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ARTIFICIAL BOUNDARY METHOD FOR THE THREE-DIMENSIONAL EXTERIOR PROBLEM OF ELASTICITY*1)

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Abstract

The exact boundary condition on a spherical artificial boundary is derived for the three-dimensional exterior problem of linear elasticity in this paper. After this boundary condition is imposed on the artificial boundary, a reduced problem only defined in a bounded domain is obtained. A series of approximate problems with increasing accuracy can be derived if one truncates the series term in the variational formulation, which is equivalent to the reduced problem. An error estimate is presented to show how the error depends on the finite element discretization and the accuracy of the approximate problem. In the end, a numerical example is given to demonstrate the performance of the proposed method.

Mathematics subject classification: 65N30
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1. Introduction

Numerical approximation to the solutions of PDEs in unbounded domains has attracted much attention of engineers and mathematicians in the last three decades. Many effective and efficient methods have been proposed for different problems arising from various research areas. Among them is the so-called artificial boundary method. The key point of this method is to limit the computational domain by introducing a proper artificial boundary in the exterior unbounded domain and imposing a suitable boundary condition on the artificial boundary, to ensure the well-posedness of the reduced problem.

Engquist and Majda [7], Bayliss and Turkel [4] considered first-order hyperbolic equations and other wave-like equations; Han and Wu [19], Yu [25] designed various types of artificial boundary conditions for the exterior Laplace equation; Feng [8], Goldstein [13], Deakin and Rasmussen [6] obtained the nonreflecting boundary conditions for reduced wave equation; Halpern and Schatzman [16], Han and Bao [17] discussed the incompressible flow in a channel; Grote and Keller [14], Alpert, Greengard and Hagstrom [1] considered the exterior problem of time-dependent hyperbolic equation.

For linear elastic problem, Givoli and Keller [11], Han and Wu [19, 20] designed artificial boundary condition on a circular artificial boundary for two-dimensional case. In addition, Han and Bao gave an error analysis in [18] for this problem. For the time-harmonic elastic wave in two dimensions, Givoli and Keller [12] derived the artificial boundary conditions. Grote

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and Keller [15] obtained the exact nonreflecting boundary conditions for elastic waves in three dimensions.

In this paper, we concentrate on the exterior problem of elasticity in three dimensions. We obtain an exact boundary condition on the spherical artificial boundary. This boundary condition is usually called D_tN mapping or D_tN artificial boundary condition. This approach has been used by Han and Wu to solve the exterior Laplace equation and elastic problem in two dimensions (see formulation 2.10 on page 181 of [19]). In the paper by Givoli and Keller [11], they presented the D_tN artificial boundary condition for the exterior problem of Laplace equation in three-dimension and linear elastic problem in two dimensions. For more information on this approach, the reader is referred to the review papers by Givoli [9, 10] and Tsynkov [24].

This paper is organized as follows. In section 2, some results on vectorial spherical harmonics in [21] are listed. In section 3, an exact artificial boundary condition is designed on the spherical artificial boundary. The equivalent variational problem to the reduced problem is formulated in section 4. In section 5, the error analysis is presented. This error estimate is dependent not only on finite element discretization, but on the accuracy of approximate variational formulation. A numerical example is presented in section 6 to show the performance of our method. This paper concludes in section 7.

2. Some Results on Vectorial Spherical Harmonics

It is well-known that the spherical harmonic functions $\{Y_l^m, l \geq 0, -l \leq m \leq l\}$ constitutes an orthogonal basis of space $L^2(S)$, where S denotes the surface of unit sphere (see page 24 in [21]). Let $\mathbf{x}$ be the location vector, $r = |\mathbf{x}|$ and $H_l^m = r^l Y_l^m$, then $\{H_l^m, -l \leq m \leq l\}$ constitutes a basis of all l-order homogeneous harmonic polynomials. We define

$$
\begin{aligned}
\mathcal{I}_l^m &\equiv \nabla H_{l+1}^m, \quad l \geq 0, \quad -(l+1) \leq m \leq l+1, \\
\mathcal{T}_l^m &\equiv \nabla H_l^m \times \mathbf{x}, \quad l \geq 1, \quad -l \leq m \leq l, \\
\mathcal{N}_l^m &\equiv (2l-1) H_{l-1}^m \mathbf{x} - r^2 \nabla H_{l-1}^m, \quad l \geq 1, \quad -(l-1) \leq m \leq (l-1)
\end{aligned}
$$

and denote by $\mathbf{I}_l^m$, $\mathbf{T}_l^m$ and $\mathbf{N}_l^m$ the traces of these functions on S, i.e.

$$
\mathbf{I}_l^m = \frac{\mathcal{I}_l^m}{r^l}, \ \mathbf{T}_l^m = \frac{\mathcal{T}_l^m}{r^l}, \ \mathbf{N}_l^m = \frac{\mathcal{N}_l^m}{r^l}.
$$

These functions are called l-order *vectorial spherical harmonics.*

Lemma 2.1. *Let* $\mathbf{n} = \dfrac{\mathbf{x}}{r}$ *be the unit vector in the radial direction, then the following hold*

$$
\begin{aligned}
&\nabla\left(\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}}\right) = \frac{(l+1)(2l+1)}{r^{l+3}} \mathbf{N}_{l+2}^m, \\
&\nabla\left(\nabla \cdot \frac{\mathbf{T}_l^m}{r^{l+1}}\right) = 0, \\
&\nabla\left(\nabla \cdot \frac{\mathbf{N}_l^m}{r^{l+1}}\right) = 0, \\
&\left(\nabla \cdot \frac{\mathbf{I}_l^m}{r^{l+1}}\right)\mathbf{n} = -\frac{1}{r^{l+2}}\left\{\frac{(l+1)(2l+1)}{2l+3}\mathbf{I}_l^m + \frac{(l+1)(2l+1)}{2l+3}\mathbf{N}_{l+2}^m\right\}, \\
&\left(\nabla \cdot \frac{\mathbf{T}_l^m}{r^{l+1}}\right)\mathbf{n} = 0, \\
&\left(\nabla \cdot \frac{\mathbf{N}_l^m}{r^{l+1}}\right)\mathbf{n} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&\nabla \frac{\mathbf{I}_l^m}{r^{l+1}} \cdot \mathbf{n} = -\frac{1}{r^{l+2}} \left\{ \frac{1}{2l+3} \mathbf{I}_l^m + \frac{(l+1)(2l+1)}{2l+3} \mathbf{N}_{l+2}^m \right\}, \\
&\nabla \frac{\mathbf{T}_l^m}{r^{l+1}} \cdot \mathbf{n} = -\frac{1}{r^{l+2}} \mathbf{T}_l^m, \\
&\nabla \frac{\mathbf{N}_l^m}{r^{l+1}} \cdot \mathbf{n} = -\frac{l+1}{r^{l+2}} \mathbf{N}_l^m.
\end{aligned}
$$

The proof can be found in the appendix of [26].

Lemma 2.2. *The families* $(\mathbf{I}_l^m, \mathbf{T}_l^m, \mathbf{N}_l^m)$ *for all* $l \geq 0$ *form an orthogonal basis of multiplied space* $\mathbf{L}^2(S) \equiv L^2(S) \times L^2(S) \times L^2(S)$. *Moreover, we have*

$$
\begin{aligned}
\int_S |\mathbf{I}_l^m|^2 ds &= (l+1)(2l+3), \ l \geq 0, \\
\int_S |\mathbf{T}_l^m|^2 ds &= l(l+1), \ l \geq 1, \\
\int_S |\mathbf{N}_l^m|^2 ds &= l(2l-1), \ l \geq 1.
\end{aligned}
$$

This lemma can be found on page 37 of [21].

3. Exact Artificial Boundary Condition

We consider the following exterior problem of elasticity in three dimensions

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \mathbf{f}, \quad \text{in } \Omega_e, \tag{3.1}$$

$$\mathbf{u} = \mathbf{g}, \quad \text{on } \Gamma, \tag{3.2}$$

$$\mathbf{u} \longrightarrow \mathbf{0}, \quad \text{when} \quad r \longrightarrow +\infty. \tag{3.3}$$

Here Γ is a closed surface with sufficient regularity; Ω_e is the unbounded domain with boundary Γ; λ and μ are two Láme constants; $\mathbf{f}$, the body force function, is assumed to be compact; $\mathbf{g}$ is a given vectorial function on Γ. Let $\Gamma_\rho = \{\mathbf{x} | r = \rho\}$ and $S_\rho = \{\mathbf{x} | r < \rho\}$ denote the spherical surface and solid spherical domain with radius ρ in the whole paper. Since $\mathbf{f}$ is compact, there exists a constant $R_0 > 0$, such that $\Gamma_{R_0} \subset \Omega_e$ and $Supp(\mathbf{f}) \subset S_{R_0}$.

Introducing a spherical artificial boundary Γ_R with $R > R_0$, we have $\Gamma_R \subset \Omega_e$. Γ_R divides Ω_e into two parts: the bounded part $\Omega_R = \Omega_e \cap S_R$ and the unbounded part $\Omega_R^e = \Omega_e \backslash \bar{\Omega}_R$.

To derive the artificial boundary condition of the given problem (3.1)-(3.3), we consider its solution $\mathbf{u}$ on Ω_R^e. Apparently, by our assumption, it satisfies

$$-\mu \Delta \mathbf{u} - (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) = \mathbf{0}, \quad \text{in } \Omega_R^e, \tag{3.4}$$

$$\mathbf{u} \longrightarrow \mathbf{0}, \quad \text{when} \quad r \longrightarrow +\infty. \tag{3.5}$$

Since no boundary condition is imposed on S_R, problem (3.4)-(3.5) is incomplete and cannot be solved independently. But if $\mathbf{u}(R, \theta, \varphi)$ is prescribed, its solution can be derived.

Analogous to the idea of Kelvin (see page 351 in [23]), we decompose $\mathbf{u}$ as

$$\mathbf{u} = \sum_{l=0}^{+\infty} \mathbf{u}_l, \tag{3.6}$$

where

$$\mathbf{u}_l = \mathbf{G}_l + c_l (r^2 - R^2) \nabla (\nabla \cdot \mathbf{G}_l) \tag{3.7}$$

is supposed to satisfy Navier equation (3.4), and $\mathbf{G}_l$ is a vectorial harmonic function, such that $r^{2l+1} \mathbf{G}_l$ is an l-order homogeneous vectorial harmonic polynomial. c_l is some constant to be determined. Since

$$\begin{aligned}\nabla\cdot\mathbf{u}_l &= \nabla\cdot\mathbf{G}_l + c_l\Big\{2\mathbf{r}\cdot\nabla(\nabla\cdot\mathbf{G}_l) + (r^2-R^2)\Delta(\nabla\cdot\mathbf{G}_l)\Big\}\\ &= \nabla\cdot\mathbf{G}_l + 2c_l\Big\{-(l+2)\Big\}\nabla\cdot\mathbf{G}_l = \Big\{1-(2l+4)c_l\Big\}\nabla\cdot\mathbf{G}_l \end{aligned}\tag{3.8}$$

and

$$\begin{aligned}\Delta\mathbf{u}_l &= \Delta\mathbf{G}_l + c_l\Big\{\Delta(r^2-R^2)\nabla(\nabla\cdot\mathbf{G}_l)\\ &\qquad +2\nabla(r^2-R^2)\cdot\nabla\nabla(\nabla\cdot\mathbf{G}_l) + (r^2-R^2)\Delta\nabla(\nabla\cdot\mathbf{G}_l)\Big\}\\ &= c_l\Big\{6\nabla(\nabla\cdot\mathbf{G}_l) + 4\mathbf{r}\cdot\nabla\nabla(\nabla\cdot\mathbf{G}_l)\Big\}\\ &= c_l\Big\{6\nabla(\nabla\cdot\mathbf{G}_l) - 4(l+3)\nabla(\nabla\cdot\mathbf{G}_l)\Big\}\\ &= -(4l+6)c_l\nabla(\nabla\cdot\mathbf{G}_l), \end{aligned}\tag{3.9}$$

from (3.4), we have

$$-\mu\Delta\mathbf{u}_l - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}_l) = \Big\{(4l+6)c_l\mu - \Big[1-(2l+4)c_l\Big](\lambda+\mu)\Big\}\nabla(\nabla\cdot\mathbf{G}_l) = 0.$$

Thus then

$$c_l = \frac{\lambda+\mu}{(2l+4)\lambda+(6l+10)\mu}\ .$$

We expand $\mathbf{u}$ on Γ_R as

$$\mathbf{u} = \sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1} A_l^m\mathbf{I}_l^m + \sum_{l=1}^{+\infty}\sum_{m=-l}^{l} B_l^m\mathbf{T}_l^m + \sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1} C_l^m\mathbf{N}_l^m,$$

where

$$\begin{aligned} A_l^m &= \frac{1}{(l+1)(2l+3)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\overline{\mathbf{I}_l^m}ds,\\ B_l^m &= \frac{1}{l(l+1)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\overline{\mathbf{T}_l^m}ds,\\ C_l^m &= \frac{1}{l(2l-1)}\int_S \mathbf{u}(R,\theta,\phi)\cdot\overline{\mathbf{N}_l^m}ds \end{aligned}$$

are named as *Fourier coefficients* of $\mathbf{u}$ on Γ_R. Set

$$\mathbf{G}_l = \begin{cases} \left(\dfrac{R}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1} A_l^m\mathbf{I}_l^m, l=0,\\ \left(\dfrac{R}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1} A_l^m\mathbf{I}_l^m + \left(\dfrac{R}{r}\right)^{l+1}\sum_{m=-l}^{l} B_l^m\mathbf{T}_l^m + \left(\dfrac{R}{r}\right)^{l+1}\sum_{m=-(l-1)}^{l-1} C_l^m\mathbf{N}_l^m, l>0, \end{cases}\tag{3.10}$$

and it is direct to verify that formula (3.6) gives the solution of problem (3.4)-(3.5) with boundary value $\mathbf{u}(R,\theta,\varphi)$.

Now we compute the normal stress on Γ_R. Let $\sigma(\mathbf{u})$ be the stress tensor. It is related with the displacement field $\mathbf{u}$ by

$$\sigma(\mathbf{u}) = \lambda(\nabla\cdot\mathbf{u})\mathbf{I} + 2\mu\epsilon(\mathbf{u}),$$

where $\mathbf{I}$ denotes the second order unit tensor and $\epsilon(\mathbf{u})$ is the strain tensor defined by

$$\epsilon(\mathbf{u}) = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2}.$$

On Γ_R, we have

$$\begin{aligned}
\nabla\mathbf{u}_l &= \nabla\mathbf{G}_l + c_l\Big\{\nabla(r^2-R^2)\otimes\nabla(\nabla\cdot\mathbf{G}_l) + (r^2-R^2)\nabla\nabla(\nabla\cdot\mathbf{G}_l)\Big\}\\
&= \nabla\mathbf{G}_l + 2c_l\mathbf{r}\otimes\nabla(\nabla\cdot\mathbf{G}_l),\\
2\epsilon(\mathbf{u}_l) &= \nabla\mathbf{u}_l + (\nabla\mathbf{u}_l)^T = \nabla\mathbf{G}_l + (\nabla\mathbf{G}_l)^T + 2c_l\mathbf{r}\otimes\nabla(\nabla\cdot\mathbf{G}_l) + 2c_l\nabla(\nabla\cdot\mathbf{G}_l)\otimes\mathbf{r}.
\end{aligned}$$

Thus, by the definition of stress tensor, we have

$$\begin{aligned}
\mathbf{n}\cdot\sigma(\mathbf{u}_l)|_{\Gamma_R} &= \lambda(\nabla\cdot\mathbf{u}_l)\mathbf{n} + \mu\mathbf{n}\cdot 2\epsilon(\mathbf{u}_l)\\
&= \lambda\Big\{1-(2l+4)c_l\Big\}(\nabla\cdot\mathbf{G}_l)\mathbf{n} - \frac{(l+1)\mu}{R}\mathbf{G}_l + \mu\nabla\mathbf{G}_l\cdot\mathbf{n}\\
&\qquad +2c_l\mu R\nabla(\nabla\cdot\mathbf{G}_l) - c_l(2l+4)\mu(\nabla\cdot\mathbf{G}_l)\mathbf{n}\\
&= \frac{(2l+2)\lambda\mu-(2l+4)\mu^2}{(6l+10)\mu+(2l+4)\lambda}(\nabla\cdot\mathbf{G}_l)\mathbf{n} - \frac{(l+1)\mu}{R}\mathbf{G}_l\\
&\qquad + \frac{2\lambda\mu+2\mu^2}{(6l+10)\mu+(2l+4)\lambda}R\nabla(\nabla\cdot\mathbf{G}_l) + \mu\nabla\mathbf{G}_l\cdot\mathbf{n}.
\end{aligned}$$

From lemma 2.1, a simple computation shows that

1. if $\mathbf{G}_l = \dfrac{\mathbf{I}_l^m}{r^{l+1}}$, we have

$$\begin{aligned}
\mathbf{n}\cdot\sigma(\mathbf{u}_l)|_{\Gamma_R} &= \frac{(2l+2)\lambda\mu-(2l+4)\mu^2}{(6l+10)\mu+(2l+4)\lambda}\Big\{-\frac{(l+1)(2l+1)}{2l+3}\frac{\mathbf{I}_l^m}{R^{l+2}} - \frac{(l+1)(2l+1)}{2l+3}\frac{\mathbf{N}_{l+2}^m}{R^{l+2}}\Big\}\\
&\quad -\frac{(l+1)\mu}{R^{l+2}}\mathbf{I}_l^m + \frac{2\lambda\mu+2\mu^2}{(6l+10)\mu+(2l+4)\lambda}\frac{(l+1)(2l+1)}{R^{l+2}}\mathbf{N}_{l+2}^m\\
&\quad +\frac{\mu}{R^{l+2}}\Big\{-\frac{1}{2l+3}\mathbf{I}_l^m - \frac{(l+1)(2l+1)}{2l+3}\mathbf{N}_{l+2}^m\Big\}\\
&= -\frac{1}{R^{l+2}}\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu\mathbf{I}_l^m;
\end{aligned}$$

2. if $\mathbf{G}_l = \dfrac{\mathbf{T}_l^m}{r^{l+1}}$, we have

$$\mathbf{n}\cdot\sigma(\mathbf{u}_l)|_{\Gamma_R} = -\frac{l+2}{R^{l+2}}\mu\mathbf{T}_l^m;$$

3. if $\mathbf{G}_l = \dfrac{\mathbf{N}_l^m}{r^{l+1}}$, we have

$$\mathbf{n}\cdot\sigma(\mathbf{u}_l)|_{\Gamma_R} = -\frac{2l+2}{R^{l+2}}\mu\mathbf{N}_l^m.$$

Combining all the components, we have

$$\begin{aligned}\mathbf{n}\cdot\sigma(\mathbf{u})|_{\Gamma_R} &= -\frac{1}{R}\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu A_l^m\mathbf{I}_l^m\\ &\quad -\frac{1}{R}\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+2)\mu B_l^m\mathbf{T}_l^m\\ &\quad -\frac{1}{R}\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(2l+2)\mu C_l^m\mathbf{N}_l^m\\ &\equiv \mathcal{K}_\infty(\mathbf{u}).\end{aligned}$$

This is an exact non-local artificial boundary condition for the considered exterior problem (3.1)-(3.3). Imposing it on the spherical artificial boundary Γ_R, we obtain a reduced problem only defined in the bounded domain Ω_R

$$-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})=\mathbf{f},\quad \text{in } \Omega_R, \tag{3.11}$$

$$\mathbf{u}=\mathbf{g},\quad \text{on } \Gamma, \tag{3.12}$$

$$\mathbf{n}\cdot\sigma(\mathbf{u})=\mathcal{K}_\infty(\mathbf{u}),\quad \text{on } \Gamma_R. \tag{3.13}$$

Its solution is the same as that of problem (3.1)-(3.3) in Ω_R.

4. Variational Form and Its Approximation

For brevity of analysis, we suppose $\mathbf{g}=0$ in the following. We denote

$$\mathbf{V}\equiv\left\{\mathbf{u}\in\mathbf{H}^1(\Omega_R)\equiv H^1(\Omega_R)\times H^1(\Omega_R)\times H^1(\Omega_R)\,\middle|\,\mathbf{u}=\mathbf{0},\ \text{on } \Gamma\right\}.$$

The reduced problem (3.11)-(3.13) is equivalent to the following variational problem:
Find $\mathbf{u}\in\mathbf{V}$, such that

$$\mathcal{A}(\mathbf{u},\mathbf{v})+\mathcal{B}_\infty(\mathbf{u},\mathbf{v})=\mathcal{F}(\mathbf{v}),\ \forall\mathbf{v}\in\mathbf{V}, \tag{4.1}$$

where

$$\begin{aligned}\mathcal{A}(\mathbf{u},\mathbf{v}) &= \int_{\Omega_R}\left\{\lambda(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{v})+2\mu\epsilon(\mathbf{u}):\epsilon(\mathbf{v})\right\}d\sigma,\\ \mathcal{B}_\infty(\mathbf{u},\mathbf{v}) &= R\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)(2l+3)\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu A_l^m\overline{D_l^m}\\ &\quad +R\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}l(l+1)(l+2)\mu B_l^m\overline{E_l^m}\\ &\quad +R\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}2l(l+1)(2l-1)\mu C_l^m\overline{F_l^m},\\ \mathcal{F}(\mathbf{v}) &= \int_{\Omega_R}\mathbf{f}\cdot\mathbf{v}d\sigma.\end{aligned}$$

(A_l^m,B_l^m,C_l^m) and (D_l^m,E_l^m,F_l^m) are the *Fourier coefficients* of vector $\mathbf{u}$ and $\mathbf{v}$ on the artificial boundary Γ_R.

In the numerical implementation, the bilinear form $\mathcal{B}_\infty(\cdot,\cdot)$ must be approximated by truncating its series terms. We denote the approximate bilinear form by

$$\begin{aligned}\mathcal{B}_N(\mathbf{u},\mathbf{v}) = R&\sum_{l=0}^{N}\sum_{m=-(l+1)}^{l+1}(l+1)(2l+3)\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu A_l^m\overline{D_l^m}\\ &+R\sum_{l=1}^{\max(1,N)}\sum_{m=-l}^{l} l(l+1)(l+2)\mu B_l^m\overline{E_l^m}\\ &+R\sum_{l=1}^{N}\sum_{m=-(l-1)}^{l-1}2l(l+1)(2l-1)\mu C_l^m\overline{F_l^m}.\end{aligned}$$

Intuitively, N is a parameter to indicate how well $\mathcal{B}_N(\mathbf{u},\mathbf{v})$ approximates $\mathcal{B}_\infty(\mathbf{u},\mathbf{v})$. The larger is N, the more accurate is $\mathcal{B}_N(\mathbf{u},\mathbf{v})$. Now suppose $\mathbf{V}_h$ is a conforming finite element space of $\mathbf{V}$ and replace $\mathbf{V}$ with $\mathbf{V}_h$ and $\mathcal{B}_\infty(\cdot,\cdot)$ with $\mathcal{B}_N(\cdot,\cdot)$ in variational problem (4.1), we obtain an approximate discrete variational problem:
Find $\mathbf{u}_N^h \in \mathbf{V}_h$, such that

$$\mathcal{A}(\mathbf{u}_N^h,\mathbf{v})+\mathcal{B}_N(\mathbf{u}_N^h,\mathbf{v})=\mathcal{F}(\mathbf{v}),\ \forall \mathbf{v}\in\mathbf{V}_h. \tag{4.2}$$

5. Error Estimate

We note that $\mathcal{B}_\infty(\mathbf{u},\mathbf{v})$ and $\mathcal{B}_N(\mathbf{u},\mathbf{v})$ are real for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$. Besides, it is easy to verify that

1. The bilinear forms $\mathcal{A}(\cdot,\cdot)$, $\mathcal{B}_\infty(\cdot,\cdot)$ and $\mathcal{B}_N(\cdot,\cdot)$ are symmetric and

$$\begin{aligned}&B_\infty(\mathbf{v},\mathbf{v})\geq 0, \quad \forall \mathbf{v}\in V,\\ &B_N(\mathbf{v},\mathbf{v})\geq 0, \quad \forall \mathbf{v}\in V, N=0,1,2,\cdots\end{aligned}$$

2. By Körn inequality (see [22]), there exists a positive constant M such that

$$\mathcal{A}(\mathbf{v},\mathbf{v})+\mathcal{B}_0(\mathbf{v},\mathbf{v})\geq M\|\mathbf{v}\|^2_{1,\Omega_R},\ \forall \mathbf{v}\in\mathbf{V}.$$

Thus by defining an equivalent norm of space $\mathbf{V}$ as

$$\|\mathbf{v}\|_* \equiv \Big\{\mathcal{A}(\mathbf{v},\mathbf{v})+\mathcal{B}_0(\mathbf{v},\mathbf{v})\Big\}^{1/2},$$

we have

$$|\mathcal{A}(\mathbf{u},\mathbf{v})+\mathcal{B}_0(\mathbf{u},\mathbf{v})|\leq \|\mathbf{u}\|_*\cdot\|\mathbf{v}\|_*, \quad \forall \mathbf{u}\in\mathbf{V},\mathbf{v}\in\mathbf{V}.$$

We denote

$$\hat{\mathcal{A}}(\hat{\mathbf{u}},\hat{\mathbf{v}})=\int_{S_R}\Big\{\lambda(\nabla\cdot\hat{\mathbf{u}})(\nabla\cdot\hat{\mathbf{v}})+2\mu\epsilon(\hat{\mathbf{u}}):\epsilon(\hat{\mathbf{v}})\Big\}d\sigma,\ \forall\hat{\mathbf{u}}\in\mathbf{H}^1(S_R),\forall\hat{\mathbf{v}}\in\mathbf{H}^1(S_R).$$

Since space $\mathbf{V}$ can be naturally embedded into $\mathbf{H}^1(S_R)$ by extending $\mathbf{v}$ to $\mathbf{0}$ in the domain interior to surface Γ, let $\hat{\mathbf{w}}$ be the corresponding function of $\mathbf{w}\in\mathbf{V}$, and we have

$$\mathcal{A}(\mathbf{w},\mathbf{w})=\hat{\mathcal{A}}(\hat{\mathbf{w}},\hat{\mathbf{w}}).$$

Lemma 5.1. *For any* $\mathbf{v} \in \mathbf{H}^1(S_R)$ *with Fourier coefficients* (D_l^m, E_l^m, F_l^m) *on* Γ_R*, we suppose* $\hat{\mathbf{v}}$ *is the solution to the following problem:*
Find $\hat{\mathbf{v}} \in \mathbf{H}^1(S_R)$ *such that*

$$\begin{cases} \hat{\mathbf{v}}|_{\Gamma_R} = \mathbf{v}|_{\Gamma_R}, \\ \hat{\mathcal{A}}(\hat{\mathbf{v}}, \hat{\mathbf{w}}) = 0, \ \forall \hat{\mathbf{w}} \in \mathbf{H}_0^1(S_R). \end{cases} \tag{5.1}$$

Then for any $\hat{\mathbf{u}} \in \mathbf{H}^1(S_R)$ *with Fourier coefficients* (A_l^m, B_l^m, C_l^m) *on* Γ_R*, we have*

$$\begin{aligned} \hat{\mathcal{A}}(\hat{\mathbf{v}}, \hat{\mathbf{u}}) = R &\sum_{l=1}^{+\infty} \sum_{m=-(l+1)}^{l+1} 2l(l+1)(2l+3)\mu \overline{A_l^m} D_l^m \\ &+ R \sum_{l=2}^{+\infty} \sum_{m=-l}^{l} l(l-1)(l+1)\mu \overline{B_l^m} E_l^m \\ &+ R \sum_{l=1}^{+\infty} \sum_{m=-(l-1)}^{l-1} l(2l-1) \frac{(2l^2+1)\lambda + (2l^2-2l+2)\mu}{(l-1)\lambda + (3l-2)\mu} \mu \overline{C_l^m} F_l^m. \end{aligned}$$

Proof. the variational problem (5.1) is equivalent to the following problem

$$\begin{aligned} &-\mu \Delta \hat{\mathbf{v}} - (\lambda + \mu) \nabla (\nabla \cdot \hat{\mathbf{v}}) = \mathbf{0}, \quad \text{in } S_R, \\ &\hat{\mathbf{v}} = \mathbf{v}, \quad \text{on } \Gamma_R. \end{aligned}$$

Thus then

$$\hat{\mathcal{A}}(\hat{\mathbf{v}}, \hat{\mathbf{u}}) = \int_{\Gamma_R} \mathbf{n} \cdot \sigma(\hat{\mathbf{v}}) \cdot \hat{\mathbf{u}} ds. \tag{5.2}$$

We expand $\mathbf{v}$ on Γ_R as

$$\mathbf{v} = \sum_{l=0}^{+\infty} \sum_{m=-(l+1)}^{l+1} D_l^m \mathbf{I}_l^m + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} E_l^m \mathbf{T}_l^m + \sum_{l=1}^{+\infty} \sum_{m=-(l-1)}^{l-1} F_l^m \mathbf{N}_l^m,$$

and set

$$\mathbf{G}_l^* = \begin{cases} \left(\frac{r}{R}\right)^l \displaystyle\sum_{m=-(l+1)}^{l+1} D_l^m \mathbf{I}_l^m, l = 0, \\ \left(\frac{r}{R}\right)^l \displaystyle\sum_{m=-(l+1)}^{l+1} D_l^m \mathbf{I}_l^m + \left(\frac{r}{R}\right)^l \sum_{m=-l}^{l} E_l^m \mathbf{T}_l^m + \left(\frac{r}{R}\right)^l \sum_{m=-(l-1)}^{l-1} F_l^m \mathbf{N}_l^m, l > 0. \end{cases}$$

Analogous to the idea to handle the exterior elastic problem, we can obtain

$$\hat{\mathbf{v}} = \sum_{l=0}^{+\infty} \Big[\mathbf{G}_l^* + c_l^* (r^2 - R^2) \nabla (\nabla \cdot \mathbf{G}_l^*) \Big]$$

with

$$c_l^* = -\frac{\lambda + \mu}{(2l-2)\lambda + (6l-4)\mu}.$$

A computation shows that the normal stress on Γ_R has the following expression

$$\begin{aligned}\mathbf{n}\cdot\sigma(\hat{\mathbf{v}}) &= \frac{1}{R}\sum_{l=1}^{+\infty}\sum_{m=-(l+1)}^{l+1} 2l\mu D_l^m \mathbf{I}_l^m \\ &+\frac{1}{R}\sum_{l=2}^{+\infty}\sum_{m=-l}^{l}(l-1)\mu E_l^m\mathbf{T}_l^m \\ &+\frac{1}{R}\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}\frac{(2l^2+1)\lambda+(2l^2-2l+2)\mu}{(l-1)\lambda+(3l-2)\mu}\mu F_l^m\mathbf{N}_l^m.\end{aligned}$$

Substituting it into equation (5.2), we finish the answer.

Lemma 5.2. *There is a constant c independent of N and R such that*

$$\begin{aligned}0\le \mathcal{B}_N(\mathbf{v},\mathbf{v})-\mathcal{B}_0(\mathbf{v},\mathbf{v})\le \mathcal{B}_\infty(\mathbf{v},\mathbf{v})-\mathcal{B}_0(\mathbf{v},\mathbf{v})\le c\|\mathbf{v}\|_*^2,\ \forall \mathbf{v}\in\mathbf{V},\\ |\mathcal{B}_\infty(\mathbf{u},\mathbf{v})-\mathcal{B}_0(\mathbf{u},\mathbf{v})|\le c\|\mathbf{u}\|_*\cdot\|\mathbf{v}\|_*,\ \forall \mathbf{u}\in\mathbf{V},\mathbf{v}\in\mathbf{V},\\ |\mathcal{B}_N(\mathbf{u},\mathbf{v})-\mathcal{B}_0(\mathbf{u},\mathbf{v})|\le c\|\mathbf{u}\|_*\cdot\|\mathbf{v}\|_*,\ \forall \mathbf{u}\in\mathbf{V},\mathbf{v}\in\mathbf{V}.\end{aligned}$$

Proof. For any $\mathbf{v}\in\mathbf{V}$, we have

$$\|\mathbf{v}\|_*^2\ge \inf_{\mathbf{w}\in\mathbf{V},\mathbf{w}|_{\Gamma_R}=\mathbf{v}|_{\Gamma_R}}\mathcal{A}(\mathbf{w},\mathbf{w})\ge \inf_{\hat{\mathbf{w}}\in\mathbf{H}^1(S_R),\hat{\mathbf{w}}|_{\Gamma_R}=\mathbf{v}|_{\Gamma_R}}\hat{\mathcal{A}}(\hat{\mathbf{w}},\hat{\mathbf{w}}).$$

We denote by $\hat{\mathbf{v}}$ the solution to the minimization problem:
Find $\hat{\mathbf{v}}\in\mathbf{H}^1(S_R)$ and $\hat{\mathbf{v}}|_{\Gamma_R}=\mathbf{v}|_{\Gamma_R}$ such that

$$\hat{\mathcal{A}}(\hat{\mathbf{v}},\hat{\mathbf{v}})=\inf_{\hat{\mathbf{w}}\in\mathbf{H}^1(S_R),\hat{\mathbf{w}}|_{\Gamma_R}=\mathbf{v}|_{\Gamma_R}}\hat{\mathcal{A}}(\hat{\mathbf{w}},\hat{\mathbf{w}}).$$

Then $\hat{\mathbf{v}}$ is also the solution to the following variational problem:
Find $\hat{\mathbf{v}}\in\mathbf{H}^1(S_R)$ such that

$$\begin{cases}\hat{\mathbf{v}}|_{\Gamma_R}=\mathbf{v}|_{\Gamma_R},\\ \hat{\mathcal{A}}(\hat{\mathbf{v}},\hat{\mathbf{w}})=0,\ \forall\hat{\mathbf{w}}\in\mathbf{H}_0^1(S_R).\end{cases}$$

From lemma 5.1, there exists a positive constant c, independent of N and R, such that

$$\begin{aligned}\hat{\mathcal{A}}(\hat{\mathbf{v}},\hat{\mathbf{v}})&=R\sum_{l=1}^{+\infty}\sum_{m=-(l+1)}^{l+1}2l(l+1)(2l+3)\mu|D_l^m|^2\\ &+R\sum_{l=2}^{+\infty}\sum_{m=-l}^{l}l(l-1)(l+1)\mu|E_l^m|^2\\ &+R\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}l(2l-1)\frac{(2l^2+1)\lambda\mu+(2l^2-2l+2)\mu^2}{(l-1)\lambda+(3l-2)\mu}|F_l^m|^2\\ &\ge\frac{1}{c}\Bigg\{R\sum_{l=1}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)(2l+3)\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu|D_l^m|^2\\ &+R\sum_{l=2}^{+\infty}\sum_{m=-l}^{l}l(l+1)(l+2)\mu|E_l^m|^2+R\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}2l(l+1)(2l-1)\mu|F_l^m|^2\Bigg\}\\ &=\frac{1}{c}\Big\{\mathcal{B}_\infty(\mathbf{v},\mathbf{v})-\mathcal{B}_0(\mathbf{v},\mathbf{v})\Big\}\ge\frac{1}{c}\Big\{\mathcal{B}_N(\mathbf{v},\mathbf{v})-\mathcal{B}_0(\mathbf{v},\mathbf{v})\Big\}.\end{aligned}$$

Thus the first result is proved. The other two follow directly by Schwartz inequality.

By Lax-Milgram theorem, we have

Theorem 5.1. *For any $\mathbf{f} \in \mathbf{V}'$, the variational problem (4.1) and variational problem (4.2) have unique solutions $\mathbf{u} \in \mathbf{V}$ and $\mathbf{u}_N^h \in \mathbf{V}_h$.*

For any $s \geq 0$ and any $\mathbf{v} \in \mathbf{H}^s(\Gamma_\rho)$, we suppose

$$\mathbf{v} = \sum_{l=0}^{+\infty} \mathbf{v}_l$$

where

$$\mathbf{v}_l = \begin{cases} \displaystyle\sum_{m=-(l+1)}^{l+1} D_l^m \mathbf{I}_l^m, l = 0, \\ \displaystyle\sum_{m=-(l+1)}^{l+1} D_l^m \mathbf{I}_l^m + \sum_{m=-l}^{l} E_l^m \mathbf{T}_l^m + \sum_{m=-(l-1)}^{l-1} F_l^m \mathbf{N}_l^m, l > 0. \end{cases}$$

We define its norm $\|\mathbf{v}\|_{s,\Gamma_\rho}$ by (see page 41 in [21])

$$\begin{aligned} \|\mathbf{v}\|_{s,\Gamma_\rho}^2 \quad = \quad \rho^2 \Bigg\{ & \sum_{l=0}^{+\infty} \sum_{m=-(l+1)}^{l+1} (l+1)^{2s+2} \left| D_l^m \right|^2 + \sum_{l=1}^{+\infty} \sum_{m=-l}^{l} (l+1)^{2s+2} \left| E_l^m \right|^2 \\ & + \sum_{l=1}^{+\infty} \sum_{m=-(l-1)}^{l-1} (l+1)^{2s+2} \left| F_l^m \right|^2 \Bigg\}. \end{aligned}$$

In the following of the this paper, if no ambiguity can be induced, the constant c is used to represent different values in different places.

Lemma 5.3. *Suppose $\mathbf{u}$ is the solution to the exterior problem (3.1)-(3.3), then there is a constant c independent of N and radius R of the spherical artificial boundary, such that*

$$|\mathcal{B}(\mathbf{u},\mathbf{v}) - \mathcal{B}_N(\mathbf{u},\mathbf{v})| \leq c \left(\frac{R_0}{R} \right)^{\frac{\max(1,2N-1)}{2}} \|\mathbf{u}\|_{1/2,\Gamma_{R_0}} \cdot \|\mathbf{v}\|_*, \ \forall \mathbf{v} \in \mathbf{V}.$$

Proof. suppose (A_l^m, B_l^m, C_l^m) are Fourier coefficients of $\mathbf{u}$ on Γ_R and (D_l^m, E_l^m, F_l^m) are Fourier coefficients of $\mathbf{v}$ on Γ_R, then we have

$$\begin{aligned} & \mathcal{B}_\infty(\mathbf{u},\mathbf{v}) - \mathcal{B}_N(\mathbf{u},\mathbf{v}) \\ = \quad & R \sum_{l=N+1}^{+\infty} \sum_{m=-(l+1)}^{l+1} (l+1)(2l+3) \frac{(4l^2+12l+12)\mu + (4l^2+8l+6)\lambda}{(6l+10)\mu + (2l+4)\lambda} \mu A_l^m \overline{D_l^m} \\ + \quad & R \sum_{l=\max(2,N+1)}^{+\infty} \sum_{m=-l}^{l} l(l+1)(l+2)\mu B_l^m \overline{E_l^m} \\ + \quad & R \sum_{l=N+1}^{+\infty} \sum_{m=-(l-1)}^{l-1} 2l(l+1)(2l-1)\mu C_l^m \overline{F_l^m}, \end{aligned}$$

and

$$
\begin{aligned}
&|\mathcal{B}(\mathbf{u},\mathbf{v})-\mathcal{B}_N(\mathbf{u},\mathbf{v})|^2 \\
\leq\ & \Bigg\{R\sum_{l=N+1}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)(2l+3)\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu|A_l^m|^2 \\
+\ & R\sum_{l=\max(2,N+1)}^{+\infty}\sum_{m=-l}^{l} l(l+1)(l+2)\mu|B_l^m|^2+R\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}2l(l+1)(2l-1)\mu|C_l^m|^2\Bigg\} \\
\times\ & \Bigg\{R\sum_{l=N+1}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)(2l+3)\frac{(4l^2+12l+12)\mu+(4l^2+8l+6)\lambda}{(6l+10)\mu+(2l+4)\lambda}\mu|D_l^m|^2 \\
+\ & R\sum_{l=\max(2,N+1)}^{+\infty}\sum_{m=-l}^{l} l(l+1)(l+2)\mu|E_l^m|^2+R\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}2l(l+1)(2l-1)\mu|F_l^m|^2\Bigg\} \\
\leq\ & cR\Bigg\{\sum_{l=N+1}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)^3|A_l^m|^2+\sum_{l=\max(2,N+1)}^{+\infty}\sum_{m=-l}^{l}(l+1)^3|B_l^m|^2 \\
&\qquad\qquad +\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^3|C_l^m|^2\Bigg\}\cdot\|\mathbf{v}\|_*^2.
\end{aligned}
$$

We suppose on Γ_{R_0}, $\mathbf{u}$ has the following expansion

$$
\mathbf{u}=\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}\tilde{A}_l^m\mathbf{I}_l^m+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}\tilde{B}_l^m\mathbf{T}_l^m+\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}\tilde{C}_l^m\mathbf{N}_l^m,
$$

and set

$$
\tilde{\mathbf{G}}_l=\begin{cases}\left(\dfrac{R_0}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1}\tilde{A}_l^m\mathbf{I}_l^m, l=0,\\[2ex] \left(\dfrac{R_0}{r}\right)^{l+1}\displaystyle\sum_{m=-(l+1)}^{l+1}\tilde{A}_l^m\mathbf{I}_l^m+\left(\dfrac{R_0}{r}\right)^{l+1}\sum_{m=-l}^{l}\tilde{B}_l^m\mathbf{T}_l^m+\left(\dfrac{R_0}{r}\right)^{l+1}\sum_{m=-(l-1)}^{l-1}\tilde{C}_l^m\mathbf{N}_l^m, l>0.\end{cases}
$$

Then the solution outside of Γ_{R_0} is

$$
\mathbf{u}=\sum_{l=0}^{+\infty}\Big[\tilde{\mathbf{G}}_l+c_l(r^2-R_0^2)\nabla(\nabla\cdot\tilde{\mathbf{G}}_l)\Big].
$$

After a trivial computation, on the artificial boundary Γ_R, we can prove $\mathbf{u}$ has the following expression

$$
\begin{aligned}
\mathbf{u}\ =\ & \sum_{l=0}^{+\infty}\left(\frac{R_0}{R}\right)^{l+1}\sum_{m=-(l+1)}^{l+1}\tilde{A}_l^m\mathbf{I}_l^m+\sum_{l=1}^{+\infty}\left(\frac{R_0}{R}\right)^{l+1}\sum_{m=-l}^{l}\tilde{B}_l^m\mathbf{T}_l^m+\sum_{l=1}^{+\infty}\left(\frac{R_0}{R}\right)^{l+1}\sum_{m=-(l-1)}^{l-1}\tilde{C}_l^m\mathbf{N}_l^m \\
+\ & \sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}\frac{c_l(R^2-R_0^2)R_0^{l+1}}{R^{l+3}}(l+1)(2l+1)\tilde{A}_l^m\mathbf{N}_{l+2}^m.
\end{aligned}
$$

Thus we have

$$A_l^m = \left(\frac{R_0}{R}\right)^{l+1} \tilde{A}_l^m,$$

$$B_l^m = \left(\frac{R_0}{R}\right)^{l+1} \tilde{B}_l^m,$$

$$C_l^m = \left(\frac{R_0}{R}\right)^{l+1} \tilde{C}_l^m + \frac{c_{l-2}(R^2-R_0^2)R_0^{l-1}}{R^{l+1}}(l-1)(2l-3)\tilde{A}_{l-2}^m,$$

and

$$\begin{aligned}\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^3|C_l^m|^2 \leq\ & 2\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^3\left(\frac{R_0}{R}\right)^{2l+2}|\tilde{C}_l^m|^2 \\ &+ 2\sum_{l=\max(0,N-1)}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+3)^3(l+1)^2(2l+1)^2\frac{c_l^2(R^2-R_0^2)^2R_0^{2l+2}}{R^{2l+6}}|\tilde{A}_l^m|^2 \\ \leq\ & c\Bigg\{\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^5\left(\frac{R_0}{R}\right)^{2l+2}|\tilde{C}_l^m|^2 \\ &+ \sum_{l=\max(0,N-1)}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^5\left(\frac{R_0}{R}\right)^{2l+2}|\tilde{A}_l^m|^2\Bigg\}.\end{aligned}$$

Finally,

$$\begin{aligned}&|\mathcal{B}_\infty(\mathbf{u},\mathbf{v})-\mathcal{B}_N(\mathbf{u},\mathbf{v})|^2 \\ \leq\ & cR\Bigg\{\sum_{l=N+1}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)^3|A_l^m|^2+\sum_{l=\max(2,N+1)}^{+\infty}\sum_{m=-l}^{l}(l+1)^3|B_l^m|^2 \\ &+\sum_{l=N+1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^3|C_l^m|^2\Bigg\}\cdot\|\mathbf{v}\|_*^2 \\ \leq\ & c\left(\frac{R_0}{R}\right)^{\max(1,2N-1)}\Bigg\{\sum_{l=0}^{+\infty}\sum_{m=-(l+1)}^{l+1}(l+1)^5|\tilde{A}_l^m|^2+\sum_{l=1}^{+\infty}\sum_{m=-l}^{l}(l+1)^5|\tilde{B}_l^m|^2 \\ &+\sum_{l=1}^{+\infty}\sum_{m=-(l-1)}^{l-1}(l+1)^5|\tilde{C}_l^m|^2\Bigg\}\cdot\|\mathbf{v}\|_*^2 \\ \leq\ & c\left(\frac{R_0}{R}\right)^{\max(1,2N-1)}\|\mathbf{u}\|_{3/2,\Gamma_{R_0}}^2\cdot\|\mathbf{v}\|_*^2,\end{aligned}$$

and we derive

$$|\mathcal{B}(\mathbf{u},\mathbf{v})-\mathcal{B}_N(\mathbf{u},\mathbf{v})| \leq c\left(\frac{R_0}{R}\right)^{\frac{\max(1,2N-1)}{2}}\|\mathbf{u}\|_{3/2,\Gamma_{R_0}}\cdot\|\mathbf{v}\|_*.$$

The proof is complete.

Theorem 5.2. *Suppose* $\mathbf{u}$ *is the solution to problem (3.1)-(3.3) and* $\mathbf{u}_N^h$ *is the solution to problem (4.2), then there exists a constant* c *independent of* N, R *and* $\mathbf{V}_h$, *such that*

$$\|\mathbf{u}-\mathbf{u}_N^h\|_* \leq c\Bigg\{\inf_{\mathbf{v}^h\in\mathbf{V}_h}\|\mathbf{u}-\mathbf{v}^h\|_* + \left(\frac{R_0}{R}\right)^{\frac{\max(1,2N-1)}{2}}\|\mathbf{u}\|_{3/2,\Gamma_{R_0}}\Bigg\}. \tag{5.3}$$

Proof. Denote $\mathbf{e} = \mathbf{u} - \mathbf{u}_N^h$, $\mathbf{e}^h = \mathbf{v}^h - \mathbf{u}_N^h$, and Subtract (4.2) from (4.1), then we have

$$\mathcal{A}(\mathbf{e}, \mathbf{e}^h) + \mathcal{B}_N(\mathbf{e}, \mathbf{e}^h) = \mathcal{B}_N(\mathbf{u}, \mathbf{e}^h) - \mathcal{B}_\infty(\mathbf{u}, \mathbf{e}^h).$$

From lemma 1 and lemma 3, it holds

$$\begin{aligned} \|\mathbf{e}^h\|_*^2 &\leq \mathcal{A}(\mathbf{e}^h, \mathbf{e}^h) + \mathcal{B}_N(\mathbf{e}^h, \mathbf{e}^h) \\ &= \mathcal{A}(\mathbf{e}^h - \mathbf{e}, \mathbf{e}^h) + \mathcal{B}_N(\mathbf{e}^h - \mathbf{e}, \mathbf{e}^h) + \mathcal{B}_N(\mathbf{u}, \mathbf{e}^h) - \mathcal{B}_\infty(\mathbf{u}, \mathbf{e}^h) \\ &\leq c\left\{ \|\mathbf{e}^h - \mathbf{e}\|_* \cdot \|\mathbf{e}^h\|_* + \left(\frac{R_0}{R}\right)^{\frac{\max(1,2N-1)}{2}} \|\mathbf{u}\|_{3/2,\Gamma_{R_0}} \cdot \|\mathbf{e}^h\|_* \right\}. \end{aligned}$$

By triangle inequality, we have

$$\|\mathbf{e}\|_* \leq \|\mathbf{e}^h\|_* + \|\mathbf{e}^h - \mathbf{e}\|_* \leq 2c\left\{ \|\mathbf{u} - \mathbf{v}^h\|_* + \left(\frac{R_0}{R}\right)^{\frac{\max(1,2N-1)}{2}} \|\mathbf{u}\|_{3/2,\Gamma_{R_0}} \right\}.$$

The result follows.

Remark 5.1. We note that the analogous idea to obtain Theorem 5.2 has been used [2, 3].

Denoting by $\mathbf{V}_h$ a conforming finite element subspace of $\mathbf{V}$ with regular tetrahedral elements of type L (see [5]) employed to divide the computational domain Ω_R, by the interpolation theorem, we have

$$\inf_{v_h \in V_h} ||\mathbf{u} - \mathbf{v}_h||_* \leq Ch^L ||\mathbf{u}||_{L+1,\Omega_R},$$

where h is the mesh size and C is some constant independent of h. In this case, formula (5.3) turns to

$$||\mathbf{u} - \mathbf{u}_N^h||_* \leq C\left\{ h^L ||\mathbf{u}||_{L+1,\Omega_R} + \left(\frac{R_0}{R}\right)^{\frac{\max(1,2N-1)}{2}} ||\mathbf{u}||_{3/2,\Gamma_0} \right\},$$

where C is a constant independent of N, h. This reveals clearly how the error depends on the mesh size and the accuracy of the approximate variational formulation.

6. Numerical Test

As a numerical test, we consider the following inhomogeneous Dirichlet problem:

$$\begin{aligned} &-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = 0, \ |\mathbf{x}| > 1, \\ &\mathbf{u} = \mathbf{u}_e, \ |\mathbf{x}| = 1, \\ &\mathbf{u} \longrightarrow \mathbf{0}, \quad \text{when } |\mathbf{x}| \longrightarrow +\infty, \end{aligned}$$

where

$$\mathbf{u}_e = \frac{\lambda + 3\mu}{8\pi(\lambda + 2\mu)} \frac{\mathbf{F}}{|\mathbf{x} - \mathbf{x}_0|} + \frac{\lambda + \mu}{8\pi\mu(\lambda + 2\mu)} \frac{[(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{F}]}{|\mathbf{x} - \mathbf{x}_0|^3} (\mathbf{x} - \mathbf{x}_0)$$

with $\mathbf{F}$ and $\mathbf{x}_0$ being two fixed vectors. It can be verified that $\mathbf{u}_e$ is a solution to Navier equations with zero infinity condition. We set $\lambda = 2$, $\mu = 1$, $\mathbf{F} = [1\ 0\ 0]$ and $\mathbf{x}_0 = [0.1\ 0\ 0]$ in our computation. Three different meshes are used in our computation. The basic information is listed in Table 1. To evaluate the numerical performance, we define the error function as

$$\epsilon = \left[\mathcal{A}(\mathbf{u} - \mathbf{u}_N^h, \mathbf{u} - \mathbf{u}_N^h)\right]^{1/2}.$$

We first study how N, an indicator of the accuracy of approximate variational formulation, influences the error function. We introduce a spherical artificial boundary with radius of 2, and

Table 1: Basic information on the finite element meshes

	Characteristic size h	Element number	Node number
Mesh A	1	252	527
Mesh B	0.5	2006	3418
Mesh C	0.25	16048	24322

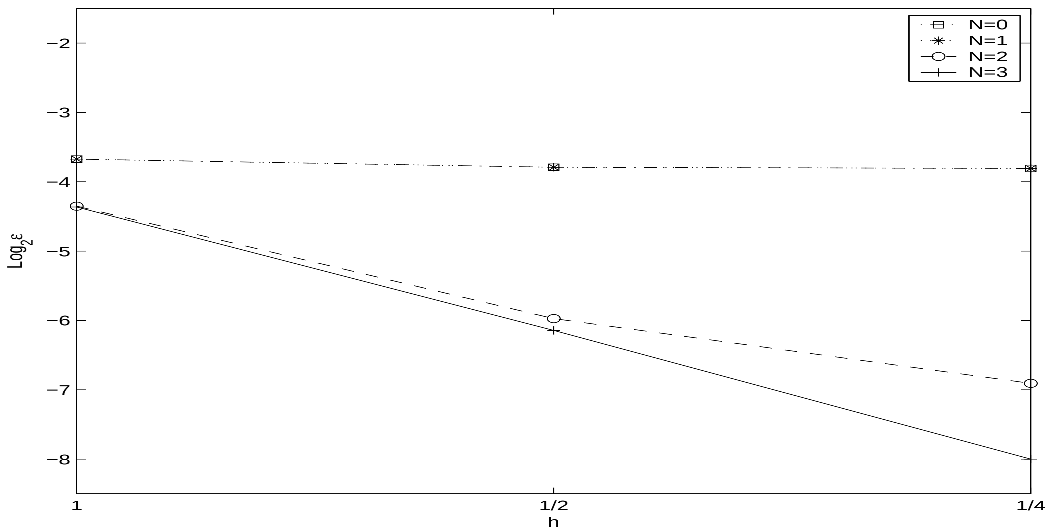

Figure 1: Log-log plot of the error function. R is fixed to be 2

employ tetrahedral elements of type 2 (see [5]) to form the finite element space $\mathbf{V}_h$ is adopted in the numerical implementation. Three meshes are used. Figure 1 plots the errors.

It can be observed that if the mesh is coarse, higher-accuracy variational formulation is not superior to the lower-accuracy ones. For instance, the errors for $N = 2, 3$ with mesh size 0.5 and 0.25 do not differ much. On the other hand, if the mesh is sufficiently refined, the error can

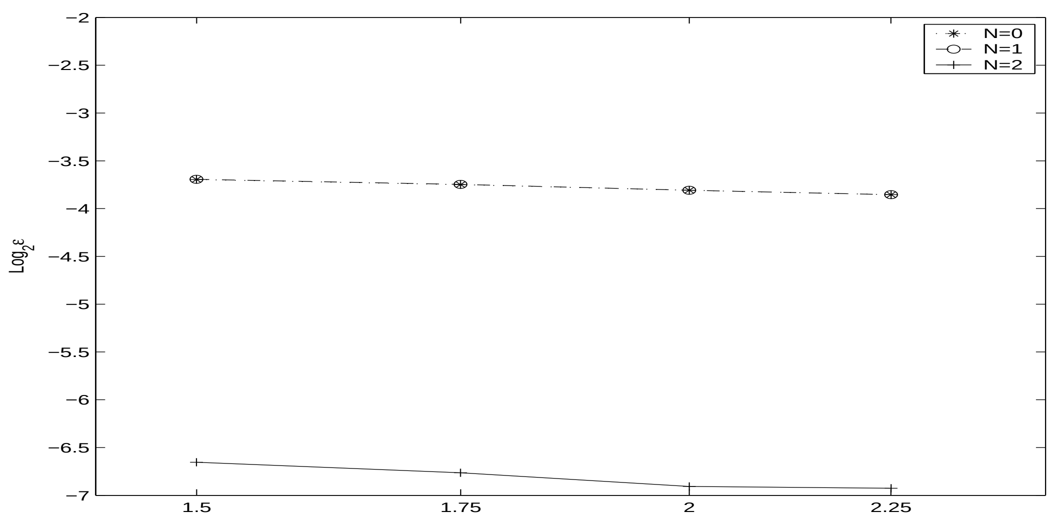

Figure 2: Log-log plot of the error function. $h = 0.25$.

only be significantly decreased when higher-accuracy variational formulation is adopted in the computation. In fact, when $N = 3$, a second order degeneracy of the error function is showed in Figure 1. This is in good agreement with our error estimate in theorem 5.2. Since the overall error consists of two parts: one is the finite element error, the other is from the approximation of variational formulation. If the mesh is coarse, the former dominates the error, but if the mesh is sufficiently refined, the latter does.

Next, we study the influence of the location of the artificial boundary. We vary the value of R in $\{1.5, 1.75, 2, 2.25\}$ and let the mesh size be 0.25. Figure 2 plots the error function. We see that generally, the errors decrease when R increases. This agrees with our theoretical analysis. But the order is hard to detect since the finite element error still plays a significant role in the overall error.

7. Conclusion

The numerical method for the exterior problem of elasticity in three dimensions has been considered in this paper. An exact artificial boundary condition with the normal stress and the displacement field involved is derived on the introduced spherical artificial boundary. Imposing this relation on the artificial boundary leads to a reduced problem defined only in a finite computational domain. Error estimate is obtained in this paper. This estimate depends not only on the finite element space, but on the accuracy of approximate variational formulation.

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