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THIRD ORDER ITERATIVE METHODS WITHOUT USING SECOND FRÉCHET DERIVATIVE [*1)]

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Abstract

A modification of classical third order methods is proposed. The main advantage of these methods is they do not need evaluate any second order Fréchet derivative. A convergence theorem in Banach spaces is analyzed. Finally, some preliminary numerical results are presented.

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1. Introduction

Great quantity of general problems may be reduced to finding zeros. The roots of a nonlinear equation cannot in general be expressed in closed form. Thus, in order to solve nonlinear equations, we have to use approximate methods. One of the most important techniques to study these equations is the use of iterative processes [9], starting from an initial approximation x_0, called pivot, successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed , $i = 1, 2, \ldots$, with the help of certain iteration function $\Phi : X \to X$,

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2 \ldots \tag{1}$$

Certainly Newton's method is the most useful iteration for this purpose. The advance of computational techniques has allowed the development of some more complicated iterative methods in order to obtain greater order of convergence as Chebyshev and Halley methods [3], [4]. In these methods we have to evaluate first and overall second derivatives. These difficulties are usually harder than the advantage because of the order of these methods. So, two order iterative methods are widely used.

In this paper, we present a modification of classical third order iterative methods. The main advantage of these methods is they do not need evaluate any second derivative, but having the same properties of convergence than the classical third order methods. The methods will depend, in each iteration, of a parameter α_n. These parameters will be a control of the good approximation to the second derivatives. We will use second order divided differences. We will study their convergence by recurrence relations and we will test their competitively with respect the classical methods. They seemed to work very well in our preliminary numerical results.

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Let $F : B \subset X \to X$ a nonlinear operator, X a Banach space and B an open convex set. If we are interesting to approximate a solution of the nonlinear equation

$$F(x) = 0, \tag{2}$$

Chebyshev method can be written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_F(x_n)\right) (F'(x_n))^{-1} F(x_n), \tag{3}$$

where

$$L_F(x_n) = (F'(x_n))^{-1} F''(x_n) (F'(x_n))^{-1} F(x_n).$$

Our idea is to approximate F'' on the classical third order methods. We will consider the case of Chebyshev method, but it is possible consider other third order methods [2].

In order to conserve the order we consider the following approximation (in the scalar case $X = \mathbb{R}$)

$$f''(x_n) \sim \frac{f(x_n + \alpha_n f(x_n)) - 2f(x_n) + f(x_n - \alpha_n f(x_n))}{(\alpha_n f(x_n))^2}. \tag{4}$$

Thus our modification (in a Banach space) will be

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} \mathcal{L}_F(x_n)\right) (F'(x_n))^{-1} F(x_n), \tag{5}$$

where

$$\mathcal{L}_F(x_n) = (F'(x_n))^{-1} \mathcal{D}_F(x_n) (F'(x_n))^{-1} F(x_n),$$

$$\mathcal{D}_F(x_n) = [x_n - \alpha_n F(x_n), x_n, x_n + \alpha_n F(x_n); F],$$

and $[\cdot,\cdot,\cdot;F]$ denotes the second divided difference of the operator F, that is, a bilinear operator from $X \times X$ to X such that

$$\mathcal{D}_F(x_n)(\alpha_n F(x_n))(\alpha_n F(x_n)) = F(x_n + \alpha_n F(x_n)) - 2F(x_n) + F(x_n - \alpha_n F(x_n)).$$

The method will depend, in each iteration, of a parameter α_n. This parameter will be a control of the good approximation to the second derivative. In practice, $\{\alpha_n\}$ will be a increasing sequence in $(0, 1]$, and $||\alpha_n F(x_n)||$ will be small enough.

Remark 1 In order to control the stability in practice, the α_n can be computed such that

$$tol_c << ||\alpha_n F(x_n)|| \leq tol_u$$

where tol_c is related with the computer precision and tol_u is a free parameter for the user.

Taylor series expansions show that with these approximations the method (in the scalar case, $f : \mathbb{R} \to \mathbb{R}$) can be written as

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_f(x_n) + O(L_f^2(x_n))\right) (f'(x_n))^{-1} f(x_n), \tag{6}$$

thus, if the method converges, it has order three [1].

We are interesting to obtain sufficient conditions of convergence. In next section we establish a convergence theorem using recurrence relations in a similar way as Gutiérrez and Hernández in [7].

2. Convergence Study

Our goal in this section is to prove the convergence of (5). Usually the convergence of third order iterative methods is established assuming that the second Fréchet derivative $F^{''}$ satisfies a Lipschitz condition. Gutiérrez and Hernández [7] obtain the convergence just assuming $F^{''}$ bounded. In our case, since we don't evaluate any second derivative, we can reduce this hypothesis. We will assume the second divided difference of F is bounded. In fact, we could consider not twice Fréchet differentiable operators.

Recurrence relations, using a similar strategy as in [7], are considered. Thus the initial problem in a Banach space can be reduced to a simpler problem with real sequences. Moreover, our real sequences will be the same than in [7].

Theorem 1. *Let be X a Banach space and B an open convex set. Let be $F : B \subset X \longrightarrow X$ a nonlinear Fréchet differentiable operator with second order divided differences in B. Let us assume that $\Gamma_0 = (F^{'}(x_0))^{-1} \in \mathcal{L}(X, X)$ exists at some $x_0 \in B$, where $\mathcal{L}(X, X)$ is the set of bounded linear operators from X into X.*

We assume that

(1) $||[x^{'}, x^{''}, x^{'''}; F]|| \leq K$, *for all* $x^{'}, x^{''}, x^{'''} \in B$

(2) $||\Gamma_0|| \leq M$

(3) $||\Gamma_0 F(x_0)|| \leq \eta$

Let us denote $a = KM\eta$. We define the sequences

$a_0 = b_0 = 1; \quad c_0 = a; \quad d_0 = 1 + \frac{a}{2}$

$a_{n+1} = \frac{a_n}{1 - a a_n d_n};$

$b_{n+1} = \frac{a a_{n+1} d_n^2}{2}(1 + \frac{4}{(2+c_n)^2});$

$c_{n+1} = a a_{n+1} b_{n+1};$

$d_{n+1} = (1 + \frac{c_{n+1}}{2}) b_{n+1}$

Suppose that $0 < a \leq s_0 = 0.32664...$, where s_0 is the smallest positive root of $2x^4 + 7x^3 - 4x^2 - 24x + 8 = 0$. Then, if $\overline{B(x_0, r\eta)} \subset B$, where $r = \sum_{n=0}^{+\infty} d_n$, the sequence (5) is well defined and converges to the unique fixed point x^* of F in $B(x_0, \frac{2}{KM} - r\eta) \bigcap B$.

Furthermore, we can obtain the following error estimates

$$||x^* - x_n|| \leq \frac{d_0}{\gamma} \sum_{k=n}^{+\infty} \gamma^{2^k}; \quad \gamma = \frac{c_1}{c_0}.$$

Proof. We are going to prove

(I_n) $||\Gamma_n|| \leq a_n M$

(II_n) $||\Gamma_n F(x_n)|| \leq b_n \eta$

(III_n) $||\mathcal{L}_F(x_n)|| \leq c_n$

(IV_n) $||x_{n+1} - x_n|| \leq d_n \eta$

(I_0), (II_0) and (III_0) follow immediately from the hypothesis.

$$\begin{aligned} ||x_1 - x_0|| &\leq (1 + \frac{1}{2}||\mathcal{L}_F(x_0)|| \cdot ||\Gamma_0 F(x_0)||) \\ &\leq (1 + \frac{1}{2} c_0)\eta \\ &= d_0 \eta \end{aligned}$$

and (IV_0) holds.

Since $aa_nd_n < 1$ see [7], we obtain

$$\begin{aligned} ||I - \Gamma_n F'(x_{n+1})|| &\leq ||\Gamma_n|| \cdot ||F'(x_n) - F'(x_{n+1})|| \\ &\leq aa_nd_n < 1 \end{aligned}$$

thus Γ_{n+1} is defined and

$$||\Gamma_{n+1}|| \leq a_{n+1}M.$$

On the other hand, we deduce from (5) that

$$F'(x_n)(x_{n+1} - x_n) = -F(x_n) - \frac{1}{2}\mathcal{D}_F(x_n)\Gamma_n F(x_n)\Gamma_n F(x_n)$$

and then

$$F(x_{n+1}) = \int_{x_n}^{x_{n+1}} (F'(x) - F'(x_n))dx - \frac{1}{2}\mathcal{D}_F(x_n)\Gamma_n F(x_n)\Gamma_n F(x_n)$$

Consequently,

$$||\Gamma_{n+1}F(x_{n+1})|| \leq b_{n+1}\eta$$

Finally,

$$||\mathcal{L}_F(x_{n+1})|| \leq aa_{n+1}b_{n+1} = c_{n+1}$$

and

$$||x_{n+2} - x_{n+1}|| \leq d_{n+1}\eta$$

We refer [7] for the rest of the details, because the real sequences are the same. In particular, they proof that $aa_nd_n < 1$ and that $\{d_n\}$ is a Cauchy sequence.

3. Numerical Experiments

In order to see the performance of the introduced iterative method, we have tested it on some nonlinear equations. We present a comparison with the classical Chebyshev method.

To test numerically the order, in table 1 we consider the 1-D equation

$$\sin(2\pi x) = 0.$$

Numerically we observe they are third order schemes.

Table 1: $\sin(2\pi x) = 0$, Error, $x_0 = 0.1$

Iterations	Chebyshev	Modified-$tol_u = 0.001$
1	$1.49e-02$	$1.49e-02$
2	$2.18e-05$	$2.18e-05$
3	$7.13e-14$	$6.40e-14$
4	$0.00e+00$	$0.00e+00$

In table 2 we analyze the following equation in 2-D:

$$(y^2 - 4, x^2 - y - 1) = (0, 0).$$

We obtain the same good convergence properties.

Table 2: Number of iterations until exact solution

(x_0, y_0)	Chebyshev	Modified-$tol_u = 0.001$
$(3,4)$	4	4
$(3,6)$	4	4

Now we consider in $[0,1]$ the ordinary differential equation:

$$\begin{aligned} y' &= -4 \cdot 10^4\, y^3, \\ y(0) &= 1. \end{aligned} \tag{7}$$

We approximate the solution of the stiff problem (7) at $t = 1$ by applying implicit Euler method,

$$y_{k+1} = y_k + h \cdot f(t_{k+1}, y_{k+1}), \quad k = 0, 1, \ldots, n.$$

The different iterative processes are used for the nonlinear equation of the implicit method. In each step we consider five iterations of the iterative methods. We compute the final error $|y(1) - y_{n+1}|$. The modified method produces an error similar as Chebyshev, see table 3.

Table 3: Cauchy problem, Error

Euler Met.	Chebyshev	Modified-$tol_u = 0.001$
$n = 5$	$3.05e-03$	$6.38e-03$
$n = 10$	$1.52e-03$	$2.39e-03$
$n = 20$	$7.81e-04$	$2.06e-04$

Finally, let $X = C([0,1])$ with the norm $||x|| = \max_{s\in[0,1]} |x(s)|$, and the equation $F(x) = 0$ where the operator $F : X \to X$ is given by

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s cos(x(t))dt, \qquad x \in C[0,1], \quad s \in [0,1]$$

in table 4 we can see the good convergence properties.

Table 4: Number of iterations until exact solution, $x_0 = x_0(s) = s$

Chebyshev	Modified-$tol_u = 0.001$
4	4

We have presented a family of iterative methods. We have studied their convergence. The theoretically analysis provides information about the existence and uniqueness of solution. We have tested their competitively with respect the classical methods and they seemed to work very well in our preliminary numerical results.

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