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MIXED LEGENDRE–HERMITE PSEUDOSPECTRAL METHOD FOR HEAT TRANSFER IN AN INFINITE PLATE *1)

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Abstract

A new mixed Legendre-Hermite interpolation is introduced. Some approximation results are established. Mixed Legendre-Hermite pseudospectral method is proposed for non-isotropic heat transfer in an infinite plate. Its convergence is proved. Numerical results show the efficiency of this approach.

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Key words: Non-isotropic heat transfer, Infinite plate, Mixed Legendre-Hermite pseudospectral method.

1. Introduction

Spectral methods have been successfully used for numerical simulations of various problems in science and engineering, such as the Fourier spectral method for periodic problems, and the Legendre and Chebyshev spectral methods for bounded rectangular domains, see [2, 3, 5, 7, 9, 10]. Some authors also studied the Hermite spectral method for the whole line and the Laguerre spectral method for the half line, see [4, 6, 8, 11, 12, 16, 18, 19, 21].

There are three kinds of Hermite polynomial approximations. If the exact solutions grow fast at the infinity, then we usually take the Hermite polynomials $H_n(z)$ as the base functions as in [11], which are mutually orthogonal associated with the weight function e^{-z^2}. But in many cases, such as nonlinear wave equations, the solutions decay to zero at the infinity, and possess certain conservations which play important role in theoretical analysis. Thereby it seems better to use the Hermite functions $e^{-\frac{z^2}{2}}H_n(z)$ as in [13], which form the $L^2(-\infty,\infty)$-orthogonal system. Accordingly, the numerical solutions also keep certain conservations as in continuous cases. Furthermore, for some problems, such as heat transfer process, the solutions usually decay exponentially at the infinity. In this case, we prefer to the generalized Hermite functions $e^{-z^2}H_n(z)$, which are mutually orthogonal with respect to the weight function e^{z^2}, see [8].

In this paper, we consider non-isotropic heat transfer process in an infinite plate. The simplest way is to confine our calculation to a sufficiently large subdomain with certain artificial boundary condition. However, it causes additional errors. The authors proposed a mixed Legendre-Hermite spectral method for solving this problem, see [15]. However, it is more convenient to use pseudospectral method in actual computation, since we only need to evaluate unknown functions at the interpolation nodes. Especially, it is much easier to deal with nonlinear heat transfer process.

The aim of this paper is to develop the mixed pseudospectral method for non-isotropic heat transfer in an infinite plate, by using the Legendre interpolation in a direction, and the Hermite

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interpolation in another direction. As we know, there have been sharp results on the Jacobi interpolation, see [14]. Thus it suffices to study the Hermite interpolation precisely. Like the Hermite polynomial approximations, there have been also two kinds of Hermite interpolations which have the same interpolation nodes, but different weights of numerical quadratures, corresponding to the base functions $H_n(z)$ and $e^{-\frac{z^2}{2}}H_n(z)$, respectively, see [13, 16]. Whereas the solutions of transfer process decay exponentially at the infinity. Thus we introduce a new Hermite interpolation corresponding to the base functions $e^{-z^2}H_n(z)$, which could fit the asymptotic behavior of such solutions properly. Then we propose a new mixed Legendre-Hermite pseudospectral method for non-isotropic heat transfer in an infinite plate. Numerical results demonstrate the high accuracy in the space of this approach. We also establish some basic results on this new mixed Legendre-Hermite interpolation, from which the convergence of proposed scheme follows. These results also play important roles in forming and analyzing other related spectral methods for an infinite strip.

The paper is organized as follow. In the next section, we introduce the new mixed Legendre-Hermite interpolation and establish some basic approximation results. Then we describe the mixed Legendre-Hermite pseudospectral method and its implementation, and present some numerical results in Section 3. We prove the convergence of proposed scheme in Section 4. The final section is for concluding remarks.

2. Mixed Legendre-Hermite Interpolation

In this section, we introduce the new mixed Legendre-Hermite interpolation.

2.1 Legendre-Gauss-Lobatto interpolation

We first recall the Legendre-Gauss-Lobatto interpolation. Let $I = \{x|\ |x| < 1\}$. For any integer $r \geq 0$, we define the Sobolev space $H^r(I)$ as usual, with the inner product $(u, w)_{r,I}$, the semi-norm $|u|_{r,I}$ and the norm $||u||_{r,I}$. In particular, $(u, w)_I = (u, w)_{0,I}$ and $||u||_I = ||u||_{0,I}$. Moreover, $H^1_0(I) = \{\ u|\ u \in H^1(I) \text{ and } u(1) = u(-1) = 0\}$.

Denote by $L_m(x)$ the standard Legendre polynomial of degree $m, m = 0, 1, \cdots$. They satisfy the recurrence relation

$$(2m+1)L_m(x) = \partial_x L_{m+1}(x) - \partial_x L_{m-1}(x), \quad m \geq 1, \tag{2.1}$$

and form the $L^2(I)$-orthogonal system, i.e.,

$$\int_I L_m(x)L_{m'}(x)dx = \frac{2}{2m+1}\delta_{m,m'} \tag{2.2}$$

For any $u \in L^2(I)$, we have that

$$u(x) = \sum_{m=0}^{\infty} \hat{u}_m L_m(x), \qquad \hat{u}_m = (m+\frac{1}{2})\int_I u(x)L_m(x)dx. \tag{2.3}$$

For any integer $M \geq 0$, $\mathcal{P}_M$ stands for the set of all polynomials of degree at most M. Furthermore, $\mathcal{P}^0_M = \{v \mid v \in \mathcal{P}_M,\ v(1) = v(-1) = 0\}$.

The orthogonal projection P_M: $L^2(I) \to \mathcal{P}_M$ is defined by

$$(P_M u - u,\ \phi)_I = 0, \qquad \forall\ \phi \in \mathcal{P}_M. \tag{2.4}$$

For description of approximation results, we introduce the space $H^r_A(I)$ with integer $r \geq 0$, equipped with the following semi-norm and norm

$$|u|_{r,A,I} = ||(1-x^2)^{\frac{r}{2}}\partial^r_x u||_I, \qquad ||u||_{r,A,I} = (\sum_{k=0}^{r}|u|^2_{k,A,I})^{\frac{1}{2}}.$$

Furthermore, let $H^r_*(I) = \{u \mid \partial_x u \in H^{r-1}_A(I)\}$, with the semi-norm $|u|_{r,*,I} = |\partial_x u|_{r-1,A,I}$ and the norm $||u||_{r,*,I} = ||\partial_x u||_{r-1,A,I}$. By Theorem 2.1 of [14], for any $u \in H^r_A(I)$ and integer $r \geq 0$,

$$||P_M u - u||_I \leq c_1 M^{-r} |u|_{r,A,I}. \tag{2.5}$$

Hereafter c_1 denotes a generic positive constant independent of any function and M.

We now turn to the Legendre-Gauss-Lobatto interpolation. Let $x_{M,l}$ be the roots of $(1 - x^2)\partial_x L_M(x)$. The corresponding Christoffel numbers

$$\rho_{M,l} = \frac{2}{M(M+1)} \frac{1}{L^2_M(x_{M,l})}, \quad 0 \leq l \leq M.$$

The related discrete inner product and norm are defined by

$$(u,w)_{M,I} = \sum_{l=0}^{M} u(x_{M,l}) w(x_{M,l}) \rho_{M,l}, \quad ||u||_{M,I} = (u,u)^{\frac{1}{2}}_{M,I}.$$

We have that (see [5, 10])

$$(u,w)_{M,I} = (u,w)_I, \quad \forall\ uw \in \mathcal{P}_{2M-1}, \tag{2.6}$$

$$||\phi||_I \leq ||\phi||_{M,I} \leq \sqrt{2 + \frac{1}{M}}\ ||\phi||_I, \quad \forall\ \phi \in \mathcal{P}_M. \tag{2.7}$$

For any $u \in C(\bar{I})$, the Legendre-Gauss-Lobatto interpolation $I_M u \in \mathcal{P}_M$ is determined by $I_M u(x_{M,l}) = u(x_{M,l})$ for $0 \leq l \leq M$. Equivalently,

$$(I_M u - u, \phi)_{M,I} = 0, \quad \forall\ \phi \in C(\bar{I}). \tag{2.8}$$

By Theorem 4.10 of [14], for any $u \in H^r_*(I)$, integer $r \geq 1$ and $0 \leq \mu \leq 1$,

$$||I_M u - u||_{\mu,I} \leq c_1 M^{\mu - r} |u|_{r,*,I}. \tag{2.9}$$

It is also noted that for any $u \in H^r_*(I)$, $\phi \in \mathcal{P}_M$ and integer $r \geq 1$,

$$|(u,\phi)_I - (u,\phi)_{M,I}| \leq c_1 M^{-r} |u|_{r,*,I} ||\phi||_I. \tag{2.10}$$

Indeed, by virtue of (2.5)-(2.9), we have that

$$\begin{aligned} |(u,\phi)_I - (u,\phi)_{M,I}| &\leq |(u,\phi)_I - (P_{M-1}u,\phi)_I| + |(P_{M-1}u,\phi)_{M,I} - (I_M u,\phi)_{M,I}| \\ &\leq c_1(||u - P_{M-1}u||_I + ||P_{M-1}u - I_M u||_I)||\phi||_I \\ &\leq c_1(||u - P_{M-1}u||_I + ||u - I_M u||_I)||\phi||_I \leq c_1 M^{-r} |u|_{r,*,I} ||\phi||_I. \end{aligned}$$

The results (2.9) and (2.10) improve the corresponding results in [5, 10].

2.2 Hermite-Gauss interpolation

We next introduce the new Hermite-Gauss interpolation. Let $R = \{z \mid -\infty < z < \infty\}$ and $\omega(z) = e^{z^2}$. For any integer $r \geq 0$, we define the weighted Sobolev space $H^r_\omega(R)$ in the usual way, with the inner product $(u,w)_{r,\omega,R}$, the semi-norm $|u|_{r,\omega,R}$ and the norm $||u||_{r,\omega,R}$. In particular, $(u,w)_{\omega,R} = (u,w)_{0,\omega,R}$ and $||u||_{\omega,R} = ||u||_{0,\omega,R}$. For any $r > 0$, we define the space $H^r_\omega(R)$ and its norm $||u||_{r,\omega,R}$ by space interpolation as in [1].

Denote by $H_n(z)$ the standard Hermite polynomial of degree n. The generalized Hermite functions are defined by

$$\tilde{H}_n(z) = \frac{1}{\sqrt{2^n n!}} e^{-z^2} H_n(z) = \frac{(-1)^n}{\sqrt{2^n n!}} \partial^n_z (e^{-z^2}), \qquad n \geq 0. \tag{2.11}$$

They satisfy the recurrence relation (see [6])

$$\partial_z \tilde{H}_n(z) = -\sqrt{2(n+1)}\ \tilde{H}_{n+1}(z), \qquad n \geq 0, \tag{2.12}$$

and form the $L^2_\omega(R)-$orthogonal system, namely,

$$\int_R \tilde{H}_n(z)\tilde{H}_{n'}(z)\omega(z)dz = \sqrt{\pi}\delta_{n,n'}. \tag{2.13}$$

For any $u \in L^2_\omega(R)$, we have that

$$u(z) = \sum_{n=0}^{\infty} \hat{u}_n \tilde{H}_n(z), \qquad \hat{u}_n = \frac{1}{\sqrt{\pi}} \int_R u(z)\tilde{H}_n(z)\omega(z)dz. \tag{2.14}$$

Remark 2.1. It is shown in [15] that for any integer $r \geq 0$, the semi-norm $|u|_{r,\omega,R}$ and the norm $||u||_{r,\omega,R}$ are equivalent to $(\sum\limits_{n=0}^{\infty} n^r |\hat{u}_n|^2)^{\frac{1}{2}}$, see Appendix of this paper.

For any integer $N \geq 0$, $\mathcal{P}_N$ denotes the set of all polynomials of degree at most N. Furthermore, $V_N = \{\ e^{-z^2}q(z) \mid q(z) \in \mathcal{P}_N\}$.

The orthogonal projection $\tilde{P}_N : L^2_\omega(R) \to V_N$ is defined by

$$(\tilde{P}_N u - u,\ \phi)_{\omega,R} = 0, \qquad \forall\ \phi \in V_N. \tag{2.15}$$

Due to Theorem 2.2 of [6] and Remark 2.1, for any $u \in H^r_\omega(R)$, integer r and $0 \leq \mu \leq r$,

$$||\tilde{P}_N u - u||_{\mu,\omega,R} \leq c_2 N^{\frac{\mu-r}{2}} |u|_{r,\omega,R}. \tag{2.16}$$

Hereafter, c_2 denotes a generic positive constant independent of any function and N.

We now turn to the new Hermite-Gauss interpolation corresponding to the weight $\omega(z)$. Let $\sigma_{N,j}$ $(0 \leq j \leq N)$ be the zeros of $\tilde{H}_{N+1}(z)$, arranged as $\sigma_{N,N} < \sigma_{N,N-1} < \cdots < \sigma_{N,0}$. The corresponding Christoffel numbers

$$\omega_{N,j} = \frac{2^N N! \sqrt{\pi} e^{2\sigma^2_{N,j}}}{(N+1)H^2_N(\sigma_{N,j})}, \quad 0 \leq j \leq N. \tag{2.17}$$

In fact, $\sigma_{N,j}$ are exactly the same as the zeros of $H_{N+1}(z)$, and $\omega_{N,j} = \omega^*_{N,j} e^{2\sigma^2_{N,j}}$, $\omega^*_{N,j}$ being the Christoffel numbers of the standard Hermite-Gauss interpolation. Let $a_N = \sqrt{2N}$ be the $N-$th Mhaskar-Rahmanov-Saff number. According to (2.7) of [16],

$$\omega_{N,j} \sim \frac{1}{\sqrt{N}} e^{\sigma^2_{N,j}} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}}. \tag{2.18}$$

The related discrete inner product and norm are defined by

$$(u, w)_{\omega,N,R} = \sum_{j=0}^{N} u(\sigma_{N,j}) w(\sigma_{N,j}) \omega_{N,j}, \quad ||u||_{\omega,N,R} = (u, u)^{\frac{1}{2}}_{\omega,N,R}.$$

As is well known, for any $q \in \mathcal{P}_{2N+1}$,

$$\int_R q(z) e^{-z^2} dz = \sum_{j=0}^{N} q(\sigma_{N,j}) \omega^*_{N,j}. \tag{2.19}$$

Now, for any $\phi \in V_m$, $\psi \in V_{2N+1-m}$ and non-negative integer $m \leq 2N+1$, there exist $q_1(z) \in \mathcal{P}_m$ and $q_2(z) \in \mathcal{P}_{2N+1-m}$ such that $\phi(z) = e^{-z^2} q_1(z)$ and $\psi(z) = e^{-z^2} q_2(z)$. Thus we use (2.19) and the relation between $\omega_{N,j}$ and $\omega^*_{N,j}$ to verify that

$$(\phi, \psi)_{\omega,R} = \int_R q_1(z) q_2(z) e^{-z^2} dz = \sum_{j=0}^{N} q_1(\sigma_{N,j}) q_2(\sigma_{N,j}) \omega^*_{N,j} = (\phi, \psi)_{\omega,N,R}. \tag{2.20}$$

For any $u \in C(R)$, the Hermite-Gauss interpolation $\tilde{I}_N u \in V_N$ is determined by $\tilde{I}_N u(\sigma_{N,j}) = u(\sigma_{N,j})$ for $0 \le j \le N$. Equivalently,

$$(\tilde{I}_N u - u, \phi)_{\omega,N,R} = 0, \quad \forall\ \phi \in C(R). \tag{2.21}$$

In order to derive better approximation results, we need the following lemma, which will be proved in Appendix of this paper.

Lemma 2.1. *For any* $u \in H^1_\omega(R)$,

$$||u||_{\omega,N,R} \le c_2(||u||_{\omega,R} + N^{-\frac{1}{6}}|u|_{1,\omega,R}).$$

The main result on the new Hermite interpolation is stated below.

Lemma 2.2. *For any* $u \in H^r_\omega(R)$, *integer* $r \ge 1$ *and* $0 \le \mu \le r$,

$$||\tilde{I}_N u - u||_{\mu,\omega,R} \le c_2 N^{\frac{1}{3}+\frac{\mu-r}{2}}|u|_{r,\omega,R}.$$

Proof. We know from Lemma 2.1 of [6] that for any $\phi \in V_N$, $|\phi|_{\mu,\omega,R} \le c_2 N^{\frac{\mu}{2}}||\phi||_{\omega,R}$. Therefore, by using (2.16), (2.20), Lemma 2.1 and the fact $\tilde{I}_N\tilde{P}_N u = \tilde{P}_N u$, we deduce that

$$\begin{aligned}||\tilde{I}_N u - \tilde{P}_N u||_{\mu,\omega,R} &\le c_2 N^{\frac{\mu}{2}}||\tilde{I}_N(\tilde{P}_N u - u)||_{\omega,R} = c_2 N^{\frac{\mu}{2}}||(\tilde{P}_N u - u)||_{\omega,N,R}\\ &\le c_2 N^{\frac{\mu}{2}}||\tilde{P}_N u - u||_{\omega,R} + c_2 N^{\frac{\mu}{2}-\frac{1}{6}}|\tilde{P}_N u - u|_{1,\omega,R} \le c_2 N^{\frac{1}{3}+\frac{\mu-r}{2}}|u|_{r,\omega,R}.\end{aligned}$$

Using (2.16) again yields that

$$||\tilde{I}_N u - u||_{\mu,\omega,R} \le ||\tilde{P}_N u - u||_{\mu,\omega,R} + ||\tilde{I}_N u - \tilde{P}_N u||_{\mu,\omega,R} \le c_2 N^{\frac{1}{3}+\frac{\mu-r}{2}}|u|_{r,\omega,R}.$$

It is noted that by (2.21) and Lemma 2.2, for any $u \in H^r_\omega(R)$, $r \ge 1$ and $\phi \in V_N$,

$$\begin{aligned}|(u,\phi)_{\omega,R} - (u,\phi)_{\omega,N,R}| &= |(\tilde{I}_N u - u, \phi)_{\omega,R}| \le c_2||\tilde{I}_N u - u||_{\omega,R}||\phi||_{\omega,R}\\ &\le c_2 N^{\frac{1}{3}-\frac{r}{2}}|u|_{r,\omega,R}||\phi||_{\omega,R}.\end{aligned} \tag{2.22}$$

2.3 Mixed Legendre-Hermite Interpolation.

We are now in position of studying the mixed Legendre-Hermite interpolation. Let $\Omega = I \times R$ and define the weighted space $L^2_\omega(\Omega)$ in the usual way, with the inner product $(u,w)_\omega$ and the norm $||u||_\omega$. For any $u \in L^2_\omega(\Omega)$,

$$u(x,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \hat{u}_{m,n} L_m(x)\tilde{H}_n(z) \tag{2.23}$$

where

$$\hat{u}_{m,n} = \frac{1}{\sqrt{\pi}}(m+\frac{1}{2})\int\int_\Omega u(x,z)L_m(x)\tilde{H}_n(z)\omega(z)dzdx.$$

Let $V_{M,N} = \mathcal{P}_M \otimes V_N$ and

$$V^0_{M,N}(\Omega) = \{\phi \mid \phi \in V_{M,N} \text{ and } \phi(-1,y) = \phi(1,y) = 0\}.$$

The orthogonal projection $P_{M,N} : L^2_\omega(\Omega) \to V_{M,N}(\Omega)$ is defined by

$$(P_{M,N}u - u,\ \phi)_\omega = 0, \qquad \forall\ \phi \in V_{M,N}.$$

In order to estimate $||P_{M,N}u - u||_\omega$, we introduce a non-isotropic space. For any integers $r,\ q \ge 0$,

$$H^{r,q}_{\omega,A}(\Omega) = L^2_\omega(R; H^r_A(I)) \cap H^q_\omega(R; L^2(I)),$$

equipped with the norm

$$||u||_{H^{r,q}_{\omega,A}} = (||u||^2_{L^2_\omega(R;H^r_A(I))} + ||u||^2_{H^q_\omega(R;L^2(I))})^{\frac{1}{2}}.$$

The corresponding semi-norm $|u|_{H^{r,q}_{\omega,A}}$ is given by

$$|u|_{H^{r,q}_{\omega,A}} = (|u|^2_{L^2_\omega(R;H^r_A(I))} + |u|^2_{H^q_\omega(R;L^2(I))})^{\frac{1}{2}}.$$

It is proved in [15] that for any $u \in H^{r,q}_{\omega,A}(\Omega)$ and integer $r, q \geq 0$,

$$||P_{M,N}u - u||_\omega \leq c_2(M^{-r} + N^{-\frac{q}{2}})|u|_{H^{r,q}_{\omega,A}}. \tag{2.24}$$

We now turn to the mixed Legendre-Hermite interpolation. Let $x_{M,l}, \rho_{M,l}, \sigma_{N,j}$ and $\omega_{N,j}$ be the same as in the previous two subsections, and $\Lambda_{M,N} = \{(x_{M,l}, \sigma_{N,j}),\ 0 \leq l \leq M,\ 0 \leq j \leq N\}$. The corresponding discrete inner product and norm are given by

$$(u,w)_{\omega,M,N} = \sum_{l=0}^{M}\sum_{j=0}^{N} u(x_{M,l},\sigma_{N,j})w(x_{M,l},\sigma_{N,j})\rho_{M,l}\omega_{N,j}, \quad ||u||_{\omega,M,N} = (u,u)^{\frac{1}{2}}_{\omega,M,N}.$$

By (2.6) and (2.20), for any $\phi \in V_{2M-1,2N+1}$,

$$\int\int_\Omega \phi(x,z)\omega(z)dxdz = \sum_{l=0}^{M}\sum_{j=0}^{N}\phi(x_{M,l},\sigma_{N,j})\rho_{M,l}\omega_{N,j}. \tag{2.25}$$

In particular, for any $u, w \in V_{M-1,N}$, $(u,w)_\omega = (u,w)_{\omega,M,N}$. Moreover, by (2.7) and (2.20),

$$||\phi||_\omega \leq ||\phi||_{\omega,M,N} \leq \sqrt{2 + \frac{1}{M}}\ ||\phi||_\omega, \quad \forall\ \phi \in V_{M,N}. \tag{2.26}$$

For any $u \in C(\bar{\Omega})$, the mixed Legendre-Hermite interpolation $I_{M,N}u \in V_{M,N}$ is determined by $I_{M,N}u(x,z) = u(x,z)$ for all $(x,z) \in \Lambda_{M,N}$. Equivalently,

$$(I_{M,N}u - u, \phi)_{\omega,M,N} = 0, \quad \forall\ \phi \in V_{M,N}. \tag{2.27}$$

We now present the main result of this section.

Theorem 2.1. *Let $0 \leq \alpha, \beta \leq 1$ and integers $r, \lambda, q, \sigma \geq 1$. Then for any $u \in H^\beta_\omega(R;H^r_*(I)) \cap H^q_\omega(R;H^\alpha(I)) \cap H^\sigma_\omega(R;H^\lambda_*(I))$,*

$$||I_{M,N}u - u||_{H^\beta_\omega(R;H^\alpha(I))} \leq cM^{\alpha-r}|u|_{H^\beta_\omega(R;H^r_*(I))} + cN^{\frac{1}{3}+\frac{\beta-q}{2}}|u|_{H^q_\omega(R;H^\alpha(I))}$$
$$+cM^{\alpha-\lambda}N^{\frac{1}{3}+\frac{\beta-\sigma}{2}}|u|_{H^\sigma_\omega(R;H^\lambda_*(I))}.$$

Hereafter c is a generic positive constant independent of M, N and any function.

Proof. Let ϑ be the identity operator. Clearly, $I_{M,N}u = I_M(\tilde{I}_N u) = \tilde{I}_N(I_M u)$ where I_M and $\tilde{I}_N$ are given by (2.8) and (2.21), respectively. Therefore, $||I_{M,N}u - u||_{H^\beta_\omega(R;H^\alpha(I))} \leq D_1 + D_2$ where

$$D_1 = ||I_M u - u||_{H^\beta_\omega(R;H^\alpha(I))} + ||\tilde{I}_N u - u||_{H^\beta_\omega(R;H^\alpha(I))}, \quad D_2 = ||(I_M - \vartheta)(\tilde{I}_N - \vartheta)u||_{H^\beta_\omega(R;H^\alpha(I))}.$$

Using (2.9), Lemma 2.2 and the Poincaré inequality, we deduce that

$$D_1 \leq cM^{\alpha-r}|u|_{H^\beta_\omega(R;H^r_*(I))} + cN^{\frac{1}{3}+\frac{\beta-q}{2}}|u|_{H^q_\omega(R;H^\alpha(I))},$$

$$D_2 \leq cM^{\alpha-\lambda}|\tilde{I}_N u - u|_{H^\beta_\omega(R;H^\lambda_*(I))} \leq cM^{\alpha-\lambda}N^{\frac{1}{3}+\frac{\beta-\sigma}{2}}|u|_{H^\sigma_\omega(R;H^\lambda_*(I))}.$$

Corollary 2.1. *For any* $u \in L^2_\omega(R; H^r_*(I)) \cap H^q_\omega(R; L^2(I)) \cap H^\sigma_\omega(R; H^\lambda_*(I))$, *integer* $r, \lambda, s, \sigma \geq 1$ *and* $\phi \in V_{M,N}$,

$$\begin{aligned}&|(u,\phi)_\omega - (u,\phi)_{\omega,M,N}|\\ &\leq c(M^{-r}|u|_{L^2_\omega(R;H^r_*(I))} + N^{\frac{1}{3}-\frac{q}{2}}|u|_{H^q_\omega(R;L^2(I))} + M^{-\lambda}N^{\frac{1}{3}-\frac{\sigma}{2}}|u|_{H^\sigma_\omega(R;H^\lambda_*(I))})||\phi||_\omega.\end{aligned}$$

Proof. By (2.25) and (2.26),

$$\begin{aligned}|(u,\phi)_\omega - (u,\phi)_{\omega,M,N}| &\leq |(u,\phi)_\omega - (P_{M-1,N}u,\phi)_\omega| + |(P_{M-1,N}u,\phi)_{\omega,M,N} - (I_{M,N}u,\phi)_{\omega,M,N}|\\ &\leq c(||P_{M-1,N}u - u||_\omega + ||I_{M,N}u - u||_\omega)||\phi||_\omega.\end{aligned}$$

Due to $H^r_*(I) \subseteq H^r_A(I)$, the desired result follows from the above, (2.24) and Theorem 2.1 with $\alpha = \beta = 0$.

3. Mixed Legendre-Hermite Pseudospectral Method

In this section, we propose the mixed Legendre-Hermite pseudospectral method for non-isotropic heat transfer in an infinite plate.

Let $\tilde{R} = \{y \mid -\infty < y < \infty\}$ and $\tilde{\Omega} = I \times \tilde{R}$ with the boundary $\tilde{\Gamma} = \{(x,y) |\ |x| = 1\}$. $W(x,y,t)$ is the temperature. The positive constants ν and μ stand for the conductivities. a and b are convective constants. $F(x,y,t)$ and $W_0(x,y)$ describe the heat source and the initial state, respectively. We consider the following initial-boundary value problem,

$$\begin{cases}\partial_t W(x,y,t) + a\partial_x W(x,y,t) + b\partial_y W(x,y,t) & \\ \qquad -\nu\partial_x^2 W(x,y,t) - \mu\partial_y^2 W(x,y,t) = F(x,y,t), & (x,y)\in\tilde{\Omega},\quad 0<t\leq T,\\ W(x,y,t) = W_1(x,y,t), & (x,y) \text{ on } \tilde{\Gamma},\quad 0<t\leq T,\\ W(x,y,0) = W_0(x,y), & (x,y)\in\bar{\tilde{\Omega}}.\end{cases} \tag{3.1}$$

As we know, if $F(x,y,t)$ and $W_0(x,y)$ decays exponentially as $|y| \to \infty$, then the solution of (3.1) also decays exponentially as $|y| \to \infty$. To fit this behavior, it seems reasonable to approximate (3.1) in the y-direction directly by the Hermite interpolation with the base functions $e^{-\alpha y^2}H_n(y), \alpha > 0$, which are mutually orthogonal associated with the weight function $e^{(2\alpha-1)y^2}$. However it does not works in our case. To show this, we multiply (3.1) by $ve^{(2\alpha-1)y^2}$ and integrate the result over $\tilde{\Omega}$. Then we obtain the following term corresponding to the last term at the left side of (3.1),

$$\mu\int_{\tilde{\Omega}}\partial_y W(x,y,t)\partial_y v(x,y,t)e^{(2\alpha-1)y^2}dxdy + 2\mu(2\alpha-1)\int_{\tilde{\Omega}} y\partial_y W(x,y,t)v(x,y,t)e^{(2\alpha-1)y^2}dxdy.$$

Clearly, the above quantity with $W(x,y,t) = v(x,y,t)$ might be negative. In other words, the leading term in (3.1), $-\nu\partial_x^2 W(x,y,t) - \mu\partial_y^2 W(x,y,t)$, loses the ellipticity in the weighted Sobolev space. To remedy this deficiency, we make the transformation

$$z = \frac{y}{2\sqrt{\mu(t+1)}}, \qquad s = \ln(t+1). \tag{3.2}$$

Accordingly,

$$U(x,z,s) = W(x,y,t), \quad U_0(x,z) = W_0(x,y), \quad f(x,z,s) = (t+1)F(x,y,t), \quad S = \ln(T+1).$$

Then the equation in (3.1) becomes

$$\begin{aligned}\partial_s U(x,z,s) &+ ae^s\partial_x U(x,z,s) + \tfrac{b}{2\sqrt{\mu}}e^{s/2}\partial_z U(x,z,s) - \tfrac{1}{2}z\partial_z U(x,z,s)\\ &-\nu e^s\partial_x^2 U(x,z,s) - \tfrac{1}{4}\partial_z^2 U(x,z,s) = f(x,z,s), \qquad (x,z)\in\Omega,\ 0<s\leq S.\end{aligned} \tag{3.3}$$

Next, we try to choose a suitable weight function so that the corresponding weak formulation of (3.3) is well-posed in the related weighted Sobolev space. To do this, we multiply (3.3) by

$ve^{(2\alpha-1)z^2}$ and integrate the resulting equation over Ω. Then we obtain the following term corresponding to the last two term at the left side of (3.3),

$$\frac{1}{4}\int_\Omega \partial_y U(x,z,s)\partial_z v(x,z,s)e^{(2\alpha-1)z^2}dxdz + (\alpha-1)\int_\Omega z\partial_z W(x,z,s)v(x,z,s)e^{(2\alpha-1)z^2}dxdz.$$

We find that the above term is always non-negative, if and only if $\alpha = 1$. This is the main reason why we use the Hermite interpolation with the base functions $e^{-z^2}H_n(z)$. Moreover, in this case, the relation (2.12) simplifies actual computation and numerical analysis essentially.

Remark 3.1. The transformation (3.2) is similar to the similarity transformation used in [8, 11]. But they are not exactly the same, since the domain $\tilde{\Omega}$ is not the whole space. As a result, the reformed equation (3.3) is no longer a simple heat equation. Thereby, we have to deal with it very carefully.

In the forthcoming discussions, we use the same notations as in the last section, such as $I, R, \Omega, \Gamma, \omega, L^2_\omega(\Omega), (u,w)_\omega$ and $||u||$, etc.. For any $r > 0$, we define the space $H^r_\omega(\Omega)$ and its norm $||u||_{r,\omega}$ as usual. Furthermore, $H^r_{0,\omega}(\Omega)$ denotes the closure in $H^r_\omega(\Omega)$ of the set consisting of all infinitely differentiable functions with compact support in Ω. For $r < 0$, $H^r_\omega(\Omega) = (H^{-r}_{0,\omega}(\Omega))'$. We also let $V = L^\infty(0,T;L^2_\omega(\Omega)) \cap L^2(0,T;H^1_{0,\omega}(\Omega))$, equipped with the norm

$$||u||_V = (ess \sup_{0\le s\le S} ||u(s)||^2_\omega + \int_0^S |u(\eta)|^2_{1,\omega}d\eta)^{\frac{1}{2}}.$$

For simplicity of statements, we assume $W_1(x,y,t) \equiv 0$. Let $u(x,z) \in H^1_{0,\omega}(\Omega)$. By multiplying (3.3) by $u(x,z)\omega(z)$, integrating the result over Ω, and noticing that

$$-2(\partial_z U(s),\ zu)_\omega - (\partial_z^2 U(s),\ u)_\omega = (\partial_z U(s),\ \partial_z u)_\omega,$$

we derive a weak formulation of (3.3). It is to seek $U(s) \in V$ for $0 \le s \le S$, such that

$$\begin{aligned}(\partial_s U(s),u)_\omega &+ ae^s(\partial_x U(s),u)_\omega + \frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z U(s),u)_\omega + \nu e^s(\partial_x U(s),\partial_x u)_\omega \\ &+\frac{1}{4}(\partial_z U(s),\partial_z u)_\omega = (f(s),u)_\omega, \qquad \forall\ u \in H^1_{0,\omega}(\Omega),\ 0 < s \le S. \end{aligned} \tag{3.4}$$

It is shown in [15] that if $U_0 \in H^r_\omega(\Omega), f \in L^2(0,S;H^{r-1}_\omega(\Omega))$ and $r \ge 0$, then (3.4) has a unique solution $U \in L^\infty(0,S;H^r_\omega(\Omega)) \cap L^2(0,S;H^{r+1}_\omega(\Omega))$.

A mixed Legendre-Hermite pseudospectral scheme for (3.4) is to find $u_{M,N}(s) \in V^0_{M,N}$ for $0 \le s \le S$, such that

$$\begin{cases} (\partial_s u_{M,N}(s),\phi)_{\omega,M,N} + ae^s(\partial_x u_{M,N}(s),\phi)_{\omega,M,N} \\ \quad + \frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z u_{M,N}(s),\phi)_{\omega,M,N} + \nu e^s(\partial_x u_{M,N}(s),\partial_x\phi)_{\omega,M,N} \\ \quad + \frac{1}{4}(\partial_z u_{M,N}(s),\partial_z\phi)_{\omega,M,N} = (f(s),\phi)_{\omega,M,N}, \quad \forall\ \phi \in V^0_{M,N},\ 0 < s \le S, \\ u_{M,N}(0) = I_{M,N}U_0. \end{cases} \tag{3.5}$$

Thanks to (2.25), (3.5) is equivalent to

$$\begin{cases} (\partial_s u_{M,N}(s),\phi)_{\omega,M,N} + ae^s(\partial_x u_{M,N}(s),\phi)_\omega \\ \quad + \frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z u_{M,N}(s),\phi)_{\omega,M,N} + \nu e^s(\partial_x u_{M,N}(s),\partial_x\phi)_\omega \\ \quad + \frac{1}{4}(\partial_z u_{M,N}(s),\partial_z\phi)_{\omega,M,N} = (f(s),\phi)_{\omega,M,N}, \quad \forall\ \phi \in V^0_{M,N},\ 0 < s \le S, \\ u_{M,N}(0) = I_{M,N}U_0. \end{cases} \tag{3.6}$$

Now, we describe the implementation of algorithm of (3.6). Let (see [20])

$$\phi_m(x) = c_m\ (L_m(x) - L_{m+2}(x)), \quad c_m = \frac{1}{\sqrt{4m+6}}, \qquad 0 \le m \le M-2. \tag{3.7}$$

Clearly

$$V^0_{M,N} = span\{\phi_m(x)\tilde{H}_n(z), \quad 0 \le m \le M-2, \quad 0 \le n \le N\}.$$

We expand the numerical solution as

$$u_{M,N}(x,z,s) = \sum_{m=0}^{M-2}\sum_{n=0}^{N} v_{m,n}(s)\phi_m(x)\tilde{H}_n(z).$$

Moreover, we set

$$V(s) = (v_{0,0}(s),\ v_{1,0}(s),\ \cdots,\ v_{M-2,0}(s),\ \ v_{0,1}(s),\ v_{1,1}(s),\ \cdots,\ v_{M-2,1}(s),$$
$$\cdots, v_{0,N}(s), v_{1,N}(s), \cdots, v_{M-2,N}(s))^T,$$
$$F(s) = (f_{0,0}(s),\ f_{1,0}(s),\ \cdots,\ f_{M-2,0}(s),\ \ f_{0,1}(s),\ f_{1,1}(s),\ \cdots,\ f_{M-2,1}(s),$$
$$\cdots,\ f_{0,N}(s),\ f_{1,N}(s),\ \cdots,\ f_{M-2,N}(s))^T,$$
$$f_{m,n} = (f(s), \phi_m\tilde{H}_n)_{\omega,M,N}, \qquad 0 \le m \le M-2,\ 0 \le n \le N.$$

Taking $\phi = \phi_{m'}(x)\tilde{H}_{n'}(z)$ in (3.6), we obtain the following set of ordinary differential equations,

$$(B_1^T \otimes A_1)\frac{d}{ds}V(s) + ae^s(B_2^T \otimes A_2)V(s) + \frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(B_3^T \otimes A_3)\ V(s)$$
$$+\nu e^s(B_4^T \otimes A_4)V(s) + \frac{1}{4}(B_5^T \otimes A_5)V(s) = F(s), \tag{3.8}$$

where the matrices $A_q = (a^{(q)}_{m',m})$ and $B_q = (b^{(q)}_{n,n'})$, with the entries

$$a^{(1)}_{m',m} = a^{(3)}_{m',m} = a^{(5)}_{m',m} = \begin{cases} (\phi_{m'}, \phi_m)_M, & m' = m = M-2, \\ \int_I \phi_{m'}(x)\phi_m(x)dx, & \text{otherwise,} \end{cases}$$

$$a^{(2)}_{m',m} = \int_I \phi_{m'}(x)\partial_x\phi_m(x)dx, \quad a^{(4)}_{m',m} = \int_I \partial_x\phi_{m'}(x)\partial_x\phi_m(x)dx, \quad 0 \le m', m \le M-2,$$
$$b^{(1)}_{n,n'} = b^{(2)}_{n,n'} = b^{(4)}_{n,n'} = \int_R \tilde{H}_n(z)\tilde{H}_{n'}(z)\omega(z)dz, \quad b^{(3)}_{n,n'} = \int_R \partial_z\tilde{H}_n(z)\tilde{H}_{n'}(z)\omega(z)dz,$$
$$b^{(5)}_{n,n'} = \int_R \partial_z\tilde{H}_n(z)\partial_z\tilde{H}_{n'}(z)\omega(z)dz, \quad 0 \le n, n' \le N.$$

We next calculate the entries of A_q and B_q. Firstly, by (2.2) and (3.7), we obtain that for $0 \le m', m \le M-2$,

$$a^{(1)}_{m',m} = a^{(3)}_{m',m} = a^{(5)}_{m',m} = \begin{cases} (\phi_{m'}, \phi_m)_M, & m' = m = M-2, \\ c_{m'}c_m(\frac{2}{2m+1} + \frac{2}{2m+5}), & m' = m < M-2, \\ -c_{m'}c_m\frac{2}{2m'+1}, & m' = m+2, \\ -c_{m'}c_m\frac{2}{2m+1}, & m' = m-2, \\ 0, & \text{otherwise.} \end{cases}$$

Further, we use (2.1) and (2.2) to obtain that for $0 \le m', m \le M-2$,

$$a^{(2)}_{m',m} = -a^{(2)}_{m,m'} = \begin{cases} 2c_{m'}c_m, & m = m'+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$a^{(4)}_{m',m} = \begin{cases} 1, & m' = m, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, using (2.12) and (2.13) yields that for $0 \le n, n' \le N$,

$$b^{(1)}_{n,n'} = b^{(2)}_{n,n'} = b^{(4)}_{n,n'} = \begin{cases} \sqrt{\pi}, & n = n', \\ 0, & \text{otherwise}, \end{cases}$$

$$b^{(3)}_{n,n'} = \begin{cases} -\sqrt{2(n+1)\pi}, & n = n'-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$b^{(5)}_{n,n'} = \begin{cases} 2(n+1)\sqrt{\pi}, & n = n', \\ 0, & \text{otherwise}. \end{cases}$$

Obviously, A_1, A_3 and A_5 are five-diagonal matrices, A_2 and B_3 are three-diagonal matrices, A_4, B_1, B_2, B_4 and B_5 are diagonal matrices. This feature simplifies the actual calculation.

Now,

let

$$\mathbf{f}_n(s) = (f_{0,n}(s),\ f_{1,n}(s),\ \cdots,\ f_{M-2,n}(s))^T, \quad 0 \le n \le N,$$

$$\mathbf{v}_n(s) = (v_{0,n}(s),\ v_{1,n}(s),\ \cdots,\ v_{M-2,n}(s))^T, \quad 0 \le n \le N.$$

Then by the definition of Kronecker product, (3.8) reads that for $s > 0$,

$$\begin{cases} A_1 \frac{d}{ds}\mathbf{v}_0(s) + ae^s A_2 \mathbf{v}_0(s) + \nu e^s I_{(M-1)\times(M-1)}\mathbf{v}_0(s) + \frac{1}{2}A_5\mathbf{v}_0(s) = \frac{1}{\sqrt{\pi}}\mathbf{f}_0, \\ A_1 \frac{d}{ds}\mathbf{v}_j(s) + ae^s A_2 \mathbf{v}_j(s) + \nu e^s I_{(M-1)\times(M-1)}\mathbf{v}_j(s) + \frac{j+1}{2}A_5\mathbf{v}_j(s) \\ \qquad = \frac{1}{\sqrt{\pi}}\mathbf{f}_j + \sqrt{\frac{j}{2\mu}}\ be^{\frac{s}{2}}A_3\mathbf{v}_{j-1}(s), \qquad 1 \le j \le N. \end{cases} \tag{3.9}$$

In addition, $\mathbf{v}_n(0)$ is determined by the system $A_1\mathbf{v}_n(0) = \mathbf{c}_n$, $0 \le n \le N$, where

$$\mathbf{c}_n = (c_{0,n}, c_{1,n}, \cdots, c_{M-2,n})^T, \quad c_{m,n} = \frac{1}{\sqrt{\pi}}(U_0(x,z), \phi_m(x)\tilde{H}_n(z))_{\omega,M,N}.$$

In actual calculation, we firstly resolve the first equation of (3.9) to obtain $\mathbf{v}_0(s)$. Then we use the forward substitution procedure to evaluate other vectors $\mathbf{v}_j(s)$, $1 \le j \le N$, successively.

By the variable transformation (3.2), the numerical solution of the original problem (3.1) is given by

$$w_{M,N}(x,y,t) = u_{M,N}(x, \frac{y}{2\sqrt{\mu(t+1)}}, \ln(t+1)). \tag{3.10}$$

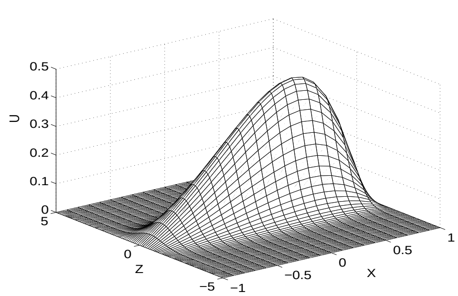

Figure 1: The exact solution

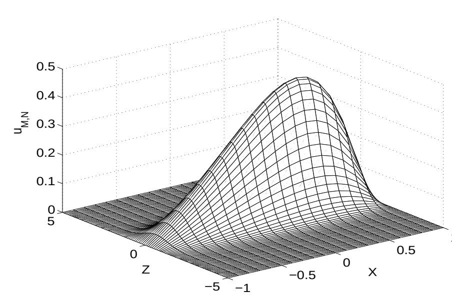

Figure 2: The numerical solution

Now, we present some numerical results. We take the test function $U(x,z,s) = (1 - x^2)e^{x-\frac{1}{2}s-z^2}$. Let $a = 1$, $b = \sqrt{2}$, $\nu = 1$, $\mu = \frac{1}{2}$ in (3.9). We use the explicit Runge-Kutta method of fourth order in time s with the step size τ.

We plot the exact solution $U(x,z,s)$ at $s=2$ in Figure 1, and the numerical solution $u_{M,N}(x,z,s)$ at $s=2$, with $M=7, N=49$ and $\tau=0.001$ in Figure 2. They demonstrate that the numerical solution fits the exact solution very well.

4. Error Estimate

In this section, we deal with the convergence of scheme (3.6). In order to obtain the optimal error estimate, we introduce a special orthogonal projection. Let $\beta,\gamma>0$ and

$$A_{\omega,\beta,\gamma}(u,w)=\beta(\partial_x u,\ \partial_x w)_\omega+\gamma(\partial_z u,\ \partial_z w)_\omega.$$

The orthogonal projection $P^{1,0}_{M,N,\beta,\gamma}: H^1_{0,\omega}(\Omega)\to V^0_{M,N}$ is defined by

$$A_{\omega,\beta,\gamma}(P^{1,0}_{M,N,\beta,\gamma}u-u,\ \phi)=0,\qquad \forall\ \phi\in V^0_{M,N}. \tag{4.1}$$

For describing approximation results, we introduce the space

$$M^{r,q}_{\omega,*}(\Omega)=L^2_\omega(R;H^r_*(I))\cap H^q_\omega(R;L^2(I))\cap H^1_\omega(R;H^{r-1}_*(I))\cap H^{q-1}_\omega(R;H^1(I)),\quad r,q\ge 1,$$

with the norm

$$||u||_{M^{r,q}_{\omega,*}}=(||u||^2_{L^2_\omega(R;H^r_*(I))}+||u||^2_{H^q_\omega(R;L^2(I))}+||u||^2_{H^1_\omega(R;H^{r-1}_*(I))}+||u||^2_{H^{q-1}_\omega(R;H^1(I))})^{\frac12}.$$

In particular, for integers $r,q\ge1$, we use the notations $|u|_{M^{r,q}_{\omega,*}}=B^{r,q}_{1,1}(u)$ and

$$B^{r,q}_{\beta,\gamma}(u)=(\beta|u|^2_{L^2_\omega(R;H^r_*(I))}+\gamma|u|^2_{H^q_\omega(R;L^2(I))}+\gamma|u|^2_{H^1_\omega(R;H^{r-1}_*(I))}+\beta|u|^2_{H^{q-1}_\omega(R;H^1(I))})^{\frac12}.$$

It is proved in [15] that for any $u\in H^1_{0,\omega}(\Omega)\cap M^{r,q}_{\omega,*,\beta,\gamma}(\Omega)$, integers $r,\ q\ge1$ and $0\le\mu\le1$,

$$||P^{1,0}_{M,N,\beta,\gamma}u-u||_{\mu,\omega}\le\frac{c}{(\min(\beta,\gamma))^{\frac12}}(M^{1-r}+N^{\frac{1-q}{2}})(M^{-1}+N^{-\frac12})^{1-\mu}B^{r,q}_{\beta,\gamma}(u). \tag{4.2}$$

Now, let U be the solution of (3.4) and $U_{M,N}=P^{1,0}_{M,N,\beta,\gamma}U$ with $\beta(s)=\nu e^s$ and $\gamma=\frac14$. By using (2.25) and (4.1), we have from (3.4) that for $0<s\le S$,

$$\begin{cases}(\partial_s U_{M,N}(s),\phi)_{\omega,M,N}+ae^s(\partial_x U_{M,N}(s),\phi)_\omega\\ \quad+\frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z U_{M,N}(s),\phi)_{\omega,M,N}+\nu e^s(\partial_x U_{M,N}(s),\partial_x\phi)_\omega\\ \quad+\frac14(\partial_z U_{M,N}(s),\partial_z\phi)_{\omega,M,N}+\sum\limits_{j=1}^{6}G_j(s,\phi)=(f(s),\phi)_{\omega,M,N},\quad\forall\ \phi\in V^0_{M,N},\\ U_{M,N}(0)=P^{1,0}_{M,N,\beta,\gamma}U_0\end{cases} \tag{4.3}$$

where

$$\begin{aligned}&G_1(s,\phi)=(\partial_s U(s),\phi)_\omega-(\partial_s U_{M,N}(s),\phi)_{\omega,M,N},\\ &G_2(s,\phi)=ae^s(\partial_x U(s)-\partial_x U_{M,N}(s),\phi)_\omega,\\ &G_3(s,\phi)=\nu e^s(\partial_x U(s)-\partial_x U_{M,N}(s),\partial_x\phi)_\omega,\\ &G_4(s,\phi)=\tfrac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}((\partial_z U(s),\phi)_\omega-(\partial_z U_{M,N}(s),\phi)_{\omega,M,N}),\\ &G_5(s,\phi)=\tfrac14((\partial_z U(s),\partial_z\phi)_\omega-(\partial_z U_{M,N}(s),\partial_z\phi)_{\omega,M,N}),\\ &G_6(s,\phi)=(f(s),\phi)_{\omega,M,N}-(f(s),\phi)_\omega.\end{aligned}$$

Let $u_{M,N}$ be the solution of (3.6) and $\tilde U_{M,N}=u_{M,N}-U_{M,N}$. Subtracting (4.3) from (3.6) yields that

$$\begin{cases}(\partial_s\tilde U_{M,N}(s),\phi)_{\omega,M,N}+ae^s(\partial_x\tilde U_{M,N}(s),\phi)_\omega\\ \quad+\frac{b}{2\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z\tilde U_{M,N}(s),\phi)_{\omega,M,N}+\nu e^s(\partial_x\tilde U_{M,N}(s),\partial_x\phi)_\omega\\ \quad+\frac14(\partial_z\tilde U_{M,N}(s),\partial_z\phi)_{\omega,M,N}=\sum\limits_{j=1}^{6}G_j(s,\phi),\quad\forall\ \phi\in V^0_{M,N},\ 0<s\le S,\\ \tilde U_{M,N}(0)=I_{M,N}U_0-P^{1,0}_{M,N,\beta,\gamma}U_0.\end{cases} \tag{4.4}$$

Taking $\phi = 2\tilde{U}_{M,N}$ in (4.4), we deduce that

$$\partial_s||\tilde{U}_{M,N}(s)||^2_{\omega,M,N} + 2\nu e^s||\partial_x\tilde{U}_{M,N}(s)||^2_\omega + \frac{1}{2}||\partial_z\tilde{U}_{M,N}(s)||^2_{\omega,M,N} = 2\sum_{j=1}^{7} G_j(s,\tilde{U}_{M,N}(s)) \quad (4.5)$$

where

$$G_7(s,\tilde{U}_{M,N}(s)) = -\frac{b}{\sqrt{\mu}}e^{\frac{s}{2}}(\partial_z\tilde{U}_{M,N}(s),\tilde{U}_{M,N}(s))_{\omega,M,N}.$$

Therefore, it remains to estimate the terms $|G_j(s,\tilde{U}_{M,N}(s))|$. Firstly, by (2.24)-(2.26) and (4.2),

$$\begin{aligned} &2|G_1(s,\tilde{U}_{M,N}(s))| = 2|(\partial_sU(s) - P_{M-1,N}\partial_sU(s),\tilde{U}_{M,N}(s))_\omega \\ &\qquad +(P_{M-1,N}\partial_sU(s) - \partial_sU_{M,N}(s),\tilde{U}_{M,N}(s))_{\omega,M,N}| \\ &\le c(||\partial_sU(s) - P_{M-1,N}\partial_sU(s)||_\omega + ||P_{M-1,N}\partial_sU(s) - \partial_sU_{M,N}(s)||_\omega)||\tilde{U}_{M,N}(s)||_\omega \\ &\le c(||\partial_sU(s) - P_{M-1,N}\partial_sU(s)||_\omega + ||\partial_sU(s) - \partial_sU_{M,N}(s)||_\omega)||\tilde{U}_{M,N}(s)||_\omega \\ &\le c(M^{2-2r} + N^{1-q})(M^{-2} + N^{-1})(B^{r,q}_{\beta,\gamma}(\partial_sU(s)))^2 \\ &\qquad +c(M^{-2r} + N^{-q})|\partial_sU(s)|^2_{H^{r,q}_{\omega,A}} + \tfrac{1}{2}||\tilde{U}_{M,N}(s)||^2_\omega. \end{aligned} \quad (4.6)$$

Next, by the Cauchy inequality,

$$\begin{aligned} &2|G_2(s,\tilde{U}_{M,N}(s))| + 2|G_3(s,\tilde{U}_{M,N}(s))| \\ &\le (a+\nu)e^s||\partial_x(U(s) - U_{M,N}(s))||^2_\omega + ae^s||\tilde{U}_{M,N}(s)||^2_\omega + \nu e^s||\partial_x\tilde{U}_{M,N}(s)||^2_\omega. \end{aligned} \quad (4.7)$$

In virtue of (2.25) and (2.26), we deduce that

$$\begin{aligned} &2|G_4(s,\tilde{U}_{M,N}(s))| \\ &= \tfrac{b}{\sqrt{\mu}}e^{\frac{s}{2}}|(\partial_zU(s) - P_{M-1,N}\partial_zU(s),\tilde{U}_{M,N}(s))_\omega \\ &\quad +(P_{M-1,N}\partial_zU(s) - \partial_zU_{M,N}(s),\tilde{U}_{M,N}(s))_{\omega,M,N}| \\ &\le \tfrac{cb}{\sqrt{\mu}}e^{\frac{s}{2}}(||P_{M-1,N}\partial_zU(s) - \partial_zU(s)||_\omega + ||\partial_z(U(s) - U_{M,N}(s))||_\omega)||\tilde{U}_{M,N}(s)||_\omega \\ &\le c||P_{M-1,N}\partial_zU(s) - \partial_zU(s)||^2_\omega + c||\partial_z(U(s) - U_{M,N}(s))||^2_\omega + \tfrac{b^2}{2\mu}e^s||\tilde{U}_{M,N}(s)||^2_\omega. \end{aligned} \quad (4.8)$$

Similarly,

$$2|G_5(s,\tilde{U}_{M,N}(s))| \le c||P_{M-1,N}\partial_zU(s) - \partial_zU(s)||^2_\omega + c||\partial_z(U(s) - U_{M,N}(s))||^2_\omega + \frac{1}{8}||\partial_z\tilde{U}_{M,N}(s)||^2_\omega. \quad (4.9)$$

Putting (4.7)-(4.9) together, and using (2.24) and (4.2), we derive that

$$\begin{aligned} 2\sum_{j=2}^{5}|G_j(s,\tilde{U}_{M,N}(s))| &\le c(M^{-2r}+N^{-q})|\partial_zU(s)|^2_{H^{r,q}_{\omega,A}} + c(1+\frac{a}{\nu})(M^{-2r}+N^{-q})(B^{r+1,q+1}_{\beta,\gamma}(U(s)))^2 \\ &\quad + \frac{1}{8}||\partial_z\tilde{U}_{M,N}(s)||^2_\omega + \nu e^s||\partial_x\tilde{U}_{M,N}(s)||^2_\omega + (ae^s + \frac{b^2}{2\mu}e^s)||\tilde{U}_{M,N}(s)||^2_\omega. \end{aligned} \quad (4.10)$$

Using Corollary 2.1 yields that

$$\begin{aligned} 2|G_6(s,\tilde{U}_{M,N}(s))| &\le c(M^{-2r}|f|^2_{L^2_\omega(R;H^r_*(I))} + N^{-q-\frac{1}{3}}|f|^2_{H^{q+1}_\omega(R;L^2(I))} \\ &\quad + M^{-2\lambda}N^{\frac{2}{3}-\sigma}|f|^2_{H^\sigma_\omega(R;H^\lambda_*(I))}) + ||\tilde{U}_{M,N}(s)||^2_\omega. \end{aligned} \quad (4.11)$$

Furthermore, by (2.26) and the Cauchy inequality,

$$2|G_7(s,\tilde{U}_{M,N}(s))| \le \frac{3b}{\sqrt{\mu}}e^{\frac{s}{2}}||\partial_z\tilde{U}_{M,N}(s)||_\omega||\tilde{U}_{M,N}(s))||_\omega \le \frac{1}{8}||\partial_z\tilde{U}_{M,N}(s)||^2_\omega + \frac{18b^2}{\mu}e^s||\tilde{U}_{M,N}(s)||^2_\omega. \quad (4.12)$$

It is observed that $\beta(s)=\nu$ at $s=0$ and $\gamma=\frac{1}{4}$. Therefore, using (4.2) and Theorem 2.1 with $\alpha=\beta=0$ gives that

$$||\tilde{U}_{M,N}(0)||_\omega^2 \le c||P^{1,0}_{M,N,\nu,\frac{1}{4}}U_0-U_0||_\omega^2+c||I_{M,N}U_0-U_0||_\omega^2$$

$$\le c(M^{-2r}+N^{-q-\frac{1}{3}}+M^{-2\lambda}N^{\frac{2}{3}-\sigma})(|U_0|^2_{L^2_\omega(R;H^r_*(I))}+|U_0|^2_{H^{q+1}_\omega(R;L^2(I))}+|U_0|^2_{H^\sigma_\omega(R;H^\lambda_*(I))})$$

$$+c(M^{2-2r}+N^{1-q})(M^{-2}+N^{-1})|U_0|^2_{M^{r,q}_{\omega,*}}. \tag{4.13}$$

Now, we use the following notation to present the average numerical errors,

$$E(u(s))=||u(s)||_\omega^2+\int_0^s(\nu e^\eta||\partial_x u(\eta)||_\omega^2+\frac{1}{4}||\partial_z u(\eta)||_\omega^2)d\eta.$$

By inserting (4.6) and (4.10)-(4.12) into (4.5) and using the fact $H^r_*(I)\subseteq H^r_A(I)$, we find that

$$\frac{d}{ds}(E(\tilde{U}_{M,N}(s)))\le D_{M,N}(s)+\phi(s)E(\tilde{U}_{M,N}(s)), \tag{4.14}$$

where

$$\phi(s)=2+ae^s+\frac{37b^2}{2\mu}e^s,$$

$$\begin{aligned}D_{M,N}(s)=&\,c(M^{-2r}+N^{-q})|\partial_s U(s)|^2_{H^{r,q}_{\omega,A}}+c(M^{2-2r}+N^{1-q})(M^{-2}+N^{-1})(B^{r,q}_{\beta,\gamma}(\partial_s U(s)))^2\\&+c(1+\tfrac{a}{\nu})(M^{-2r}+N^{-q})(B^{r+1,q+1}_{\beta,\gamma}(U(s)))^2\\&+c(M^{-2r}|f|^2_{L^2_\omega(R;H^r_*(I))}+N^{-q-\frac{1}{3}}|f|^2_{H^{q+1}_\omega(R;L^2(I))}+M^{-2\lambda}N^{\frac{2}{3}-\sigma}|f|^2_{H^\sigma_\omega(R;H^\lambda_*(I))}).\end{aligned}$$

Furthermore, (4.14) implies that

$$\frac{d}{ds}(E(\tilde{U}_{M,N}(s))\ \exp(-\int_0^s\phi(\eta)d\eta))\le D_{M,N}(s)\exp(-\int_0^s\phi(\eta)d\eta).$$

By integrating the above with respect to s and using (4.13), we reach that

$$E(\tilde{U}_{M,N}(s))\le\rho_{M,N}(s)\ \exp(\int_0^s\phi(\eta)d\eta) \tag{4.15}$$

where

$$\begin{aligned}\rho_{M,N}(s)=&\int_0^s D_{M,N}(\eta)\exp(-\int_0^\eta\phi(\xi)d\xi)d\eta+c(M^{2-2r}+N^{1-q})(M^{-2}+N^{-1})|U_0|^2_{M^{r,q}_{\omega,*}}\\&+c(M^{-2r}+N^{-q-\frac{1}{3}}+M^{-2\lambda}N^{\frac{2}{3}-\sigma})\\&(|U_0|^2_{L^2_\omega(R;H^r_*(I))}+|U_0|^2_{H^{q+1}_\omega(R;L^2(I))}+|U_0|^2_{H^\sigma_\omega(R;H^\lambda_*(I))}).\end{aligned}$$

For any weight function $\chi(s)$, we define the weighted space $H^\delta_\chi(0,S;H^{r,q}_{\omega,A}(\Omega))$ and $H^\delta_\chi(0,S;M^{r,q}_{\omega,*}(\Omega))$ in the usual way. Then the following conclusion comes form (4.15) immediately.

Theorem 4.1. *Let integer* $r,q,\lambda,\sigma,\ge 1$. *If* $U\in L^2(0,S;H^1_{0,\omega}(\Omega))\cap L^2_{e^s}(0,S;M^{r+1,q+1}_{\omega,*}(\Omega))\cap H^1_{e^s}(0,S;M^{r,q}_{\omega,*}(\Omega))\cap H^1(0,S;H^{r,q}_{\omega,A}(\Omega))$, $U_0\in L^2_\omega(R;H^r_*(I))\cap H^{q+1}_\omega(R;L^2(I))\cap H^1_\omega(R;H^{r-1}_*(I))\cap H^{q-1}_\omega(R;H^1(I))\cap H^\sigma_\omega(R;H^\lambda_*(I))$ *and* $f\in L^2(0,S;L^2_\omega(R;H^r_*(I))\cap H^{q+1}_\omega(R;L^2(I))\cap H^\sigma_\omega(R;H^\lambda_*(I)))$, *then for all* $0\le s\le S$,

$$E(\tilde{U}_{M,N}(s))\le\rho_{M,N}(s)\ \exp(\int_0^s\phi(\eta)d\eta).$$

According to (4.2),

$$E(U(s)-U_{M,N}(s)) \le c(M^{2-2r}+N^{1-q})(M^{-2}+N^{-1})(B^{r,q}_{e^s,\frac14}(U(s)))^2 \\ +c(M^{-2r}+N^{-q})\int_0^s (B^{r+1,q+1}_{e^s,\frac14}(U(\eta)))^2 d\eta.$$

The above with Theorem 4.1 leads to

$$E(U(s)-u_{M,N}(s)) \le c^*((M^{2-2r}+N^{1-q})(M^{-2}+N^{-1})+M^{-2r}+N^{-q-\frac13}+M^{-2\lambda}N^{\frac23-\sigma}) \quad (4.16)$$

where c^* is a positive constant depending only on μ, ν, a, b, S and the norms of U, U_0 and f in the space mentioned in Theorem 4.1.

We can use the above results and the transformation (3.2) to derive a sharp error estimate for the numerical solution, given by (3.10), of the original problem (3.1). In particular, the weight function e^s appearing in Theorem 4.1 becomes $t+1$. Thus the numerical errors grow slowly as t increases. Therefore, the transformation (3.2) not only ensures the proper algorithm, but also leads to the better stability of computation.

5. Concluding Remarks

In this paper, we introduced the new Hermite interpolation associated with the weight function e^{z^2}, which is very appropriate for approximations to functions decaying exponentially at the infinity. We also introduced the new mixed Legendre-Hermite interpolation on an infinite strap, based on which we proposed the Legendre-Hermite pseudospectral method for non-isotropic heat transfer in an infinite plate. This is a high order method, and is very suitable for parallel computation. The numerical results demonstrated the high accuracy in the space of this algorithm, and coincide very well with theoretical analysis.

In this paper, we established some basic results on the new Hermite interpolation and the corresponding mixed Legendre-Hermite interpolation. They play important roles in numerical analysis of the related pseudospectral methods for unbounded domains.

Although we only considered a simple model problem in this paper, the proposed method and techniques used in theoretical analysis are also applicable to other problems defined on certain unbounded domains, such as nonlinear heat transfer in an infinite plate, nonlinear wave equations in an infinite strip, some problems in statistical physics and so on.

Appendix: Proof of Lemma 2.1.

We have from [2] that for any $u \in H^1(a,b)$ and $a<b$,

$$\sup_{z\in[a:,b]} |u(z)|^2 \le \frac{c_2}{b-a}||u||^2_{L^2(a,b)} + c_2(b-a)|u|^2_{H^1(a,b)}. \quad (A1)$$

On the other hand, by Lemmas 2.2 and 2.3 of [6], for any $u \in H^1_\omega(R)$,

$$e^{z^2}u^2(z) \le 4||u||_{\omega,R}|u|_{1,\omega,R}, \qquad \forall z \in R. \quad (A2)$$

Let $\Lambda_{N,j} = (\sigma_{N,j+1}, \sigma_{N,j-1})$ and $\Delta_{N,j} = \sigma_{N,j-1} - \sigma_{N,j+1}$. It is proved in [17] that

$$-a_{N+1}(1-N^{-\frac23}) \le c_2\sigma_{N,N}, \quad \sigma_{N,0} \le c_2 a_{N+1}(1-N^{-\frac23}), \quad (A3)$$

and for $1 \le j \le N-1$,

$$\Delta_{N,j} \sim \frac{1}{\sqrt{N+1}}(1-\frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac12}. \quad (A4)$$

Now, by (A1), for $1 \le j \le N-1$,

$$e^{\sigma^2_{N,j}}u^2(\sigma_{N,j}) \le \frac{c_2}{\Delta_{N,j}}||u||^2_{L^2_\omega(\Lambda_{N,j})} + c_2\Delta_{N,j}\int_{\Lambda_{N,j}} (\partial_z(e^{\frac{z^2}{2}}u(z)))^2 dz. \quad (A5)$$

Meanwhile, due to (A2) and (A3), for $j = 0, N$,

$$e^{\sigma_{N,j}^2} u^2(\sigma_{N,j})(1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} \le c_2 N^{\frac{1}{3}} ||u||_{\omega,R} |u|_{1,\omega,R}. \qquad (A6)$$

Therefore, we use (2.18), (A5) and (A6) to obtain that

$$\begin{aligned} ||u||^2_{\omega,N,R} &\le c_2 N^{-\frac{1}{2}} \sum_{j=0}^{N} u^2(\sigma_{N,j}) e^{\sigma_{N,j}^2} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} \\ &\le c_2 N^{-\frac{1}{6}} ||u||_{\omega} |u|_{1,\omega,R} + c_2 N^{-\frac{1}{2}} \sum_{j=1}^{N-1} \frac{1}{\Delta_{N,j}} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} ||u||^2_{L^2_\omega(\Lambda_{N,j})} \\ &\quad + c_2 N^{-\frac{1}{2}} \sum_{j=1}^{N-1} \Delta_{N,j} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} \int_{\Lambda_{N,j}} (\partial_z (e^{\frac{z^2}{2}} u(z)))^2 dz. \qquad (A7) \end{aligned}$$

We next estimate the right side of (A7). Obviously,

$$\int_{\Lambda_{N,j}} (\partial_x (e^{\frac{z^2}{2}} u(z)))^2 dz \le c_2 \int_{\Lambda_{N,j}} e^{z^2} (z^2 u^2(z) + (\partial_z u(z))^2) dz. \qquad (A8)$$

Furthermore, by (A4),

$$N^{-\frac{1}{2}} \frac{1}{\Delta_{N,j}} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} \le c_2. \qquad (A9)$$

On the other hand, (A3) implies that

$$|1 - \frac{|\sigma_{N,j}|}{a_{N+1}}| \ge |1 - \frac{|\sigma_{N,0}|}{a_{N+1}}| \ge c_2 N^{-\frac{2}{3}}, \quad 1 \le j \le N-1.$$

A combination of the above estimate and (A4) leads to that

$$N^{-\frac{1}{2}} \Delta_{N,j} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-\frac{1}{2}} \le c_2 N^{-1} (1 - \frac{|\sigma_{N,j}|}{a_{N+1}})^{-1} \le c_2 N^{-\frac{1}{3}}. \qquad (A10)$$

Moreover, by Lemma 2.2 of [6],

$$\int_R e^{z^2} z^2 u^2(z) dz \le \int_R (\partial_z u(z))^2 e^{z^2} dz. \qquad (A11)$$

Substituting (A8)-(A10) into (A7), we use (A11) to obtain that

$$||u||^2_{\omega,N} \le c_2 N^{-\frac{1}{6}} ||u||_{\omega,R} |u|_{1,\omega,R} + c_2 ||u||^2_{\omega,R} + c_2 N^{-\frac{1}{3}} |u|^2_{1,\omega,R} \le c_2 ||u||^2_{\omega,R} + c_2 N^{-\frac{1}{3}} |u|^2_{1,\omega,R}. \quad \square$$

Proof of Remark 2.1. We have from (2.12) that

$$\partial_z^r u(z) = (-1)^r 2^{\frac{r}{2}} \sum_{n=0}^{\infty} \hat{u}_n (\prod_{j=1}^{r} (n+j))^{\frac{1}{2}} \ \tilde{H}_{n+r}(z).$$

Thus by (2.13),

$$2^r \sqrt{\pi} \sum_{n=0}^{\infty} (n+1)^r \hat{u}_n^2 \le |\partial_z^r u|^2_{\omega,R} \le 2^r \sqrt{\pi} \sum_{n=0}^{\infty} (n+r)^r \hat{u}_n^2.$$

This implies the first result. The second result comes from the above and the fact that $|u|_{k,\omega,R} \le |u|_{r,\omega,R}$ for all $k \le r$.

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