Journal of Computational Mathematics, Vol.23, No.6, 2005, 635–646.

ON THE ANISOTROPIC ACCURACY ANALYSIS OF ACM'S NONCONFORMING FINITE ELEMENT *1)

Dong-yang Shi Shi-peng Mao Shao-chun Chen
(*Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China*)

Abstract

The main aim of this paper is to study the superconvergence accuracy analysis of the famous ACM's nonconforming finite element for biharmonic equation under anisotropic meshes. By using some novel approaches and techniques, the optimal anisotropic interpolation error and consistency error estimates are obtained. The global error is of order $O(h^2)$. Lastly, some numerical tests are presented to verify the theoretical analysis.

Mathematics subject classification: 65N30, 65N15.
Key words: Superconvergence, Nonconforming finite element, Anisotropic interpolation error, Consistency error.

1. Introduction

There are a lot of studies on the famous ACM's nonconforming finite element (refer to [6,8,12]). It is well-known that ACM's element is often employed as solving biharmonic equation. But all the results obtained previously are based on usual admissibility conditions of meshes J_h , in which regular assumption [6](or quasi-uniform assumption or inverse assumption) plays a very important role in the error estimates. That is, denoted by h_K, h the diameter of the finite element $K \in J_h$ and $\max\limits_{K\in J_h} h_K$, and by ρ_K the superior diameter of all circles contained in K respectively, then it is assumed in the classical finite element theory that $\frac{h_K}{\rho_K} \leq C, \frac{h}{h_K} \leq C$. Here and later in this paper, C denotes a general positive constant which is independent of $\frac{h_K}{\rho_K}$ and the function under consideration. However, such assumption is no longer valid in the case of anisotropic meshes. Conversely, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$,where the limit can be considered as $h \to 0$. Recently, Zenisek[13,14] and Apel [1,2] published a series of papers concentrating on the interpolation error estimates of some Lagrange type elements (conforming elements), but nonconforming methods are hardly treated. As far as we know, it seems that there are few studies on the nonconforming elements on anisotropic meshes and the application to the fourth order equation is still an open problem.

On the other hand, the superconvergence study of the finite element methods is one of the most active research subjects both in theoretical analysis and in practical computations. Many superconvergence results about conforming finite element methods have been obtained (see [3,7,9,16]). Do the superconvergence results of conforming elements still hold for those nonconforming ones? [4,11,15] studied the superconvergence results of Wilson element, and obtained the superconvergence estimates of the gradient error at the centers , nodes and midpoints of edges of the elements. [10] obtained the same superconvergentce results of rotated Q_1 element under square meshes.

Besides the conventional error order of ACM's element for the fourth order problem is of $O(h)$[6,8], [9] and [12] obtained the optimal error estimate of ACM's element for biharmonic

* Received October 29, 2003; final revised July 17, 2004.
1) The research is supported by NSF of China (No.10371113).

equation with uniform rectangular meshes and rectangular meshes respectively. Furthermore, [9] also get the superclose result of ACM's element for biharmonic equation.

In this paper, we will consider the superconvergence of ACM's element for the biharmonic equation on anisotropic meshes. The interpolation error estimate can be regarded as an application of the anisotropic finite element theory proposed by the same authors in [5], and the consistency error estimate is a generation to anisotropic mesh of the result of [8,9]. The results obtained herein are helpful in developing a posterior error estimates for the ACM's element and then designing some adaptive algorithm for numerical solution for biharmonic equations.

2. The Anisotropic Interpolation Property of ACM's Element

Let Ω be a domain of tensor product type,which means that the domain is the union of rectangles with sides parallel to the coordinate axes. Let J_h be a rectangular subdivision of Ω without the restrictions of regular assumption and inverse assumption. Let $K \in \Gamma_h$ be a rectangle , with the central point (x_K, y_K), $2h_x$ and $2h_y$ the length of sides parallel to x axis and y axis respectively, $a_1(x_K - h_x, y_K - h_y), a_2(x_K + h_x, y_K - h_y), a_3(x_K + h_x, y_K + h_y)$ and $a_4(x_K - h_x, y_K + h_y)$ the four vertices. Let $\hat{K}$ be a reference element in $\xi - \eta$ plane with central point (0,0), and four vertices $a_1(-1,-1), a_2(1,-1), a_3(1,1)$ and $a_4(-1,1)$. Let $\hat{l}_1 = \overrightarrow{\hat{a}_1\hat{a}_2}, \hat{l}_2 = \overrightarrow{\hat{a}_2\hat{a}_3}, \hat{l}_3 = \overrightarrow{\hat{a}_3\hat{a}_4}$ and $\hat{l}_4 = \overrightarrow{\hat{a}_4\hat{a}_1}$. Then there exists an affine mapping $F_K : \hat{K} \longrightarrow K$:

$$\begin{cases} x = h_x\xi + x_K, \\ y = h_y\eta + y_K. \end{cases}$$

We define the finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ on $\hat{K}$ as follows:

$$\hat{P} = P_3(\hat{K}) \bigcup \{\xi^3\eta, \xi\eta^3\}, \quad \hat{\Sigma} = \{\hat{v}_1, \hat{v}_{1\xi}, \hat{v}_{1\eta}, \cdots, \hat{v}_4, \hat{v}_{4\xi}, \hat{v}_{4\eta}\} \tag{1}$$

where $\hat{v}_{i\xi} = \frac{\partial v}{\partial \xi}(\hat{a}_i), \hat{v}_{i\eta} = \frac{\partial v}{\partial \eta}(\hat{a}_i)$,i=1,2,3,4.

It can be easily proved that the interpolation defined above is properly posed, and the interpolation function may be written as:

$$\hat{v} = \sum_{i=1}^{4} \hat{N}_{1i}(\xi,\eta)\hat{v}_i + \sum_{i=1}^{4} \hat{N}_{2i}(\xi,\eta)\hat{v}_{i\xi} + \sum_{i=1}^{4} \hat{N}_{3i}(\xi,\eta)\hat{v}_{i\eta}, \qquad \forall \hat{v} \in \hat{P}, \tag{2}$$

where

$$N_{1i}(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)(1+\frac{\xi_i\xi+\eta_i\eta}{2} - \frac{\xi^2+\eta^2}{2}), i = 1,2,3,4,$$
$$N_{2i}(\xi,\eta) = (1+\xi_i\xi)^2(1+\eta_i\eta)(1-\xi_i\xi)(-\xi_i)/8,, i = 1,2,3,4,$$
$$N_{3i}(\xi,\eta) = (1+\xi_i\xi)(1+\eta_i\eta)^2((1-\eta_i\eta)(-\eta_i)/8, i = 1,2,3,4,$$
$$(\xi_1,\xi_2,\xi_3,\xi_4) = (-1,1,1,-1), (\eta_1,\eta_2,\eta_3,\eta_4) = (-1,1,1,-1).$$

Then we define the interpolate operator of ACM's element as

$$\hat{\Pi} : H^4(\hat{K}) \to \hat{P}, \hat{\Pi}\hat{v} = \sum_{i=1}^{4} \hat{N}_{1i}(\xi,\eta)\hat{v}_i + \sum_{i=1}^{4} \hat{N}_{2i}(\xi,\eta)\hat{v}_{i\xi} + \sum_{i=1}^{4} \hat{N}_{3i}(\xi,\eta)\hat{v}_{i\eta} \tag{3}$$

and

$$\Pi : H^4(K) \to \hat{P} \circ F_K^{-1}, \Pi_h v = (\hat{\Pi}\hat{v}) \circ F_K^{-1}.$$

Lemma 1. *The interpolation operator $\hat{\Pi}$ defined by (3) has the anisotropic interpolation property, i.e., for $|\alpha| = 2$, we have*

$$\|\hat{D}^\alpha(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \le C|\hat{D}^\alpha \hat{v}|_{2,\hat{K}}, \forall \hat{v} \in H^4(\hat{K}). \tag{4}$$

Proof. 1) if $\alpha = (2,0)$, then $\hat{D}^\alpha \hat{P} = Q_1(\hat{K})$, and

$$\hat{D}^\alpha \hat{\Pi}\hat{v} = \frac{\partial^2 \hat{\Pi}\hat{v}}{\partial \xi^2} = \sum_{i=1}^4 \frac{\partial^2 \hat{N}_{1i}}{\partial \xi^2}\hat{v}_i + \sum_{i=1}^4 \frac{\partial^2 \hat{N}_{2i}}{\partial \xi^2}\hat{v}_{i\xi} + \sum_{i=1}^4 \frac{\partial^2 \hat{N}_{3i}}{\partial \xi^2}\hat{v}_{i\eta} = \sum_{i=1}^4 \beta_{1i}(\hat{v})\hat{q}_{1i}(\xi,\eta),$$

where

$$\hat{q}_{11} = \frac{1}{4}(1-\eta),\ \hat{q}_{12} = \frac{1}{4}(1+\eta),\ \hat{q}_{13} = \frac{3}{4}\xi(1-\eta),\ \hat{q}_{14} = \frac{3}{4}\xi(1+\eta)$$

are basis of $Q_1(\hat{K})$, and

$$\begin{aligned}
\beta_{11}(\hat{v}) &= \hat{v}_{2\xi} - \hat{v}_{1\xi} = \int_{\hat{l}_1} \frac{\partial^2 \hat{v}}{\partial \xi^2} ds, \quad \beta_{12}(\hat{v}) = \hat{v}_{3\xi} - \hat{v}_{4\xi} = \int_{\hat{l}_3} \frac{\partial^2 \hat{v}}{\partial \xi^2} ds,\\
\beta_{13}(\hat{v}) &= \hat{v}_1 - \hat{v}_2 + \hat{v}_{1\xi} + \hat{v}_{2\xi} = -\int_{-1}^1 \frac{\partial \hat{v}}{\partial \xi}(\xi,-1)d\xi + \frac{\partial \hat{v}}{\partial \xi}(-1,-1) + \frac{\partial \hat{v}}{\partial \xi}(1,-1)\\
&= \frac{1}{2}[-2\int_{-1}^1 \frac{\partial \hat{v}}{\partial \xi}(\xi,-1)d\xi + \int_{-1}^1 \frac{\partial \hat{v}}{\partial \xi}(-1,-1)d\xi + \int_{-1}^1 \frac{\partial \hat{v}}{\partial \xi}(1,-1)d\xi]\\
&= \frac{1}{2}\int_{-1}^1 [\int_\xi^1 \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi,-1)d\xi - \int_{-1}^\xi \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi,-1)d\xi]d\xi.
\end{aligned}$$

By the same argument, we get that

$$\beta_{14}(\hat{v}) = \frac{1}{2}\int_{-1}^1 [\int_\xi^1 \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi,1)d\xi - \int_{-1}^\xi \frac{\partial^2 \hat{v}}{\partial \xi^2}(\xi,1)d\xi]d\xi.$$

So

$$\beta_{1i}(\hat{v}) = F_{1i}(\frac{\partial^2 \hat{v}}{\partial \xi^2}), \qquad 1 \le i \le 4,$$

where F_{1i} is a function defined over $H^2(\hat{K})$:

$$F_{11}(\hat{w}) = \int_{\hat{l}_1} \hat{w} ds, \quad F_{12}(\hat{w}) = \int_{\hat{l}_3} \hat{w} ds, \quad F_{13}(\hat{w}) = \frac{1}{2}\int_{-1}^1 [\int_\xi^1 \hat{w}(s,-1)ds - \int_{-1}^\xi \hat{w}(s,-1)ds]d\xi,$$

$$F_{14}(\hat{w}) = \frac{1}{2}\int_{-1}^1 [\int_\xi^1 \hat{w}(s,1)ds - \int_{-1}^\xi \hat{w}(s,1)ds]d\xi.$$

Employing the trace theorem[5], we have

$$|F_{11}(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}}, \quad |F_{12}(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}},$$

$$|F_{13}(\hat{w})| \le 2\int_{-1}^1 |\hat{w}(s,-1)|ds = 2\int_{\hat{l}_1} |\hat{w}| ds \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}},$$

$$|F_{14}(\hat{w})| \le 2\int_{-1}^1 |\hat{w}(s,1)|ds = 2\int_{\hat{l}_3} |\hat{w}| ds \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}}.$$

Then employing the basic anisotropic interpolation theorem[5] yields

$$\|\hat{D}^\alpha(\hat{v}-\hat{\Pi}\hat{v})\|_{0,\hat{K}} \le C|\hat{D}^\alpha\hat{v}|_{2,\hat{K}}.$$

2) if α=(0,2), we can prove that (11) is still valid similarly.

3) if $\alpha=(1,1)$, then

$$\begin{aligned}
\frac{\partial^2\hat{\Pi}\hat{v}}{\partial\xi\partial\eta} &= \sum_{i=1}^4\frac{\partial^2\hat{N}_{1i}}{\partial\xi\partial\eta}\hat{v}_i+\sum_{i=1}^4\frac{\partial^2\hat{N}_{2i}}{\partial\xi\partial\eta}\hat{v}_{i\xi}+\sum_{i=1}^4\frac{\partial^2\hat{N}_{3i}}{\partial\xi\partial\eta}\hat{v}_{i\eta}\\
&= \frac{1}{2}\sum_{i=1}^4(\xi_i\eta_i\hat{v}_i-\frac{1}{4}\eta_i\hat{v}_{i\xi}-\frac{1}{4}\xi_i\hat{v}_{i\eta})+\frac{\xi}{4}\sum_{i=1}^4\xi_i\eta_i\hat{v}_{i\xi}+\frac{\eta}{4}\sum_{i=1}^4\xi_i\eta_i\hat{v}_{i\eta}\\
&+ \frac{3\xi^2}{8}\sum_{i=1}^4(-\xi_i\eta_i\hat{v}_i+\eta_i\hat{v}_{i\xi}+\frac{3\eta^2}{8}\sum_{i=1}^4(-\xi_i\eta_i\hat{v}_i+\xi_i\hat{v}_{i\eta})\\
&= \sum_{i=1}^5\beta_{3i}(\hat{v})\hat{q}_{3i}(\xi,\eta),
\end{aligned}$$

where

$$\hat{q}_{31}=\frac{1}{2},\hat{q}_{32}=\frac{\xi}{4},\quad \hat{q}_{33}=\frac{\eta}{4},\quad \hat{q}_{34}=\frac{3}{8}\eta^2,\quad \hat{q}_{35}=\frac{3}{8}\eta^2,$$

are the basis of $\hat{D}(\hat{P})=P_1(\hat{K})\bigcup\{\xi^2,\eta^2\}$,

$$\begin{aligned}
\beta_{31}(\hat{v}) &= \hat{v}_1-\hat{v}_2+\hat{v}_3-\hat{v}_4+\frac{1}{4}(\hat{v}_{1\xi}+\hat{v}_{2\xi}-\hat{v}_{3\xi}-\hat{v}_{4\xi})+\frac{1}{4}(\hat{v}_{1\eta}-\hat{v}_{2\eta}-\hat{v}_{3\eta}+\hat{v}_{4\eta})\\
&= \int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}d\xi d\eta-\frac{1}{4}(\int_{\hat{l}_2}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_4}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds)-\frac{1}{4}(\int_{\hat{l}_1}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_3}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds)\\
&= \int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}d\xi d\eta-\frac{1}{4}\int_{\partial\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds,\\
\beta_{32}(\hat{v}) &= \hat{v}_{1\xi}-\hat{v}_{2\xi}+\hat{v}_{3\xi}-\hat{v}_{4\xi}=-\int_{\hat{l}_4}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_2}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\beta_{33}(\hat{v}) &= -\int_{\hat{l}_1}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_3}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds,\\
\beta_{34}(\hat{v}) &= -\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_4}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_2}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds,\\
\beta_{35}(\hat{v}) &= -\int_{\hat{K}}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_1}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds+\int_{\hat{l}_3}\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}ds.
\end{aligned}$$

Using the trace theorem yields

$$\begin{aligned}
&\beta_{3i}(\hat{v})=F_{3i}(\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}),\quad i=1,2,3,4,5,\\
&F_{31}(\hat{w})=\int_{\hat{K}}\hat{w}d\xi d\eta-\frac{1}{4}\int_{\partial\hat{K}}\hat{w}ds, |F_{31}(\hat{w})|\le C\|\hat{w}\|_{1,\hat{K}}\le C\|\hat{w}\|_{2,\hat{K}},\\
&F_{32}(\hat{w})=-\int_{\hat{l}_4}\hat{w}ds+\int_{\hat{l}_2}\hat{w}ds,\qquad |F_{32}(\hat{w})|\le C\|\hat{w}\|_{1,\hat{K}}\le C\|\hat{w}\|_{2,\hat{K}},
\end{aligned}$$

$$F_{33}(\hat{w}) = -\int_{\hat{l}_1} \hat{w}ds + \int_{\hat{l}_3} \hat{w}ds, \qquad |F_{33}(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}},$$
$$F_{34}(\hat{w}) = -\int_{\hat{K}} \hat{w}d\xi d\eta + \int_{\hat{l}_4} \hat{w}ds + \int_{\hat{l}_2} \hat{w}ds, \qquad |F_{34}(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}},$$
$$F_{35}(\hat{w}) = -\int_{\hat{K}} \hat{w}d\xi d\eta + \int_{\hat{l}_1} \hat{w}ds + \int_{\hat{l}_3} \hat{w}ds, \qquad |F_{35}(\hat{w})| \le C\|\hat{w}\|_{1,\hat{K}} \le C\|\hat{w}\|_{2,\hat{K}}.$$

Therefore, we have

$$\|\hat{D}^\alpha(\hat{v} - \hat{\Pi}\hat{v})\|_{0,K} \le C|\hat{D}^\alpha \hat{v}|_{2,\hat{K}}, \qquad \forall \hat{v} \in H^4(\hat{K}), \tag{5}$$

which follows the proof.

For the later use, we would like to prove the following lemma.

Lemma 2. *On the reference* $\hat{K}, \forall v_h \in V_h$*, we have*

$$\|\hat{v}_{h\xi\xi\xi\eta}\|_{0,\hat{K}} \le 3\sqrt{5}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\eta\eta\eta}\|_{0,\hat{K}} \le 3\sqrt{5}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}, \tag{6}$$

$$\|\hat{v}_{h\xi\xi\eta}\|_{0,\hat{K}} \le \sqrt{3}\|\hat{v}_{h\xi\xi}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\eta\eta}\|_{0,\hat{K}} \le \sqrt{3}\|\hat{v}_{h\eta\eta}\|_{0,\hat{K}}, \tag{7}$$

$$\|\hat{v}_{h\xi\xi\eta}\|_{0,\hat{K}} \le \sqrt{15}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\eta\eta}\|_{0,\hat{K}} \le \sqrt{15}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}. \tag{8}$$

Proof. We get from (1) that

$$\hat{v}_{h\xi\eta} \in span\{1, \xi, \xi^2, \eta, \eta^2\}. \tag{9}$$

Supposing $\hat{v}_{h\xi\eta} = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\eta + \alpha_4\eta^2$, then

$$\begin{aligned} \|\hat{v}_{h\xi\eta}\|^2_{0,\hat{K}} &= \int_{\hat{K}} |\hat{v}_{h\xi\eta}|^2 d\xi d\eta \\ &= 4\alpha_0^2 + \frac{4}{3}\alpha_1^2 + \frac{4}{5}\alpha_2^2 + \frac{4}{3}\alpha_3^2 + \frac{4}{5}\alpha_4^2 + \frac{8}{3}\alpha_0\alpha_2 + \frac{8}{3}\alpha_0\alpha_4 + \frac{8}{9}\alpha_2\alpha_4 \\ &= 4(\frac{\alpha_2}{3} + \frac{\alpha_4}{3} + \alpha_0)^2 + \frac{4}{3}\alpha_1^2 + \frac{4}{3}\alpha_3^2 + \frac{16}{45}\alpha_2^2 + \frac{16}{45}\alpha_4^2. \end{aligned}$$

Note that $\hat{v}_{h\xi\xi\xi\eta} = 2\alpha_2, \hat{v}_{h\xi\xi\eta} = \alpha_1 + 2\alpha_2\xi$, and

$$\|\hat{v}_{h\xi\xi\xi\eta}\|^2_{0,\hat{K}} = \int_{\hat{K}} |\hat{v}_{h\xi\xi\xi\eta}|^2 d\xi d\eta = 16\alpha_2^2,$$

$$\|\hat{v}_{h\xi\xi\eta}\|^2_{0,\hat{K}} = \int_{\hat{K}} |\hat{v}_{h\xi\xi\eta}|^2 d\xi d\eta = 4\alpha_1^2 + \frac{16}{3}\alpha_2^2,$$

we have

$$\|\hat{v}_{h\xi\xi\xi\eta}\|_{0,\hat{K}} \le 3\sqrt{5}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\xi\eta}\|_{0,\hat{K}} \le \sqrt{15}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}.$$

On the other hand, since $\hat{v}_{h\xi\xi} \in span\{1, \xi, \eta, \xi\eta\}$, we may assume $\hat{v}_{h\xi\xi} = \beta_0 + \beta_1\xi + \beta_2\eta + \beta_3\xi\eta$. Then

$$\|\hat{v}_{h\xi\xi}\|^2_{0,\hat{K}} = \int_{\hat{K}} |\hat{v}_{h\xi\xi}|^2 d\xi d\eta = 4\beta_0^2 + \frac{4}{3}\beta_1^2 + \frac{4}{3}\beta_2^2 + \frac{4}{9}\beta_3^2,$$

$$\|\hat{v}_{h\xi\xi\eta}\|^2_{0,\hat{K}} = \int_{\hat{K}} |\hat{v}_{h\xi\xi\eta}|^2 d\xi d\eta = 4\beta_2^2 + \frac{4}{3}\beta_3^2.$$

Therefore

$$\|\hat{v}_{h\xi\xi\eta}\|_{0,\hat{K}} \leq \sqrt{3}\|\hat{v}_{h\xi\xi}\|_{0,\hat{K}}.$$

By the same argument, we can get

$$\|\hat{v}_{h\xi\eta\eta}\|_{0,\hat{K}} \leq \sqrt{3}\|\hat{v}_{h\eta\eta}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\eta\eta\eta}\|_{0,\hat{K}} \leq 3\sqrt{5}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}, \|\hat{v}_{h\xi\eta\eta}\|_{0,\hat{K}} \leq \sqrt{15}\|\hat{v}_{h\xi\eta}\|_{0,\hat{K}}.$$

This completes the proof.

3. Anisotropic Interpolation Error Estimate

Now, we consider the following the biharmonic equation

$$\begin{cases} \triangle^2 u = f, & in \ \ \Omega, \\ u = \frac{\partial u}{\partial n} = 0, & on \ \ \partial\Omega. \end{cases} \tag{10}$$

The corresponding variational form[6] is

$$\begin{cases} Find \ u \in H_0^2(\Omega), & such \ \ that \\ a(u,v) = f(v), & \forall \ v \in H_0^2(\Omega), \end{cases} \tag{11}$$

where

$$a(u,v) = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy} + v_{yy})dxdy, \qquad f \in L^2(\Omega),$$

$$f(v) = \int_\Omega fv dxdy, \quad H_0^2(\Omega) = \{v \in H^2(\Omega), v|_{\partial\Omega} = \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}.$$

The finite element space is defined as follows

$$\begin{array}{l} V_h = \{v_h | \hat{v}_h = v_h|_K \circ F_K \in \hat{P}, \ \forall K \in J_h, and \\ v_h(a) = v_{hx}(a) = v_{hy}(a) = 0, \quad for \ \ any \ \ node \ \ a \in \partial\Omega\}. \end{array} \tag{12}$$

The finite element approximation of (11) reads as

$$\begin{cases} Find \ u_h \in V_h, & such \ \ that \\ a_h(u_h, v_h) = f(v_h), & \forall \ v_h \in V_h, \end{cases} \tag{13}$$

where

$$a_h(u_h, v_h) = \sum_{K \in J_h} \int_K (u_{hxx}v_{hxx} + 2u_{hxy}v_{hxy} + u_{hyy}v_{hyy})dxdy.$$

Define

$$\| \cdot \|_h = (\sum_{K \in J_h} | \cdot |_{2,K}^2 \ )^{\frac{1}{2}},$$

then it is easy to see that $\| \cdot \|_h$ is the norm over V_h .

Now, we begin to estimate the anisotropic interpolation error .

Theorem 1. *Supposed $u \in H^4(\Omega) \bigcap H_0^2(\Omega)$ be the exact solution of (10). Define the interpolation operator $\Pi_h : H^4(\Omega) \bigcap H_0^2(\Omega) \to V_h, \Pi_h|_K = \Pi_K, \forall K \in J_h$, then*

$$\|u - \Pi_h u\|_h \leq Ch^2|u|_{4,\Omega}. \tag{14}$$

Proof. Let $\alpha = (\alpha_1, \alpha_2), |\alpha| = 2, \quad h_K = (h_x, h_y), h_K^\alpha = h_x^{\alpha_1} h_y^{\alpha_2}$. Then by Lemma 1, we have

$$\|\hat{D}^\alpha(\hat{u} - \hat{\Pi}\hat{u})\|_{0,\hat{K}} \leq C|\hat{D}^\alpha \hat{u}|_{2,\hat{K}} \leq Ch_K^\alpha (h_x h_y)^{-\frac{1}{2}} \sum_{\beta=2} h_K^\beta \|D^{\alpha+\beta}u\|_{0,K}. \tag{15}$$

So

$$\begin{aligned}\|u-\Pi_h u\|_h &= (\sum_{K\in J_h}|u-\Pi_K u|^2_{2,K})^{\frac{1}{2}} = (\sum_{K\in J_h}\sum_{|\alpha|=2}\|D^\alpha(u-\Pi_K u)\|^2_{0,K})^{\frac{1}{2}}\\ &= (\sum_{K\in J_h}\sum_{|\alpha|=2}h_K^{-2\alpha}(h_xh_y)\|\hat{D}^\alpha(\hat{u}-\hat{\Pi}_K\hat{u})\|^2_{0,\hat{K}})^{\frac{1}{2}}\\ &\le C(\sum_{K\in J_h}\sum_{|\alpha|=2,|\beta|=2}h_K^{2\beta}\|D^{\alpha+\beta}u\|^2_{0,K})^{\frac{1}{2}}\\ &\le Ch^2|u|_{4,\Omega},\end{aligned}$$

which completes the proof.

4. Anisotropic Superclose Result and the Optimal Error Estimate

In this section we will focus on obtaining the anisotropic superclose result of ACM's element. Firstly, we prove the following lemma.

Lemma 3. *Assume u satisfies the same assumption as in theorem 1 and u_h be the ACM finite element solution of (13). Then under anisotropic rectangular meshes, $\forall v_h\in V_h$, we have*

$$a_h(u_h-u,v_h)\le Ch^2|u|_{4,\Omega}\|v_h\|_h. \tag{16}$$

Proof. By Green's formula, and noticing that $v_h\in C^0(\bar{\Omega}), v_h|_{\partial\Omega}=0$, we have

$$\begin{aligned}a_h(u_h,v_h) &= f(v_h)=(\Delta^2u,v_h)=\sum_{K\in J_h}\int_{\partial K}\frac{\partial(\triangle u)}{\partial n}v_hds-(\nabla(\triangle u),\nabla v_h)\\ &= 0+\sum_{K\in J_h}\int_K\triangle u\triangle v_hdxdy-\sum_{K\in J_h}\int_{\partial K}\triangle u\frac{\partial v_h}{\partial n}ds\\ &= \sum_{K\in J_h}\int_K(u_{xx}v_{hxx}+u_{xy}v_{hxy})dxdy-\sum_{K\in J_h}(\int_{l_2}-\int_{l_4})(u_{xx}v_{hx}+u_{xy}v_{hy})dy\\ &\quad+\sum_{K\in J_h}\int_K(u_{xy}v_{hxy}+u_{yy}v_{hyy})dxdy-\sum_{K\in J_h}(\int_{l_3}-\int_{l_1})(u_{xy}v_{hx}+u_{yy}v_{hy})dy\\ &= a_h(u,v_h)-\sum_{K\in J_h}(\int_{l_2}-\int_{l_4})(u_{xx}v_{hx}+u_{xy}v_{hy})dy\\ &\quad-\sum_{K\in J_h}(\int_{l_3}-\int_{l_1})(u_{xy}v_{hx}+u_{yy}v_{hy})dy.\end{aligned} \tag{17}$$

Let I_h be the bilinear interpolate operator. Then it is easy to see that I_hv_{hx} is continuous in Ω and $I_hv_{hx}|_{\partial\Omega}=0$. Noticing that v_{hy} is continuous on the sides parallelling to x-axis and $v_{hy}|_{\partial\Omega}=0$, we get

$$\sum_{K\in J_h}(\int_{l_2}-\int_{l_4})(u_{xx}v_{hx}+u_{xy}v_{hy})dy=\sum_{K\in J_h}(\int_{l_2}-\int_{l_4})u_{xx}(v_{hx}-I_hv_{hx})dy. \tag{18}$$

On $\hat{l}_2,\hat{l}_4$ of the reference element $\hat{K}$, we get that

$$\hat{v}_{h\xi}(\pm1,\eta)-\hat{I}_h(\hat{v}_{h\xi}(\pm1,\eta))=\frac{1}{2}(\eta^2-1)\hat{v}_{h\xi\eta\eta}(\pm1,\eta)-\frac{1}{3}(\eta^2-1)\eta\hat{v}_{h\xi\eta\eta\eta}. \tag{19}$$

So, on a general element K, we have

$$\begin{aligned} &v_{hx}(x_K \pm h_x, y) - I_h(v_{hx}(x_K \pm h_x, y)) \\ &= -\tfrac{1}{2}((y-y_K)^2 - h_y)v_{hxyy}(x_K \pm h_x, y)) - \tfrac{1}{3}((y-y_K)^2 - h_y)(y-y_K)v_{hxyyy} \\ &= F(y)v_{hxyy}(x_K \pm h_x, y)) - \tfrac{2}{3}F(y)F'(y)v_{hxyyy}. \end{aligned} \tag{20}$$

Noticing that $v_{hxxyy} = v_{hxxyyy} = 0$, then

$$\begin{aligned} (\int_{l_2} - \int_{l_4})u_{xx}(v_{hx} - I_h v_{hx})dy &= (\int_{l_2} - \int_{l_4})u_{xx}(F(y)v_{hxyy} - \frac{2}{3}F(y)F'(y)v_{hxyyy})dy \\ &= \int_K (u_{xxx}(F(y)v_{hxyy} - \frac{2}{3}F(y)F'(y)v_{hxyyy})dxdy. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{K\in J_h} (\int_{l_2} - \int_{l_4})(u_{xx}v_{hx} + u_{xy}v_{hy})dy \\ &= -\sum_{K\in J_h} \int_K F(y)u_{xxxx}v_{hyy}dxdy + \tfrac{1}{3}\sum_{K\in J_h}\int_K F^2(y)u_{xxxy}v_{hxyyy}dxdy. \end{aligned} \tag{21}$$

By the same argument, we get that

$$\begin{aligned} &\sum_{K\in J_h} (\int_{l_3} - \int_{l_1})(u_{xy}v_{hx} + u_{yy}v_{hy})dx \\ &= -\sum_{K\in J_h} \int_K E(x)u_{yyyy}v_{hxx}dxdy + \tfrac{1}{3}\sum_{K\in J_h}\int_K E^2(x)u_{xyyy}v_{hxxxy}dxdy. \end{aligned} \tag{22}$$

By (17)-(22) and (6), we can obtain that

$$\begin{aligned} a_h(u_h - u, v_h) &= \sum_{K\in J_h}\int_K F(y)u_{xxxx}v_{hyy}dxdy + \sum_{K\in J_h}\int_K E(x)u_{yyyy}v_{hxx}dxdy \\ &- \tfrac{1}{3}\sum_{K\in J_h}\int_K F^2(y)u_{xxxy}v_{hxyyy}dxdy - \tfrac{1}{3}\sum_{K\in J_h}\int_K E^2(x)u_{xyyy}v_{hxxxy}dxdy \\ &\le C\sum_{K\in J_h}(h_y^2|u|_{4,K}|v_h|_{2,K} + h_x^2|u|_{4,K}|v_h|_{2,K} \\ &+ h_y^4|u|_{4,K}\|v_{hxyyy}\|_{0,K} + h_x^4|u|_{4,K}\|v_{hxxxy}\|_{0,K}) \\ &\le C\sum_{K\in J_h}(h_y^2|u|_{4,K}|v_h|_{2,K} + h_x^2|u|_{4,K}|v_h|_{2,K} \\ &+ h_y^2|u|_{4,K}\|v_{hxy}\|_{0,K} + h_x^2|u|_{4,K}\|v_{hxy}\|_{0,K}) \le Ch^2|u|_{4,\Omega}\|v_h\|_h. \end{aligned} \tag{23}$$

The proof is completed.

Based on Lemma 3, it is not hard to get the following superclose result.

Theorem 2. *Suppose u satisfies the same assumption as in Theorem 1. Then on anisotropic meshes, we have*

$$\|\Pi_h u - u_h\|_h \le Ch^2|u|_{4,\Omega}. \tag{24}$$

Proof. It is easy to prove that[12]:

$$\|\Pi_h u - u_h\|_h^2 \leq a_h(\Pi_h u - u_h, \Pi_h u - u_h). \tag{25}$$

On the other hand,

$$a_h(\Pi_h u - u_h, \Pi_h u - u_h) = a_h(\Pi_h u - u, \Pi_h u - u_h) + a_h(u - u_h, \Pi_h u - u_h). \tag{26}$$

Noticing that $\Pi_h u - u_h \in V_h$, by Lemma 3 , we have

$$a_h(\Pi_h u - u, \Pi_h u - u_h) \leq Ch^2 |u|_{4,\Omega} \|\Pi_h u - u_h\|_h. \tag{27}$$

By Theorem 1,

$$a_h(u - u_h, \Pi_h u - u_h) \leq Ch^2 |u|_{4,\Omega} \|\Pi_h u - u_h\|_h. \tag{28}$$

Then (24) follows from (25)-(28).

Furthermore, based on the above Theorem 1 and Theorem 2, we can get the following optimal anisotropic energy norm error estimate of ACM's element.

Theorem 3. *Suppose u satisfies the same assumption as in Theorem 1. Then on anisotropic meshes we have*

$$\|u - u_h\|_h \leq Ch_{max}^2 |u|_{4,\Omega}. \tag{29}$$

Proof. By triangle inequality,

$$\|u - u_h\|_h \leq \|u - \Pi_h u\|_h + \|\Pi_h u - u_h\|_h. \tag{30}$$

Then by Theorem 1 and Theorem 2, the proof is completed.

5. Numerical Examples

In order to investigate the numerical behavior of anisotropic ACM's element, we still consider the biharmonic equation (10) with $f(x,y) = 8\pi^4 \cos(2\pi x)\cos(2\pi y) - 8\pi^4 \cos(2\pi x)$ $\sin^2(\pi y) - 8\pi^4 \cos(2\pi y)\sin^2(\pi x) \in L^2(\Omega)$, and $\Omega = (-1,1)\times(-1,1)$. It can be verified that the exact solution of problem (10) is $u(x,y) = \sin^2(\pi x)\sin^2(\pi y)$. Denote the rectangular meshes of Ω by J_h, $h = \max\limits_{K\in J_h} h_K$, $\rho = \max\limits_{K\in J_h} \rho_K$, $h_K = diam(K)$, $\rho_K = \max\limits_{M} diam(M)$, and M is an arbitrary circle contained in K. Let u_h be the the ACM's element solution of problem (13).

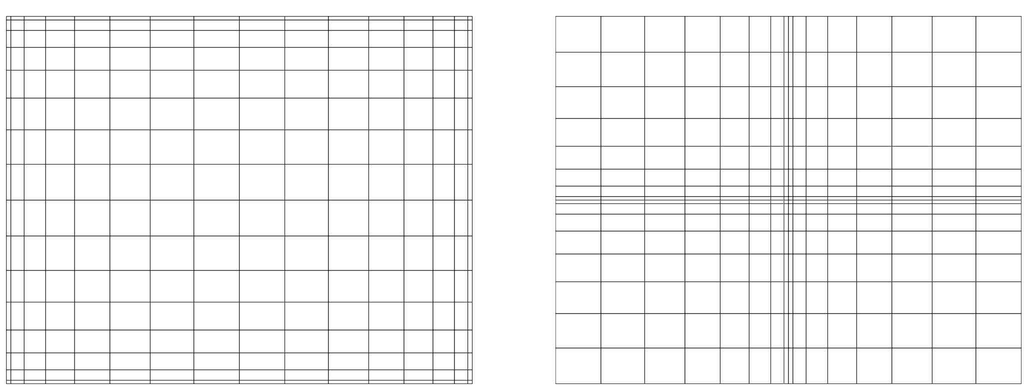

Figure 1: the anisotropic meshes of Ω for $n = 16$(left: mesh 1, right: mesh 2)

We take the following two types of mesh on $\Omega = [0,1]^2$: mesh 1 and mesh 2 (see Figure 1). To obtain mesh 1, we subdivide the boundary of Ω which parallel to x-axis into m parts by the following $m+1$ points: $sin(\frac{i\pi}{m})/2,\ i = 0, 1, ..., \frac{m}{2},\ (1 - cos(\frac{i\pi}{m}))/2,\ i = \frac{m}{2} + 1, ...,\ m$ and n intervals along $y - axis$ by the $n+1$ points: $sin(\frac{i\pi}{n})/2,\ i = 0, 1, ...,\ \frac{n}{2},\ (1 - cos(\frac{i\pi}{n}))/2,\ i = \frac{n}{2} + 1, ...,\ n$. For mesh 2, we subdivide the boundary of Ω ,which parallel to x-axis, into m parts by the following $m+1$ points: $(1 - cos(\frac{i\pi}{m}))/2,\ i = 0, 1, ..., \frac{m}{2},\ (1 + sin(\frac{i\pi}{m} - \frac{\pi}{2}))/2,\ i = \frac{m}{2} + 1, ..., m$ and n intervals along $y - axis$ by the $n+1$ points: $(1 - cos(\frac{i\pi}{n}))/2,\ i = 0, 1, ..., \frac{n}{2},\ (1 + sin(\frac{i\pi}{n} - \frac{\pi}{2}))/2,\ i = \frac{n}{2} + 1, ..., n$.

The numerical results are listed in Table 1 and Table 2. Here, α denotes the convergence order.

Table 1. The errors $\|\Pi_h u - u_h\|_h$ and $\|u - u_h\|_h$ (mesh 1)

$m \times n$	8×8	16×16	32×32	64×64	128×128
$\|\Pi_h u - u_h\|_h$	1.5598175379	0.4159846531	0.1054176569	0.0264402926	0.0066234110
α	\	1.9067749977	1.9804137945	1.99513064919	1.9979852361
$\|u - u_h\|_h$	2.0730948783	0.5518639451	0.1398930088	0.0350916579	0.0087871245
α	\	1.9094015360	1.9799888134	1.9951238632	1.9976650476
$\max_{K\in J_h} h_K$	0.270598	0.137950	0.069309	0.034696	0.017353
$\max_{K\in J_h}\{h_K/\rho_K\}$	7.109732	14.358751	28.786978	57.608674	115.234703
$\max_{K\in J_h}\{h/h_K\}$	5.027339	10.53170	20.355408	40.735484	81.483240

Table 2. The errors $\|\Pi_h u - u_h\|_h$ and $\|u - u_h\|_h$ (mesh 2)

$m \times n$	8×8	16×16	32×32	64×64	128×128
$\|\Pi_h u - u_h\|_h$	1.4101900778	0.3760484456	0.0953962667	0.0239339392	0.0059887510
α	\	1.9068992138	1.9789137840	1.9948749542	1.9987307787
$\|u - u_h\|_h$	2.0227487338	0.5348153631	0.1354684092	0.0339762842	0.0085008748
α	\	1.9192042351	1.9810844660	1.9953565598	1.9988447428
$\max_{K\in J_h} h_K$	0.270598	0.137950	0.069309	0.034696	0.017353
$\max_{K\in J_h}\{h_K/\rho_K\}$	7.109732	14.358751	28.786978	57.608674	115.234703
$\max_{K\in J_h}\{h/h_K\}$	5.027339	10.53170	20.355408	40.735484	81.483240

From the above two tables, we can see that with the increasing of the number of meshes, $\max_{K\in J_h}\{h_K/\rho_K\}$ and $\max_{K\in J_h}\{h/h_K\}$ increase sharply. However, the order of the errors $\|\Pi_h u - u_h\|_h$ and $\|u - u_h\|_h$ are approximating to 2, which show that the regularity assumption and inverse assumption on the meshes are not necessary conditions for the optimal order of the convergence of ACM's element for biharmonic equation. In fact, we can get sharper meshes than those in the above two tables as the following two tables, but the optimal orders of the errors are obtained too.

Table 3. The errors $\|\Pi_h u - u_h\|_h$ and $\|u - u_h\|_h$ (mesh 1)

$m \times n$	8×48	16×96	32×192	64×384
$\|\Pi_h u - u_h\|_h$	0.7860449921	0.1928776907	0.0477986851	0.0119093725
α	\	2.0269255638	2.0125789642	2.0048735291
$\|u - u_h\|_h$	1.0918419990	0.2696962329	0.0671754592	0.0167830123
α	\	2.0173568231	2.0053291873	2.0009325938
$\max_{K\in J_h} h_K$	0.194116	0.098907	0.049687	0.024873
$\max_{K\in J_h}\{h_K/\rho_K\}$	252.768191	515.302439	1035.522221	2073.507418
$\max_{K\in J_h}\{h/h_K\}$	5.098217	10.290977	20.629149	41.281885

Table 4. The errors $\|\Pi_h u - u_h\|_h$ and $\|u - u_h\|_h$ (mesh 2)

$m \times n$	8×48	16×96	32×192	64×384
$\|\Pi_h u - u_h\|_h$	0.6129853474	0.1439701267	0.0352935480	0.0087788713
α	\	2.0900831223	2.0282931328	2.0072972775
$\|u - u_h\|_h$	0.9927046569	0.2470434794	0.0616648662	0.0154113710
α	\	2.0065996647	2.0022442341	2.0004534721
$\max_{K\in J_h} h_K$	0.194116	0.098907	0.049687	0.024873
$\max_{K\in J_h}\{h_K/\rho_K\}$	252.768191	515.302439	1035.522221	2073.507418
$\max_{K\in J_h}\{h/h_K\}$	5.098217	10.290977	20.629149	41.281885

Acknowledgements. The authors are very thankful to the anonymous referees for their helpful suggestions and corrections of the English and typesetting mistakes.

References

[1] T. Apel, M. Dobrowolski, Anisotropic Interpolation with Applications to the Finite Element Method, *Computing*, **47** (1992), 277-293.

[2] T. Apel, Anisotropic Finite Element: Local Estimates and Approximations, B.G. Teubner Leipzig, 1999.

[3] C.M. Chen, Finite Method and Its Analysis in Improving Accuary, Hunan Science Press, 1982.

[4] H.S. Chen, B. Li, Superconvergence Analysis and Error Expansion for the Wilson Nonconforming Finite Element, *Numer. Math.*, **69** (1994), 125-140.

[5] S.C. Chen, D. Y. Shi, Y. C. Zhao, Anisotropic Interpolation and Quasi-Wilson Element for Narrow Quadrilateral Meshes, *IMA. J. Numer. Anal.*, **24** (2004), 77-95.

[6] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, New York, Oxford, 1978.

[7] M. Křížek, P. Neittaanmäki, On Superconvergence Techniques, *Acta Applicandae Mathematicae*, **9** (1987), 175-198.

[8] P. Lascaux, P. Lesaint, Some Nonconforming Finite Elements for the Plate Bending Problem, RAIRO, R-1, (1975), 9-53.

[9] Q. Lin, N. Yan, The Construction and Analysis of High Efficient Elements, HeBei University Press, 1996.

[10] P.B. Ming, Z.C. Shi, Convergence Analysis for Quadrilateral Rotated Q_1 Element, Advances in Computation: Theory and Practice, Scientific Computing and Applications, eds: Peter Minev and Yanping Lin, Nova Science Publications, Inc., **7** (2001), 115-124.

[11] Z.C. Shi, B. Jiang, W.M. Xue, A New Property of Wilson Nonconforming Finite Element, *Numer. Math.*, **78** (1997), 259-268.

[12] M. Wang, L^∞ Error Estimates of Nonconforming Finite Elements for the Biharmonic Equation, *J. Comp. Math.*, **11** (1993), 276-288.

[13] A. Zenisek, M. Vanmaele, The Interpolation Theorem for Narrow Quadrilateral Isoparametric Finite Elements, *Numer. Math.*, **72** (1995), 123-141.

[14] A. Zenisek, M. Vanmaele, Applicability of the Bramble-Hilbert Lemma in Interpolation Problems of Narrow Quadrilateral Isoparametric Finite Elements, *J. Comp. Appl. Math.*, **63** (1995), 109-122.

[15] L. Zhang, L. K. Li, Some Superconvergence Results of Wilson-like Elements. *J. Comp. Math.*, **16**:1 (1998), 81-96.

[16] Q.D. Zhu, Q. Lin, The Superconvergence Theory of Finite Element Methods, Hunan Science and Technology Press, 1989.