_k}$ with m_k being the smallest nonnegative integer m, such that

$$-F(x^k + t_k S^k)^\top S^k \ge \sigma\rho\|S^k\|^2. \tag{14}$$

Then, set $z^k := x^k + t_k S^k$, $v_k := F(z^k)$ and compute

$$\alpha_k := v_k^\top (x^k - z^k)/\|v_k\|^2.$$

Step 6 *(Projection step)*

Compute

$$x^{k+1} = [x^k - \alpha_k v_k)]^+, \tag{15}$$

set $k := k + 1$ and go to Step1.

To ensure it does not occur that $x_i^k - l_i \leq \delta_k$ and $u_i - x_i^k \leq \delta_k$ for the same index $i \in \mathcal{A}_k$, throughout this paper, we choose

$$\delta < \frac{1}{2} \min_{i=1,\cdots,n} |u_i - l_i|.$$

Before analyzing the convergence properties of Algorithm 3.1, let us give some explanations and comments on each step. First of all, the set $\mathcal{A}_k$ defined in Step 2 is used as an approximation to the set of active constraints

$$\mathcal{A}_* := \{i \mid x_i^* = l_i \text{ or } x_i^* = u_i\}. \tag{16}$$

This strategy was first investigated in [20], which has a close connection to the identification technique studied in [5] and was further used in [21]. The set $\mathcal{I}_k$ can thus be viewed as a suitable approximation of the set of inactive constraints

$$\mathcal{I}_* := \{i \mid x_i^* \in (l_i, u_i)\}. \tag{17}$$

Based on this active strategy, in Step 3, we try to compute a Newton-type search direction d^k by solving the proximal Newton equations

$$(G^k + \mu_k I)d = -F(x^k),$$

which, after a possible permutation of the rows and columns, can be rewritten as

$$\begin{pmatrix} G^k_{\mathcal{I}_k\mathcal{I}_k} & G^k_{\mathcal{I}_k\mathcal{A}_k} \\ G^k_{\mathcal{A}_k\mathcal{I}_k} & G^k_{\mathcal{A}_k\mathcal{A}_k} \end{pmatrix} \begin{pmatrix} d_{\mathcal{I}_k} \\ d_{\mathcal{A}_k} \end{pmatrix} = - \begin{pmatrix} F(x^k)_{\mathcal{I}_k} \\ F(x^k)_{\mathcal{A}_k}. \end{pmatrix} \tag{18}$$

We just take the same simple formula for the components d_i^k with $i \in \mathcal{A}_k$ as [21], which basically aims at bringing the corresponding components of the next iterate closer to the boundary. The components d_i^k with $i \in \mathcal{I}_k$, are obtained by solving the first block row in (18), which is just the system of linear equations (9).

In Step 4, we first check if the direction d^k provides us a solution for the linear mixed complementarity subproblem (10) and a sufficient descent direction (11). If this is true, we just accept this direction; otherwise, we turn to use the extragradient-type direction in Step 5, which will guarantee the global convergence of our method. It should be noted that in Step 5, our line search procedures do not use any merit function for MCP (1).

In Solodov and Svaiter's method [35], the stepsize α_k is always computed by

$$\alpha_k = v_k^\top (x^k - z^k)/\|v_k\|^2.$$

Here we take a larger stepsize

$$\alpha_k = \max\{1.0/\mu_k, v_k^\top (x^k - z^k)/\|v_k\|^2\}$$

in the case that (10) and (11) hold. This has some advantage from the viewpoint of computation, since we should take a stepsize as large as we can.

Note that the sequence $\{z^k\}$ is feasible and since the iterates are generated by projection to $[l, u]$, the whole sequence $\{x^k\}$ is also feasible.

In the following we will always assume implicitly that the termination parameter ϵ in the algorithm is equal to zero, and the algorithm does not terminate after finitely many iterates. This assumption is reasonable since Lemma 2.1 shows that otherwise the current iterate would already be a solution of MCP (1).

4. Global Convergence

This section will be devoted to the global convergence of the method. We will establish our global convergence of the proposed method under the assumptions of merely monotonicity and continuity of F. We now start our analysis with some lemmas.

Our first result states that the whole algorithm is well-defined.

Lemma 4.1. *Let F be a continuous, monotone mapping from $[l, u]$ to R^n, and let $G^k \in R^{n\times n}$ be a positive semidefinite matrix. Then the method is well-defined for any parameter $\mu_k > 0$.*

Proof. Since G^k is semidefinite and $\mu_k > 0$, the system of linear equations (9) is always solvable. Thus, to show that the algorithm is well defined, we need only to show that for any $k \geq 0$, the line search in Step 5 is finite. In fact, by setting $x := x^k - \rho^{-1}F(x^k)$ and $y := x^k \in [l, u]$ in (7), we have

$$(x^k - \rho^{-1}F(x^k) - \bar{x}^k)^\top (x^k - \bar{x}^k) \leq 0,$$

which means that

$$-F(x^k)^\top S^k \geq \rho \|S^k\|^2. \tag{19}$$

The conclusion thus follows immediately from the above inequality, the continuity of F and the fact that $\sigma < 1$.

Lemma 4.2. *For all $k \geq 0$, we have that*

$$v_k^\top (z^k - x^*) \geq 0, \tag{20}$$

where $x^* \in [l, u]$ is an arbitrary solution of MCP (1).

Proof. For Step 5, since $v_k = F(z^k)$, $z^k \in [l, u]$, (20) follows immediately from (1) and the monotonicity of F. We now pay our attention to v_k defined in Step 4. Note that in Step 4, (10) implies that $z^k = x^k + d^k$ is a solution of the linearized mixed complementarity problem (9), i.e.,

$$(F(x^k) + (G^k + \mu_k I)(z^k - x^k))^\top (y - z^k) \geq 0, \quad \forall y \in [l, u].$$

Since $x^* \in [l, u]$, we have that

$$(F(x^k) + (G^k + \mu_k I)d^k)^\top (x^* - z^k) \geq 0.$$

On the other hand, it follows from (1) and the monotonicity of F that

$$F(z^k)^\top (z^k - x^*) \geq 0.$$

Thus, from the above inequalities, (20) also holds for v_k defined in Step 4.

Lemma 4.3. *Suppose that F is continuous and monotone, $G^k \in R^{n\times n}$ is positive semidefinite. The sequence $\{x^k\}$ generated by the algorithm is bounded for all positive parameter $\mu_k > 0$.*

Proof. Let $x^* \in [l, u]$ be an arbitrary solution of MCP (1). Then, it follows from the nonexpansivity of the projection operator $[\cdot]^+$ that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \|x^k - x^* - \alpha_k v_k\|^2 \\ &= \|x^k - x^*\|^2 - 2\alpha_k v_k^\top (x^k - x^*) + \alpha_k^2 \|v_k\|^2 \\ &\leq \|x^k - x^*\|^2 - 2\alpha_k v_k^\top (x^k - z^k) + \alpha_k^2 \|v_k\|^2, \end{aligned}$$

where the second inequality follows from (20). If

$$\alpha_k = v_k^\top (x^k - z^k)/\|v_k\|^2,$$

then

$$\begin{aligned}\|x^{k+1}-x^*\|^2 &\le \|x^k-x^*\|^2-\alpha_k v_k^\top(x^k-z^k)\\ &= \|x^k-x^*\|^2-(v_k^\top(x^k-z^k))^2/\|v_k\|^2.\end{aligned} \tag{21}$$

Otherwise, if $\alpha_k = 1.0/\mu_k$, then it follows from (11) that

$$2\alpha_k v_k^\top(x^k-z^k) \ge \alpha_k^2\|v_k\|^2+(1-\sigma^2)\|x^k-z^k\|^2, \tag{22}$$

and thus

$$\|x^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2-(1-\sigma^2)\|x^k-z^k\|^2. \tag{23}$$

From (21) and (23), we have that

$$\|x^{k+1}-x^*\|^2 \le \|x^k-x^*\|^2 \le \cdots \le \|x^0-x^*\|^2. \tag{24}$$

The generated sequence $\{x^k\}$ is thus bounded.

We are now ready to prove the main result in this section, which states that the whole sequence of iterates always converges to a solution of MCP (1) under the assumptions of merely monotonicity and continuity of F. As far as we know, this is the only Newton-type method that possess this remarkable property without any regularity assumptions except the method of Solodov and Svaiter [35].

Theorem 4.4. *Suppose that F is continuous and monotone and suppose further that there exist constants c_1, c_2, $c_3 > 0$ and $t > 0$, such that for all $k \ge 0$,*

$$\|G^k\| \le c_1$$

and

$$c_2 \ge \mu_k \ge c_3\|e(x^k,\rho^{-1})\|^t.$$

Then the sequence $\{x^k\} \subseteq [l,u]$ generated by Algorithm 1 converges to a solution of MCP (1).

Proof. If the tests (10) and (11) are satisfied infinitely many times, then it follows from (23) that

$$\lim_{k\to\infty} d^k = 0. \tag{25}$$

Since $\{x^k\}$ is bounded, it has at least a cluster point. Let $\bar{x} \in [l,u]$ be a cluster point and $\{x^{k_j}\}$ be the subsequence of $\{x^k\}$ converging to $\bar{x}$. Therefore, it follows from (25) that

$$\lim_{j\to\infty} z^{k_j} = \bar{x}.$$

Then, taking limit along the subsequence and using the continuity of F and the boundedness of G_k and μ_k, we have, from (10) that

$$F(\bar{x})^\top(y-\bar{x}) \ge 0, \quad \forall y \in [l,u].$$

Thus, $\bar{x}$ is a solution of MCP (1). Since x^* is an arbitrary solution, we can just take $x^* = \bar{x}$ and it follows from (24) that

$$\|x^{k+1}-\bar{x}\| \le \|x^k-\bar{x}\|.$$

The whole sequence $\{x^k\}$ thus converges to $\bar{x}$, a solution of MCP (1).

In the rest of this proof, we can therefore assume that the tests in Step 4 are satisfied only a finite number of times. That is, there exists a constant $k_0 \ge 0$, such that for all $k \ge k_0$, (10) and (11) are not satisfied. For simplicity but without loss of generality, we can just assume that $k_0 = 0$. Under these conditions, z^k is defined in Step 5. It follows from the monotonicity of F and the line search procedures (14) that

$$F(x^k)^\top (x^k - z^k) \geq F(z^k)^\top (x^k - z^k) \geq \sigma\rho \|x^k - z^k\|^2.$$

Then, using the Cauchy-Schwarz inequality, we have

$$\|F(x^k)\| \|x^k - z^k\| \geq \sigma\rho \|x^k - z^k\|^2.$$

From the boundedness of $\{x^k\}$ and the continuity of F, it follows that $\{z^k\}$ is bounded. Using again the continuity of F, we have that there exists a constant $M > 0$, such that

$$\|v_k\| \leq M$$

for all $k \geq 0$. It then follows from (14) and (21) that

$$\lim_{k\to\infty} t_k \|e(x^k, \rho^{-1})\| = 0. \tag{26}$$

We now consider the two possible case that

$$\liminf_{k\to\infty} \|e(x^k, \rho^{-1})\| = 0$$

and

$$\liminf_{k\to\infty} \|e(x^k, \rho^{-1})\| > 0.$$

In the first case, by continuity of $e(\cdot, \cdot)$ and boundedness of $\{x^k\}$, there exists $\bar{x} \in [l, u]$, an accumulation of $\{x^k\}$, such that $e(\bar{x}, \rho^{-1}) = 0$. Thus, it follows from Lemma 2.1 that $\bar{x}$ is a solution of MCP (1) and

$$\|x^{k+1} - \bar{x}\| \leq \|x^k - \bar{x}\|.$$

The whole sequence $\{x^k\}$ thus converges to $\bar{x}$.

We now consider the second case. Because of (26), it must be the case that

$$\lim_{k\to\infty} t_k = 0,$$

which is equivalent to saying that $m_k \to \infty$. Then, for sufficiently large $k \geq 0$, $m_k \geq 2$. Hence, (14) is not satisfied for the value of $m_k - 1$, i.e.,

$$-F(x^k + t_k/\beta S^k)^\top S^k < \sigma\rho \|e(x^k, \rho^{-1})\|^2.$$

Taking into account the boundedness of $\{x^k\}$, $\{z^k\}$ and passing onto a subsequence if necessary, as $k \to \infty$, we obtain that

$$-F(\bar{x})^\top \bar{S} < \sigma\rho \|e(\bar{x}, \rho^{-1})\|^2,$$

where $\bar{x}$, $\bar{S}$ are limits of corresponding subsequences. On the other hand, (19) implies that

$$-F(\bar{x})^\top \bar{S} \geq \rho \|e(\bar{x}, \rho^{-1})\|^2.$$

Then we have $\|e(\bar{x}, \rho^{-1})\| \leq \sigma \|e(\bar{x}, \rho^{-1})\|$, which yields that $e(\bar{x}, \rho^{-1}) = 0$, i.e., $\bar{x}$ is a solution of the MCP, and that the whole sequence converges to $\bar{x}$.

This completes the proof.

5. Local Convergence

Having established the global convergence of the method, we are now ready to prove the superlinear convergence of our algorithm for solving monotone mixed complementarity problems under the assumptions of strict complementarity of a solution, positive definiteness of ∇F at the solution and its local hölder continuity. The proof relies on the fact that in a small neighborhood of such a solution, the tests (10) and (11) in Step 4 of Algorithm 3.1 always hold and the line search procedures in Step 5 is not used. We now begin our analysis with some technical results from [21].

Lemma 5.1.([21], Proposition 8) *Let ∇F be positive definite at the solution of the mixed complementarity problem MCP (1). Then, $\mathcal{A}_k = \mathcal{A}_*$ and $\mathcal{I}_k = \mathcal{I}_*$ for all $x^k \in [l, u]$ sufficiently close to x^*, where $\mathcal{A}_k$ and $\mathcal{I}_k$ denote the index sets computed in Step 3 of Algorithm 3.1 and $\mathcal{A}_*$ and $\mathcal{I}_*$ are defined by (16) and (17), respectively.*

Lemma 5.2. ([21], Lemma 3) *Let the assumptions in Lemma 5.1 hold, and let x^* be a solution of the MCP (1). Then,*

$$d^k_{\mathcal{A}_k} = x^*_{\mathcal{A}_k} - x^k_{\mathcal{A}_k}$$

for all x^k sufficiently close to x^*, where $d^k_{\mathcal{A}_k}$ denotes the vector calculated in Step 3 of Algorithm 3.1.

Lemma 5.3. *Let the assumptions in Lemma 5.1 hold, and let x^* be a solution of the MCP (1). Then,*

$$\lim_{k\to\infty} \|d^k\| = 0.$$

Proof. Since $\{x^k\}$ converges to x^*, it follows from Lemma 5.2 that

$$\lim_{k\to\infty} d^k_{\mathcal{A}_k} = 0.$$

On the other hand, since $\mathcal{I}_k = \mathcal{I}_*$ for k sufficiently large and $F(x^*)_{\mathcal{I}_*} = 0$, it follows from (9) and the boundedness of G^k and μ_k that

$$\lim_{k\to\infty} d^k_{\mathcal{I}_k} = 0.$$

The assertion thus follows immediately.

Theorem 5.4. *Suppose F is continuous and monotone on $[l, u]$. Let $\bar{x}$ be the (unique) strictly complementarity solution of MCP (1) at which F is differentiable with $\nabla F(\bar{x})$ positive definite. Suppose further that ∇F is locally Hölder continuous around $\bar{x}$ with degree p. If*

$$\mu_k = \|e(x^k, \rho^{-1})\|^t, \quad t \in (0, p),$$

and starting with some index $k_0 \geq 0$, $G^k = \nabla F(x^k)$, then

1. *Eventually, the algorithm takes only the Newton-type directions from Step 4 of Algorithm 3.1.*
2. *The rate of convergence is Q-superlinear.*

Proof. By Theorem 4.4, we know that the generated sequence $\{x^k\}$ converges to $\bar{x}$ and from the choice of μ_k and Lemma 2.1, we have that $\lim_{k\to\infty} \mu_k = 0$.

By Hölder continuity of ∇F around $\bar{x}$, we have that for all $w \in R^n$ sufficiently small and all indices k sufficiently large, there exists a constant $L > 0$, such that

$$\|\nabla F(x^k + w) - \nabla F(x^k)\| \leq L\|w\|^p, \quad p \in (0, 1].$$

Therefore, we have

$$\begin{aligned} F(x^k + w) - F(x^k) - \nabla F(x^k)w &= R^k(w) \\ \|R^k(w)\| &\leq \frac{L}{2}\|w\|^{1+p}. \end{aligned}$$

From Lemma 5.1 and Lemma 5.2, the boundedness of $\{x^k\}$ and the continuity of ∇F, it follows that there is a constant $\gamma > 0$, such that for all k sufficiently large,

$$\|d^k\| \leq \gamma \|x^k - \bar{x}\|. \tag{27}$$

It then follows from Lemma 2.2 that

$$\|d^k\| \leq \theta_2 \gamma \|e(x^k, \rho^{-1})\|.$$

Since $d^k \to 0$, we have

$$\begin{aligned} \|F(z^k) - F(x^k) - G_k d^k\| &\leq \frac{L}{2}\|d^k\|^{1+p} \\ &\leq \frac{L}{2}\theta_2^p \gamma^p \|e(x^k, \rho^{-1})\|^p \|d^k\|. \end{aligned}$$

Thus, (11) always holds if

$$\frac{L}{2}\theta_2^p \gamma^p \|e(x^k, \rho^{-1})\|^p \leq \sigma \mu_k = \sigma \|e(x^k, \rho^{-1})\|^t.$$

Since $t < p$ and $\|e(x^k, \rho^{-1})\| \to 0$, the above inequality will hold for k large enough.

Now we consider (10). From Lemma 5.1, the strict complementarity of $\bar{x}$ and the continuity of F, we have that for k sufficiently large, if $x_i^* = l_i$, then $F_i(x^k) > 0$ and if $x_i^* = u_i$, $F_i(x^k) < 0$. Since $d^k \to 0$, (10) will also hold for k sufficiently large. Thus, the tests in Step 4 will hold and eventually, the algorithm takes only the Newton-type directions.

If $\alpha_k = v_k^\top (x^k - z^k)/\|v_k\|^2$, the superlinear rate of convergence can be proved in a similar way to that of Solodov and Svaiter [35]. We thus pay our attention to the case that $\alpha_k = 1.0/\mu_k$. In this case, from Theorem 4.4, we have

$$\begin{aligned} \|x^{k+1} - \bar{x}\| &\leq \|x^k - \bar{x} - \frac{1}{\mu_k} v^k\| \\ &\leq \|z^k - \bar{x}\| + \|x^k - z^k - \frac{1}{\mu_k} v^k\|. \end{aligned} \tag{28}$$

Note that

$$\begin{aligned} &F(\bar{x}) - F(x^k) - \nabla F(x^k)(z^k - x^k) - \mu_k (z^k - x^k) \\ &= R^k(\bar{x} - x^k) - \nabla F(x^k)(z^k - \bar{x}) - \mu_k (z^k - x^k). \end{aligned} \tag{29}$$

Since $\bar{x}$ is a solution of MCP (1) and z^k satisfies (10), we have

$$(F(\bar{x}) - F(x^k) - \nabla F(x^k)(z^k - x^k) - \mu_k (z^k - x^k))^\top (\bar{x} - z^k) \leq 0.$$

Therefore, we have

$$(z^k - \bar{x})^\top \nabla F(x^k)(z^k - \bar{x}) \leq (R^k(\bar{x} - x^k))^\top (z^k - \bar{x}) - \mu_k (z^k - x^k)^\top (z^k - \bar{x}).$$

Since $\nabla F(\bar{x})$ is positive definite and continuous and $x^k \to \bar{x}$, for k large enough, there is a constant $C_4 > 0$, such that

$$C_4 \|z^k - \bar{x}\| \leq \|x^k - \bar{x}\|^{1+p} + \mu^k \|z^k - x^k\|. \tag{30}$$

Note that it follows from (27) and Lemma 2.2 that, for k large enough, there exists a constant $C_5 > 0$, such that

$$\|x^k - z^k\| \leq C_5 \|e(x^k, \rho^{-1})\|.$$

Thus,

$$\begin{aligned}\|x^k - z^k - \frac{1}{\mu_k}v^k\| &= \frac{1}{\mu_k}\|\mu_k(x^k - z^k) - v^k\| \\ &= \frac{1}{\mu_k}\|R^k(x^k - z^k)\| \\ &\leq \frac{1}{\mu_k}\|x^k - z^k\|^{1+p} \\ &\leq C_5^{1+p}\|e(x^k, \rho^{-1})\|^{1+p-t}. \end{aligned} \tag{31}$$

It follows from Lemma 2.2 and (28), (30) and (31) that

$$\|x^{k+1} - \bar{x}\| \leq (\|x^k - \bar{x}\|^p + C_5/\theta_1\mu_k + C_5^{1+p}/\theta_1\|e(x^k, \rho^{-1})\|^{p-t})\|x^k - \bar{x}\|.$$

Since $x^k \to \bar{x}$, $\mu_k \to 0$ and $\|e(x^k, \rho^{-1})\| \to 0$, $\{x^k\}$ thus converges to $\bar{x}$ superlinearly. This completes the proof.

6. Conclusions

We presented a new Newton-type method for solving the monotone mixed complementarity problem. The method has the nice property that it generates only feasible iterates, while has only to solve simple subproblems (system of linear equations with reduced dimension). The algorithm also possesses the remarkable property that it is truly globally convergent. That is, without any regularity assumption, the generated sequence converges to a solution of MCP if the solution set is nonempty. Under natural assumptions, locally superlinear rate of convergence was established.

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