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# THE MORTAR ELEMENT METHOD FOR A NONLINEAR BIHARMONIC EQUATION *1)

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**Abstract**

The mortar element method is a new domain decomposition method(DDM) with nonoverlapping subdomains. It can handle the situation where the mesh on different subdomains need not align across interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. But until now there has been very little work for nonlinear PDEs. In this paper, we will present a mortar-type Morley element method for a nonlinear biharmonic equation which is related to the well-known Navier-Stokes equation. Optimal energy and $H^1$-norm estimates are obtained under a reasonable elliptic regularity assumption.

*Mathematics subject classification:* 65F10, 65N30, 65N55.
*Key words*: Mortar method, Nonlinear biharmonic equation, $H^1$-norm error, Energy norm error.

## 1. Introduction

In recent years, the mortar finite element method as a special domain decomposition methodology appears very attractive because it can handle the situation where meshes on different subdomains need not align across interfaces, and the matching of the solutions on adjacent subdomains is only enforced weakly. We refer to[3],[5],[6]for the general presentation of the mortar element method. Recently, there have been many works in constructing efficient iterative solvers for the discrete system resulting from the mortar element method (cf. In [1],[2],[20],[17],[21],[22]). So far, many mortar element methods were presented for solving linear elliptic problems. Very little work has been done for the nonlinear problems. In this direction, a mortar finite element for quasilinear elliptic problems was considered in [14], while the mortar element methods for some variational inequalities were developed in [4], [12].

The mortar element method for biharmonic problems also attracted many authors' attentions. For instance, the mortar finite element method for some plate elements, like the conforming Hsieh-Clough-Tocher, the reduced Hsieh-Clough-Tocher and a nonconforming Morley element, was studied by Marcinkowski in [15]. But his error estimate requires that the solution is very smooth (in $H^4(\Omega)\cap H_0^2(\Omega)$) which is generally not valid, even for some convex polygonal domains. Recently, Huang, Li and Chen [13] extended this work and obtained an optimal error estimate with a weaker elliptic regularity assumption ($H^3(\Omega)\cap H_0^2(\Omega)$). An efficient multigrid for such kind of mortar element method was proposed in [23]. But till now there have been no results for the nonlinear counterparts. In this paper, we shall design an effective mortar element method for a nonlinear biharmonic equation which is related to the well known Navier-Stokes equation. Optimal energy and $H^1$-norm estimates are obtained under the weaker elliptic regularity assumption($H^3(\Omega)\cap H_0^2(\Omega)$).

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This paper is organized as follows. Section 2 introduces the model problem. In section 3, we shall present the mortar-type Morley element method, some prelimilary shall be given in this section. Optimal energy amd $H_1$ norm error estimates shall be studied in section 4.

## 2. Model Problem

We consider the following nonlinear biharmonic equation:

$$\begin{cases} \frac{1}{R_e}\triangle^2 u = Bu + f & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega$ is a convex polygonal domain in $R^2$, $n = (n_1, n_2)$ denotes the unit outward normal vector along the boundary $\partial\Omega$, and

$$Bu = \partial_x(\partial_y u \triangle u) - \partial_y(\partial_x u \triangle u) = \partial_y u \triangle \partial_x u - \partial_x u \triangle \partial_y u.$$

Let $H^r(\Omega)$ denote the standard Sobolev space of order $r \geq 0$ with respect to domain $\Omega$, equipped with the standard norm $\|\cdot\|_r$. Define the subspace

$$H_0^2(\Omega) = \{v \in H^2(\Omega) : v = \partial_n v = 0 \text{ on } \partial\Omega\}.$$

Let $|\cdot|_r$ be the seminorm over the Sobolev space $H^r(\Omega)$. It is known that $|\cdot|_2$ is a norm over the space $H_0^2(\Omega)$ and (cf. [8] for details)

$$|v|_2 = \|\triangle v\|_0, \quad \forall v \in H_0^2(\Omega).$$

The variational form of (2.1) is to find $u \in H_0^2(\Omega)$ such that

$$\frac{1}{R_e} a(u, v) = (\partial_x u \triangle u, \partial_y v) - (\partial_y u \triangle u, \partial_x v) + (f, v), \quad \forall v \in H_0^2(\Omega), \tag{2.2}$$

where $f$ is a function in $L^2(\Omega)$, and

$$a(u, v) = \int_\Omega \triangle u \triangle v dxdy,$$

$$(f, v) = \int_\Omega f v dxdy.$$

By the Sobolev embedding Theorem, we know that

$$\|\nabla v\|_{L^4} \leq C_0 |v|_2, \quad \text{and} \quad \|v\|_0 \leq C_1 |v|_2, \quad \forall v \in H_0^2(\Omega). \tag{2.3}$$

Here $\|\cdot\|_{L^4}$ is the norm over the space $L^4(\Omega)$. In this paper $C$ with or without subscript and supscript denotes a positive constant.

It is known ([7],[10]) that (2.2) has a unique solution $u \in H_0^2(\Omega)$ which satisfies

$$\|\triangle u\|_0 \leq C_1 R_e \|f\|_0$$

under the assumption

$$R_e < \sqrt{\frac{1}{C_0^2 C_1 \|f\|_0}}. \tag{2.4}$$

## 3. The Mortar-type Morley Nonconforming Element

We now introduce a mortar finite element method for solving (2.2). First, we partition $\Omega$ into nonoverlapping polygonal subdomains such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.$$

They are arranged so that the intersection of $\bar{\Omega}_i \cap \bar{\Omega}_j$ for $i \neq j$ is either an empty set, an edge or a vertex, i.e., the partition is geometrically conforming. The interface

$$\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$$

is broken into a set of disjoint open straight segments $\gamma_m (1 \leq m \leq M)$ (that are the edges of subdomains) called mortars, i.e.

$$\Gamma = \bigcup_{m=1}^{M} \bar{\gamma}_m, \quad \gamma_m \cap \gamma_n = \emptyset, \quad \text{if} \quad m \neq n.$$

We denote the common open edge to $\Omega_i$ and $\Omega_j$ by $\gamma_m$. By $\gamma_{m(i)}$ we emphasize that the edge $\gamma_m$ associated with subdomain $\Omega_i$ is a mortar, while the other edge, which geometrically occupies the same place, is denoted by $\delta_{m(j)}$. We refer to it as a nonmortar and the subdomain to which it belongs is $\Omega_j$.

Let $\Gamma_h^i$ be the triangulation of $\Omega_i$ with the mesh size $h_i$. The triangulation generally does not align at the subdomain interface. Denote the global mesh $\cup_i \Gamma_h^i$ by $\Gamma_h$ with the mesh size $h = \max_i h_i$. Moreover let $\underline{h} = \min_i h_i$.

We first define the following Morley element space locally:

$$\begin{aligned} \tilde{V}_{h,i} \quad = \quad & \{v | v|_K \in P_2(K), \ \forall K \in \Gamma_h^i, \ v \text{ is continuous} \\ & \text{at each vertex } p \text{ of } K, \text{ and } \partial_n v \text{ is continuous} \\ & \text{at midpoint } m \text{ of each edge of } K \text{ . Moreover} \\ & v(p) = \partial_n v(m) = 0, \text{ if } p, \ m \text{ also belong to } \partial\Omega\}. \end{aligned}$$

Let

$$\tilde{V}_h = \prod_{i=1}^{N} \tilde{V}_{h,i} = \{v_h | v_h|_{\Omega_i} = v_{h,i} \in \tilde{V}_{h,i}\}.$$

For any interface $\gamma_m = \gamma_{m(i)} = \delta_{m(j)}$, $1 \leq m \leq \tilde{N}$, there are two different and independent 1D triangulations $\Gamma_h(\gamma_{m(i)})$ and $\Gamma_h(\delta_{m(j)})$. Meanwhile, there are two sets of vertices belonging to $\gamma_m$: the vertices of the elements belonging to $\Gamma_h(\gamma_{m(i)})$ and to $\Gamma_h(\delta_{m(j)})$ are denoted by $\gamma_{h,m(i)}^P$ and $\delta_{h,m(j)}^P$, respectively. Similarly, there are two sets of midpoints belonging to $\gamma_m$: the midpoints of the elements belonging to $\Gamma_h(\gamma_{m(i)})$ and to $\Gamma_h(\delta_{m(j)})$ are denoted by $\gamma_{h,m(i)}^M$ and $\delta_{h,m(j)}^M$, respectively. Moreover, we need an auxiliary test space $S_h(\delta_{m(j)})$ which is a subspace of the space $L^2(\delta_{m(j)})$ such that every function in this space is piecewise constant on each element of the nonmortar triangulation $\Gamma_h(\delta_{m(j)})$. The dimension of $S_h(\delta_{m(j)})$ is equal to the number of midpoints on $\delta_{m(j)}$, i.e., to the number of elements on $\delta_{m(j)}$.

For each nonmortar $\delta_{m(j)}$, define an $L^2$-projection operator $Q_{h,\delta_{m(j)}} : L^2(\gamma_m) \rightarrow S_h(\delta_{m(j)})$ by

$$(Q_{h,\delta_{m(j)}} v, w) = (v, w), \quad \forall w \in S_h(\delta_{m(j)}), \tag{3.1}$$

where $(\cdot,\cdot)$ denotes the $L^2$ inner product over the space $L^2(\delta_{m(j)})$.

We now define the mortar-type Morley finite element space as follows:

$$\begin{aligned} V_h \quad = \quad & \{v|v \in \tilde{V}_h, \text{ and } \forall \gamma_m = \gamma_{m(i)} = \delta_{m(j)}, \\ & Q_{h,\delta_{m(j)}}(\partial_{n_\delta} v|_{\gamma_{m(i)}}) = Q_{h,\delta_{m(j)}}(\partial_{n_\delta} v|_{\delta_{m(j)}}) \\ & \text{and } v|_{\gamma_{m(i)}}(p) = v|_{\delta_{m(j)}}(p), \ \forall p \in \delta^P_{h,m(j)}\}, \end{aligned} \tag{3.2}$$

where $n_\delta$ means the unit outward normal along the interface $\gamma_m$ with the direction from $\delta_{m(j)}$ to $\gamma_{m(i)}$.

Define

$$|v|^2_{t,h,\Omega_i} \hat{=} \sum_{K\in\Gamma^i_h} |v|^2_{t,K}, \ |v|^2_{t,h} \hat{=} \sum_{i=1}^N |v|^2_{t,h,\Omega_i}, \ t = 0,1,2,$$

and

$$\|v\|^2_{t,h,\Omega_i} \hat{=} \sum_{K\in\Gamma^i_h} \|v\|^2_{t,K}, \ \|v\|^2_{t,h} \hat{=} \sum_{i=1}^N \|v\|^2_{t,h,\Omega_i}, \ t = 0,1,2,$$

where $|\cdot|_{t,K}$ and $\|\cdot\|_{t,K}$ are the usual semi-norm and norm in the Sobolev space $H^t(K)$, respectively, $\|\cdot\|_0 \hat{=} |\cdot|_{0,h} = \|\cdot\|_{0,h}$ and $\|\cdot\|_{0,\Omega_i} \hat{=} |\cdot|_{0,h,\Omega_i} = \|\cdot\|_{0,h,\Omega_i}$. From [13], we know that $|\cdot|_{2,h}$ is a norm over the space $V_h$.

Next, we give some prelimilary lemmas which will be used later.

We can construct an operator $\pi_h$ from $V_h$ to $H^1_0(\Omega)$ which holds the following approximate property (cf. [23] for details).

**Lemma 3.1.**(cf. [23]) *There exists an operator $\pi_h$ from $V_h$ to $H^1_0(\Omega)$ such that*

$$|v - \pi_h v|_{t,h,\Omega_i} \le C_2 h_i^{2-t}(|v|^2_{2,h,\Omega_i} + \sum_{\Omega_j} |v|^2_{2,h,\Omega_j})^{\frac{1}{2}}, \quad \forall v \in V_h, \quad t = 0,1,$$

where the sum is taken over all $\Omega_j$ such that $meas(\partial\Omega_j \cap \partial\Omega_i) \ne 0, \quad j \ne i$.

Based on Lemma 3.1, we can easy to check that

$$|v - \pi_h v|_{t,h} \le C_2 M h^{2-t} |v|_{2,h}, \tag{3.3}$$

where $M$ is

$$M = \max_i M_i, \quad M_i = card\{\Omega_j | meas(\partial\Omega_j \cap \partial\Omega_i) \ne 0\}.$$

**Lemma 3.2.** *For any $v \in V_h$, it holds that*

$$\|v\|_0 + |v|_{1,h} \le C_3 |v|_{2,h}.$$

**Proof:** See the proof in the appendix of this paper.

**Condition A.** *There exists a positive constant $C_a$ independent of the mesh size $h$ such that*

$$h \le C_a \underline{h}^{\frac{1}{4}}.$$

In the following of this paper, we always assume that the condition A is valid.

**Lemma 3.3.** *For any $v \in V_h$, it holds that*

$$\|\nabla_h v\|_{L^4} \le C_4 |v|_{2,h},$$

where $\|\nabla_h v\|_{L^4} = (\sum_{K\in\Gamma_h} \|\nabla v\|^4_{L^4(K)})^{\frac{1}{4}}$.

*Proof.* See the proof in the appendix of this paper.

The finite element problem corresponding to (2.2) is to find $u_h \in V_h$ such that

$$\frac{1}{R_e} a_h(u_h, v_h) = (\partial_x u_h \triangle u_h, \partial_y v_h)_h - (\partial_y u_h \triangle u_h, \partial_x v_h)_h + (f, v_h), \quad \forall v_h \in V_h, \tag{3.4}$$

where

$$\begin{aligned} a_h(u_h, v_h) = &\sum_{K\in\Gamma_h} \int_K \{\triangle u_h \triangle v_h + (1-\sigma)(2\partial_{xy} u_h \partial_{xy} v_h \\ &-\partial_{xx} u_h \partial_{yy} v_h - \partial_{yy} u_h \partial_{xx} v_h)\} dxdy, \\ &(\partial_x u_h \triangle u_h, \partial_y v_h)_h - (\partial_y u_h \triangle u_h, \partial_x v_h)_h \\ &= \sum_{K\in\Gamma_h} [(\partial_x u_h \triangle u_h, \partial_y v_h)_K - (\partial_y u_h \triangle u_h, \partial_x v_h)_K], \end{aligned}$$

here $0 < \sigma < 0.5$ . It is known that (cf. [8])

$$\begin{aligned} a_h(v, v) &\geq (1-\sigma)|v|_{2,h}, \\ a_h(v, w) &\leq (1+\sigma)|v|_{2,h}|w|_{2,h}. \end{aligned}$$

**Theorem 3.1.** *Problem (3.4) has a unique solution if condition*

$$R_e < \sqrt{\frac{(1-\sigma)^2}{\sqrt{2}C_3 C_4^2 \|f\|_0}}$$

*is valid.*

*Proof.* We use the Schauder fixed point theorem ([9]) to prove Theorem 3.1. It is easy to check that there exists an operator $A_h : V_h \to V_h$ such that

$$(A_h v, w) = a_h(v, w) \quad \forall v, w \in V_h.$$

So for any $f \in L^2(\Omega)$, there exists an $f_{0,h} \in V_h$ such that

$$f_{0,h} = A_h^{-1} Q_h f,$$

where $Q_h$ is the $L^2$-projection from $L^2(\Omega)$ to $V_h$ such that

$$(Q_h v, w) = (v, w), \quad \forall v \in L^2(\Omega),\ w \in V_h.$$

On the other hand,

$$\begin{aligned} &|(\partial_x u_h \triangle u_h, \partial_y v)_h - (\partial_y u_h \triangle u_h, \partial_x v)_h| \\ \leq &\sum_K \|\nabla u_h\|_{L^4(K)} \|\triangle u_h\|_{L^2(K)} \|\nabla v\|_{L^4(K)} \\ \leq &\sqrt{2}\|\nabla_h u_h\|_{L^4} |u_h|_{2,h} \|\nabla_h v\|_{L^4} \\ \leq &\sqrt{2}C_4^2 |u_h|_{2,h}^2 |v|_{2,h}, \quad \forall v \in V_h. \end{aligned}$$

So we know that there exists an operator $T_h : V_h \to V_h$ such that

$$a_h(T_h u_h, v_h) = (\partial_x u_h \triangle u_h, \partial_y v_h)_h - (\partial_y u_h \triangle u_h, \partial_x v_h)_h.$$

Furthermore, $T_h$ is compact since $V_h$ is finite-dimensional, and it is easy to check that $T_h$ is continuous.

Problem (3.4) can be expressed as:

$$\frac{1}{R_e}a_h(u_h,v_h)=a_h(T_hu_h,v_h)+a_h(f_{0,h},v_h),\quad \forall v_h\in V_h. \tag{3.5}$$

Then equation (3.5) can be written as:

$$\frac{1}{R_e}u_h=T_hu_h+f_{0,h}. \tag{3.6}$$

According to the Schauder fixed point theorem, (3.6) holds if we can show that the solutions of the following equation with parameter $t$ $(0\le t\le 1)$

$$\frac{1}{R_e}u_h=tT_hu_h+tf_{0,h} \tag{3.7}$$

are bounded in $V_h$.

In fact, based on (3.7), we have

$$\frac{1}{R_e}a_h(u_h,u_h)=ta_h(T_hu_h,u_h)+ta_h(f_{0,h},u_h).$$

It is easy to check that

$$a_h(T_hu_h,u_h)=(\partial_xu_h\triangle u_h,\partial_yu_h)_h-(\partial_yu_h\triangle u_h,\partial_xu_h)_h=0.$$

So by Lemma 3.2, we have

$$\begin{aligned}\frac{1-\sigma}{R_e}|u_h|_{2,h}^2 &\le \frac{1}{R_e}a_h(u_h,u_h)=t(f,u_h)\\ &\le \|f\|_0\|u_h\|_0\le C_3\|f\|_0|u_h|_{2,h}.\end{aligned}$$

Finally, we get

$$|u_h|_{2,h}\le\frac{R_eC_3}{1-\sigma}\|f\|_0, \tag{3.8}$$

which ensures that equation (3.4) has at least one solution.

We now prove that the solution of equation (3.4) is unique. Let $u_h,\ u_h'$ be two solutions of (3.4), that is,

$$\frac{1}{R_e}a_h(u_h,v_h)=(\partial_xu_h\triangle u_h,\partial_yv_h)_h-(\partial_yu_h\triangle u_h,\partial_xv_h)_h+(f,v_h),\ \forall v_h\in V_h, \tag{3.9}$$

$$\frac{1}{R_e}a_h(u_h',v_h)=(\partial_xu_h'\triangle u_h',\partial_yv_h)_h-(\partial_yu_h'\triangle u_h',\partial_xv_h)_h+(f,v_h),\ \forall v_h\in V_h. \tag{3.10}$$

Subtracting (3.10) from (3.9), we have

$$\begin{aligned}\frac{1}{R_e}a_h(u_h-u_h',v_h) &= (\partial_yu_h'\triangle u_h',\partial_xv_h)_h-(\partial_yu_h\triangle u_h,\partial_xv_h)_h\\ &\quad+(\partial_xu_h\triangle u_h,\partial_yv_h)_h-(\partial_xu_h'\triangle u_h',\partial_yv_h)_h.\end{aligned}$$

It is not difficult to check that

$$\begin{aligned}&(\partial_yu_h'\triangle u_h',\partial_xv_h)_h-(\partial_yu_h\triangle u_h,\partial_xv_h)_h\\ =\ &(\partial_yu_h\triangle(u_h'-u_h),\partial_xv_h)_h+(\partial_y(u_h'-u_h)\triangle u_h',\partial_xv_h)_h.\end{aligned}$$

Similarly,

$$\begin{aligned} & (\partial_x u_h' \triangle u_h', \partial_y v_h)_h - (\partial_x u_h \triangle u_h, \partial_y v_h)_h \\ = \quad & (\partial_x u_h \triangle(u_h' - u_h), \partial_y v_h)_h + (\partial_x (u_h' - u_h)\triangle u_h', \partial_y v_h)_h. \end{aligned}$$

Finally, we have

$$\begin{aligned} \frac{1-\sigma}{R_e}|(u_h - u_h')|_{2,h}^2 \quad & \leq \quad \frac{1}{R_e} a_h(u_h - u_h', u_h - u_h') \\ & = \quad (\partial_y u_h \partial_x (u_h' - u_h) - \partial_x u_h \partial_y (u_h' - u_h)_h, \triangle(u_h' - u_h)) \\ & \leq \quad \sqrt{2} C_3^2 |u_h|_{2,h} |u_h' - u_h|_{2,h}^2 \leq \frac{\sqrt{2} C_4^2 R_e C_3}{1-\sigma} \|f\|_0 |u_h' - u_h|_{2,h}^2, \end{aligned}$$

which implies uniqueness of the solution if condition

$$R_e < \sqrt{\frac{(1-\sigma)^2}{\sqrt{2} C_3 C_4^2 \|f\|_0}}$$

is valid.

## 4. Error Estimates

In order to obtain the error estimates of the mortar element solution $u_h$, we first introduce an interpolant in the mortar element space $V_h$. Let $\tilde{E}_h^i : H^3(\Omega_i) \to V_{h,i}$ be the local Morley element interpolation operator. Based on the local operator $\tilde{E}_h^i$, we define a global interpolation operator $\tilde{E}_h : H^3(\Omega) \cap H_0^2(\Omega) \to \tilde{V}_h$ as follows: For any $v \in H^3(\Omega) \cap H_0^2(\Omega)$,

$$\tilde{E}_h v = (\tilde{E}_h^1 v_1, ..., \tilde{E}_h^N v_N) \in \tilde{V}_h.$$

where $v_i = v|_{\Omega_i}$.

For any $v \in H^3(\Omega) \cap H_0^2(\Omega)$, we now give an approximation function over the mortar space $V_h$ as follows:

$$\Pi_h v = \tilde{E}_h v + \sum_{m=1}^{M} \Xi_{h,\delta_{m(j)}} (\tilde{E}_h v) \in V_h, \tag{4.1}$$

where the operator $\Xi_{h,\delta_{m(j)}} : \tilde{V}_h \to \tilde{V}_h$ which is defined by

$$(\Xi_{h,\delta_{m(j)}}(v))(p) = \begin{cases} (v|_{\gamma_{m(i)}} - v|_{\delta_{m(j)}})(p) & p \in \delta_{h,m(j)}^P, \\ 0 & \text{other vertices,} \end{cases}$$

and

$$(\partial_{n_\delta} \Xi_{l,\delta_{m(j)}}(v))(m) = \begin{cases} (Q_{l,\delta_{m(j)}}(\partial_{n_\delta} v|_{\gamma_{m(i)}} - \partial_{n_\delta} v|_{\delta_{m(j)}}))(m) & m \in \delta_{h,m(j)}^M, \\ 0 & \text{other midpoints.} \end{cases}$$

**Lemma 4.1.** *For the operator $\Pi_h$ defined by (4.1), we have*

$$|v - \Pi_h v|_{t,h} \leq C_5 (\sum_{i=1}^{N} h_i^{6-2t} \|v\|_{3,\Omega_i}^2)^{\frac{1}{2}}, \quad \forall v \in H^3(\Omega) \cap H_0^2(\Omega). \tag{4.2}$$

*Proof.* Please refer to [23], [13] for the detailed proof. The basic idea of the proof is to use the approximation properties of the operators $\tilde{E}_h, Q_{h,\delta_{m(j)}}$ and the mortar condition (3.2).

Similarly, we also have

$$\begin{aligned}\|\nabla_h(v-\Pi_h v)\|_{L^4} &\le C_6(\sum_{i=1}^N h_i^3\|v\|^2_{3,\Omega_i})^{\frac{1}{2}}\\ &\le C_6(\sum_{i=1}^N h_i^2\|v\|^2_{3,\Omega_i})^{\frac{1}{2}},\quad v\in H^3(\Omega)\cap H_0^2(\Omega),\end{aligned}\tag{4.3}$$

where we have used the fact $h_i<1$.

Next, we shall prove the following energy estimate.

**Theorem 4.1.** *Let $u$ and $u_h$ be the solutions of the equations (2.2), (3.4), respectively. Then if*

$$R_e<\sqrt{\frac{(1-\sigma)^2}{2\sqrt{2}C_3C_4^2\|f\|_0}},$$

*we have*

$$\begin{aligned}|u-u_h|_{2,h} \le\ & (C_{*1}\sum_{i=1}^N h_i^2(\|u\|^2_{3,\Omega_i}+R_e^2h_i^2\|f\|^2_{0,\Omega_i}\\ &+C_{*2}\sum_{i}^N h_i^2(\|\partial_x u\triangle u\|^2_{0,\Omega_i}+\|\partial_y u\triangle u\|^2_{0,\Omega_i})^{\frac{1}{2}},\end{aligned}$$

*where $C_{*1}=\frac{2\tilde{C}_1(1-\sigma)^2}{(1-\sigma)^2-2\sqrt{2}C_4^2R_e^2C_3}$, and $C_{*2}=\frac{2\tilde{C}_2(1-\sigma)^2}{(1-\sigma)^2-2\sqrt{2}C_4^2R_e^2C_3}$, $\tilde{C}_1$ and $\tilde{C}_2$ are two positive constants which will be defined later.*

*Proof.* Using Green's formula, we get (cf. [8],[16],[18])

$$a_h(u,v_h)=(-\nabla(\triangle u),\nabla v_h)_h+E_h(u,v_h),$$

where

$$E_h(u,v_h)=\sum_K(\int_{\partial K}[\triangle\theta-(1-\sigma)\partial^2_{\tau\tau}u]\partial_n v_h ds+(1-\sigma)\int_{\partial K}\partial^2_{n\tau}u\partial_\tau v_h ds),$$

here $n=(n_1,n_2)$, $\tau=(-n_2,n_1)$ denote the unit normal and tangent vector on $\partial K$, respectively. By [13],[16], we know

$$|E_h(u,v_h)|\le C_7(\sum_{i=1}^N h_i^2\|u\|^2_{3,\Omega_i})^{\frac{1}{2}}|v_h|_{2,h}.\tag{4.4}$$

Moreover, it is easy to check that for any $v_h\in V_h$, $\pi_h v_h\in H_0^1(\Omega)$, we have (cf. [8])

$$\frac{1}{R_e}(-\nabla(\triangle u),\nabla\pi_h v_h)=(\partial_x u\triangle u,\partial_y\pi_h v_h)-(\partial_y u\triangle u,\partial_x\pi_h v_h)+(f,\pi_h v_h).$$

Then

$$
\begin{aligned}
\frac{1}{R_e}a_h(u-u_h,v_h) &= \frac{1}{R_e}a_h(u,v_h)-\frac{1}{R_e}a_h(u_h,v_h)\\
&= \frac{1}{R_e}(-\nabla(\triangle u),\nabla v_h)_h+\frac{1}{R_e}E_h(u,v_h)\\
&\quad -[(\partial_x u_h\triangle u_h,\partial_y v_h)_h-(\partial_y u_h\triangle u_h,\partial_x v_h)_h+(f,v_h)]\\
&= \frac{1}{R_e}(-\nabla(\triangle u),\nabla(v_h-\pi_h v_h))_h+(f,\pi_h v_h-v_h)+\frac{1}{R_e}E_h(u,v_h)\\
&\quad +[(\partial_x u\triangle u,\partial_y\pi_h v_h)-(\partial_y u\triangle u,\partial_x\pi_h v_h)]\\
&\quad -[(\partial_x u_h\triangle u_h,\partial_y v_h)_h-(\partial_y u_h\triangle u_h,\partial_x v_h)_h]\\
&= \frac{1}{R_e}(-\nabla(\triangle u),\nabla(v_h-\pi_h v_h))_h+(f,\pi_h v_h-v_h)+\frac{1}{R_e}E_h(u,v_h) \qquad (4.5)\\
&\quad +[(\partial_x u\triangle u,\partial_y(\pi_h v_h-v_h))_h-(\partial_y u\triangle u,\partial_x(\pi_h v_h-v_h))_h]\\
&\quad +[(\partial_x u\triangle u,\partial_y v_h)_h-(\partial_y u\triangle u,\partial_x v_h)_h]\\
&\quad -[(\partial_x u_h\triangle u_h,\partial_y v_h)_h-(\partial_y u_h\triangle u_h,\partial_x v_h)_h]\\
&= \frac{1}{R_e}(-\nabla(\triangle u),\nabla(v_h-\pi_h v_h))_h+(f,\pi_h v_h-v_h)+\frac{1}{R_e}E_h(u,v_h)\\
&\quad +[(\partial_x u\triangle u,\partial_y(\pi_h v_h-v_h))_h-(\partial_y u\triangle u,\partial_x(\pi_h v_h-v_h))_h]\\
&\quad +[\partial_x(u-u_h)\triangle u,\partial_y v_h)_h+(\partial_x u_h\triangle(u-u_h),\partial_y v_h)_h\\
&\quad -(\partial_y(u-u_h)\triangle u,\partial_x v_h)_h-(\partial_y u_h\triangle(u-u_h),\partial_x v_h)_h]\\
&\hat{=} \sum_{i=1}^{5} I_i.
\end{aligned}
$$

For the terms $I_1, I_2, I_3$, we have (cf. [16])

$$
\begin{aligned}
|\sum_{i=1}^{3} I_i| &\le \frac{1}{R_e}\sum_{i=1}^{N}(\sqrt{2}\|u\|_{3,\Omega_i}|v_h-\pi_h v_h|_{1,h,\Omega_i}+R_e\|f\|_{0,\Omega_i}\|\pi_h v_h-v_h\|_{0,\Omega_i})\\
&\quad +\frac{C_7}{R_e}(\sum_{i=1}^{N} h_i^2\|u\|_{3,\Omega_i}^2)^{\frac12}|v_h|_{2,h}\\
&\le \frac{\sqrt{2}C_2M+C_7}{R_e}(\sum_{i=1}^{N} h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac12}|v_h|_{2,h}.
\end{aligned}
$$

For $I_4$,

$$
\begin{aligned}
|I_4| &\le \sum_{i=1}^{N}(\|\partial_x u\triangle u\|_{0,\Omega_i}|\pi_h v_h-v_h|_{1,h,\Omega_i}+\|\partial_y u\triangle u\|_{0,\Omega_i}|\pi_h v_h-v_h|_{1,h,\Omega_i})\\
&\le C_2M(\sum_{i}^{N} h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac12}|v_h|_{2,h},
\end{aligned}
$$

Finally, we get

$$\begin{aligned}\frac{1}{R_e}a_h(u-u_h,v_h) \quad \le \quad & \frac{C_2M+C_7}{R_e}(\sum_{i=1}^{N}h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|v_h|_{2,h} \\ & +C_2M(\sum_{i}^{N}h_i^2(\|\partial_xu\triangle u\|_{0,\Omega_i}^2+\|\partial_yu\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|v_h|_{2,h} \\ & +[\partial_x(u-u_h)\triangle u,\partial_yv_h)_h+(\partial_xu_h\triangle(u-u_h),\partial_yv_h)_h \\ & -(\partial_y(u-u_h)\triangle u,\partial_xv_h)_h-(\partial_yu_h\triangle(u-u_h),\partial_xv_h)_h].\end{aligned}$$

Let $e_h = u - u_h$, and $v_h = \phi_h - u_h$ in the above inequality, we have

$$\begin{aligned}\frac{1}{R_e}a_h(e_h,e_h) \quad \le \quad & \frac{1}{R_e}a_h(e_h,u-\phi_h)+\frac{C_2M+C_7}{R_e}(\sum_{i=1}^{N}h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|\phi_h-u_h|_{2,h} \\ & +C_2M(\sum_{i}^{N}h_i^2(\|\partial_xu\triangle u\|_{0,\Omega_i}^2+\|\partial_yu\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|\phi_h-u_h|_{2,h} \\ & +[(\partial_xe_h\triangle u,\partial_y(\phi_h-u_h))_h+(\partial_xu_h\triangle e_h,\partial_y(\phi_h-u_h))_h \\ & -(\partial_ye_h\triangle u,\partial_x(\phi_h-u_h))-(\partial_yu_h\triangle e_h,\partial_x(\phi_h-u_h)_h] \\ \le \quad & \frac{1}{R_e}a_h(e_h,u-\phi_h)+\frac{C_2M+C_7}{R_e}(\sum_{i=1}^{N}h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h} \\ & +\frac{C_2M+C_7}{R_e}(\sum_{i=1}^{N}h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|u-\phi_h|_{2,h} \\ & +C_2M(\sum_{i}^{N}h_i^2(\|\partial_xu\triangle u\|_{0,\Omega_i}^2+\|\partial_yu\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h} \\ & +C_2M(\sum_{i}^{N}h_i^2(\|\partial_xu\triangle u\|_{0,\Omega_i}^2+\|\partial_yu\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|u-\phi_h|_{2,h} \\ & +(\partial_xe_h\triangle u+\partial_xu_h\triangle e_h,\partial_ye_h)_h \\ & +(\partial_xe_h\triangle u+\partial_xu_h\triangle e_h,\partial_y(\phi_h-u))_h \\ & -(\partial_ye_h\triangle u+\partial_yu_h\triangle e_h,\partial_xe_h)_h \\ & -(\partial_ye_h\triangle u+\partial_yu_h\triangle e_h,\partial_x(\phi_h-u))_h.\end{aligned}$$

For the last four terms in the above inequality, we have

$$\begin{aligned}
& (\partial_x e_h \triangle u + \partial_x u_h \triangle e_h, \partial_y e_h)_h \\
& +(\partial_x e_h \triangle u + \partial_x u_h \triangle e_h, \partial_y(\phi_h - u))_h \\
& -(\partial_y e_h \triangle u + \partial_y u_h \triangle e_h, \partial_x e_h)_h \\
& -(\partial_y e_h \triangle u + \partial_y u_h \triangle e_h, \partial_x(\phi_h - u))_h \\
= \ & (\partial_x u_h \partial_y e_h - \partial_y u_h \partial_x e_h, \triangle e_h)_h \\
& +(\partial_x e_h \partial_y(\phi_h - u) - \partial_y e_h \partial_x(\phi_h - u), \triangle u)_h \\
& +(\partial_x u_h \partial_y(\phi_h - u) - \partial_y u_h \partial_x(\phi_h - u), \triangle e_h)_h \\
\leq \ & \sqrt{2}\|\nabla_h u_h\|_{L^4}\|\nabla_h e_h\|_{L^4}|e_h|_{2,h} \\
& +\sqrt{2}\|\nabla_h e_h\|_{L^4}\|\nabla_h(\phi_h - u)\|_{L^4}|u|_2 \\
& +\sqrt{2}\|\nabla_h u_h\|_{L^4}\|\nabla_h(\phi_h - u)\|_{L^4}|e_h|_{2,h}.
\end{aligned}$$

Then

$$\begin{aligned}
& \frac{1}{R_e} a_h(e_h, e_h) \leq \frac{1}{R_e} a_h(e_h, u - \phi_h) \\
& \quad + \frac{C_2 M + C_7}{R_e} (\sum_{i=1}^N h_i^2 (\|u\|_{3,\Omega_i}^2 + R_e^2 h_i^2 \|f\|_{0,\Omega_i}^2))^{\frac{1}{2}} |\phi_h - u_h|_{2,h} \\
& \quad + C_2 M (\sum_{i}^N h_i^2 (\|\partial_x u \triangle u\|_{0,\Omega_i}^2 + \|\partial_y u \triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}} |\phi_h - u_h|_{2,h} \qquad (4.6) \\
& \quad + \sqrt{2}\|\nabla_h u_h\|_{L^4}\|\nabla_h e_h\|_{L^4}|e_h|_{2,h} \\
& \quad + \sqrt{2}\|\nabla_h e_h\|_{L^4}\|\nabla_h(\phi_h - u)\|_{L^4}|u|_2 \\
& \quad + \sqrt{2}\|\nabla_h u_h\|_{L^4}\|\nabla_h(\phi_h - u)\|_{L^4}|e_h|_{2,h}.
\end{aligned}$$

By (4.2), (4.3) and Lemma 3.3, we have

$$\begin{aligned}
\|\nabla_h e_h\|_{L^4} & \leq \|\nabla_h(u - \Pi_h u)\|_{L^4} + \|\nabla_h(\Pi_h u - u_h)\|_{L^4} \\
& \leq C_6 (\sum_{i=1}^N h_i^2 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} + C_4 |\Pi_h u - u_h|_{2,h} \\
& \leq C_6 (\sum_{i=1}^N h_i^2 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} + C_4 (|\Pi_h u - u|_{2,h} + |e_h|_{2,h}) \\
& \leq (C_6 + C_4 C_5)(\sum_{i=1}^N h_i^2 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} + C_4 |e_h|_{2,h},
\end{aligned}$$

Taking $\phi_h = \Pi_h u$ in (4.6) and combining above inequality, we get

$$\begin{aligned}
\frac{1}{R_e}a_h(e_h,e_h) \quad \leq \quad & \frac{1}{R_e}a_h(e_h,u-\Pi_h u)+\frac{\sqrt{2}C_2M+C_7}{R_e}(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2+R_eh_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h}\\
&+\frac{\sqrt{2}C_2M+C_7}{R_e}(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|u-\Pi_h u|_{2,h}\\
&+C_2M(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h}\\
&+C_2M(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|u-\Pi_h u|_{2,h}\\
&+\sqrt{2}C_4|u_h|_{2,h}[(C_6+C_4C_5)(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}+C_4|e_h|_{2,h}]|e_h|_{2,h}\\
&+\sqrt{2}[(C_6+C_4C_5)(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}+C_4|e_h|_{2,h}]\|\nabla_h(u-\Pi_h u)\|_{L^4}|u|_2\\
&+\sqrt{2}C_4|u_h|_{2,h}\|\nabla_h(u-\Pi_h u)\|_{L^4}|e_h|_{2,h}.
\end{aligned}$$

Then by (3.8) and (4.2), (4.3), we get

$$\begin{aligned}
\frac{(1-\sigma)}{R_e}|e_h|_{2,h}^2 \quad \leq \quad & \frac{1}{R_e}a_h(e_h,e_h)\leq\frac{1}{R_e}(1+\sigma)C_5(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}|e_h|_{2,h}\\
&+\frac{\sqrt{2}C_2M+C_7}{R_e}(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h}\\
&+\frac{\sqrt{2}C_2M+C_7}{R_e}C_5\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2))\\
&+C_2M(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h}\\
&+C_2C_5M(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}\\
&+\frac{\sqrt{2}R_eC_3C_4}{1-\sigma}(C_6+C_4C_5)\|f\|_0(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}|e_h|_{2,h}\\
&+\frac{\sqrt{2}R_eC_3C_4^2}{1-\sigma}\|f\|_0|e_h|_{2,h}^2\\
&+\sqrt{2}C_6(C_6+C_4C_5)|u|_2(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)\\
&+\sqrt{2}C_6C_4(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}|u|_2|e_h|_{2,h}\\
&+\sqrt{2}C_4C_6\frac{R_eC_3}{1-\sigma}\|f\|_0(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}}|e_h|_{2,h}.
\end{aligned}$$

So we have

$$\begin{aligned}
|e_h|_{2,h}^2 \leq\ & [\frac{1+\sigma}{1-\sigma}C_5 + \frac{\sqrt{2}C_2M + C_7}{1-\sigma} + \frac{\sqrt{2}C_4C_6R_e}{1-\sigma}|u|_2 \\
&+ \frac{\sqrt{2}R_e^2C_3C_4}{(1-\sigma)^2}(C_6 + C_4C_5)\|f\|_0 + \frac{\sqrt{2}C_3C_4C_6R_e^2}{(1-\sigma)^2}\|f\|_0] \\
&\cdot(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2 + R_e^2h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h} \\
&+ \frac{C_2MR_e}{1-\sigma}(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}|e_h|_{2,h} \\
&+ [\frac{(\sqrt{2}C_2M + C_7)C_5}{1-\sigma} + \frac{\sqrt{2}C_6(C_6 + C_4C_5)R_e|u|_2}{1-\sigma}](\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2 + R_e^2h_i^2\|f\|_{0,\Omega_i}^2)) \\
&+ \frac{C_2C_5MR_e}{1-\sigma}(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}(\sum_{i=1}^N h_i^2\|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} \\
&+ \frac{\sqrt{2}R_e^2C_3C_4^2}{(1-\sigma)^2}\|f\|_0|e_h|_{2,h}^2.
\end{aligned}$$

By a simple caculation, we get

$$\begin{aligned}
&(\frac{1}{2} - \frac{\sqrt{2}R_e^2C_3C_4^2}{(1-\sigma)^2}\|f\|_0)|e_h|_{2,h}^2 \\
\leq\ & [\frac{1+\sigma}{1-\sigma}C_5 + \frac{\sqrt{2}C_2M + C_7}{1-\sigma} + \frac{\sqrt{2}C_4C_6R_e}{1-\sigma}|u|_2 \\
&+ \frac{\sqrt{2}R_e^2C_3C_4}{(1-\sigma)^2}(C_6 + C_4C_5)\|f\|_0 + \frac{\sqrt{2}C_3C_4C_6R_e^2}{(1-\sigma)^2}\|f\|_0 \\
&+ 2\frac{C_2C_5MR_e}{1-\sigma} + \frac{(\sqrt{2}C_2M + C_7)C_5}{1-\sigma} + \frac{\sqrt{2}C_6(C_6 + C_4C_5)R_e|u|_2}{1-\sigma}] \\
&\cdot(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2 + R_e^2h_i^2\|f\|_{0,\Omega_i}^2)) \\
&+ [\frac{C_2MR_e}{1-\sigma} + \frac{2C_2C_5MR_e}{1-\sigma}](\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2)) \\
\hat{=}\ & \tilde{C}_1\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2 + R_e^2h_i^2\|f\|_{0,\Omega_i}^2)) \\
&+ \tilde{C}_2(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2)),
\end{aligned}$$

which implies Theorem 4.1.

In the following, using a Aubin-Nitsche trick, we shall present an optimal $H^1$-norm estimate. First, we construct the following auxiliary equation

$$\begin{cases} \frac{1}{R_e}\triangle^2\psi = G\psi + g & \text{in } \Omega, \\ \psi = \partial_n\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where

$$\begin{aligned} G\psi &\hat{=} \triangle(\partial_x u \partial_y \psi) - \triangle(\partial_y u \partial_x \psi) + \partial_y(\triangle u)\partial_x \psi - \partial_x(\triangle u)\partial_y \psi \\ &= \partial_x u \triangle(\partial_y \psi) - \partial_y u \triangle(\partial_x \psi) + 2\nabla(\partial_x u)\cdot\nabla(\partial_y \psi) - 2\nabla(\partial_y u)\cdot\nabla(\partial_x \psi). \end{aligned}$$

For the above auxiliary problem, we have the following result.

**Lemma 4.2.** *Equation (4.7) has a unique solution $\psi$. Moreover the solution satisfies the following a prior estimate*

$$\|\psi\|_3 \le C'\|g\|_{-1},$$

*where $\|\cdot\|_{-1}\hat{=}\sup_{\phi\in H_0^1(\Omega)}\frac{(v,\phi)}{|\phi|_1}$ is the norm of the space $H^{-1}(\Omega) = H_0^1(\Omega)'$.*

*Proof.* Please see the proof in the appendix of this paper.

**Theorem 4.2.** *Let $u$ and $u_h$ be solutions of equations (2.2) and (3.4), respectively. Then*

$$|u - u_h|_{1,h} \le T_1(h, h_i),$$

*where $T_1(h, h_i)$ will be defined later.*

*Proof.* Let $e_h = u - u_h$, then $\pi_h(\Pi_h e_h) = \pi_h(\Pi_h u - u_h) \in H_0^1(\Omega)$. Consider the following problem

$$\begin{cases} \frac{1}{R_e}\triangle^2\psi = G\psi - \triangle\pi_h(\Pi_h e_h) & \text{in } \Omega, \\ \psi = \partial_n\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 4.2, we know that

$$\|\psi\|_3 \le C'\| - \triangle\pi_h(\Pi_h e_h)\|_{-1} \le C'|\pi_h(\Pi_h e_h)|_1. \tag{4.8}$$

On the other hand, by Green's formula, we obtain

$$\begin{aligned} |\pi_h(\Pi_h e_h)|_1^2 &= \frac{1}{R_e}(\triangle^2\psi, \pi_h(\Pi_h e_h)) - (G\psi, \pi_h(\Pi_h e_h)) \\ &= -\frac{1}{R_e}(\nabla(\triangle\psi), \nabla(\pi_h(\Pi_h e_h)) - (G\psi, \pi_h(\Pi_h e_h)) \\ &= \frac{1}{R_e}(\nabla(\triangle\psi), \nabla(\Pi_h e_h - \pi_h(\Pi_h e_h)) \\ &\quad -(G\psi, \pi_h(\Pi_h e_h) - e_h) \\ &\quad -\frac{1}{R_e}(\nabla(\triangle\psi), \nabla(\Pi_h e_h)) - (G\psi, e_h)\hat{=}\sum_{i=1}^4 II_i. \end{aligned}$$

For the term $II_1$,

$$\begin{aligned} |II_1| &\le \frac{1}{R_e}\|\psi\|_3\|\nabla(\Pi_h e) - \pi_h(\Pi_h e_h)\|_0 \\ &\le \frac{C_2}{R_e}\|\psi\|_3|\Pi_h e_h|_{2,h} \\ &\le \frac{C_2}{R_e}[(C_5 + C_{*1})h(\sum_{i=1}^N h_i^2(\|u\|_{3,\Omega_i}^2 + h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}} \\ &\quad +C_{*2}h(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}]\|\psi\|_3. \end{aligned}$$

For $II_2$,

$$\begin{aligned}
|II_2| &\leq \|G\psi\|_0 \|\pi_h(\Pi_h e_h) - e_h\|_0 \\
&= \|\triangle(\partial_x u \partial_y \psi) - \triangle(\partial_y u \partial_x \psi) + \partial_y(\triangle u)\partial_x \psi - \partial_x(\triangle u)\partial_y \psi\|_0 \|\pi_h(\Pi_h e_h) - e_h\|_0 \\
&\leq 4\sqrt{2}(\|u\|_3 \|\psi\|_{1,\infty} + \|u\|_{1,\infty}\|\psi\|_3)(\|\pi_h(\Pi_h e_h) - \Pi_h e_h\|_0 + \|u - \Pi_h u\|_0) \\
&\leq 8\sqrt{2} C_8 C_2 M h^2 [(C_5 + C_{*1} + 1)(\sum_{i=1}^{N} h_i^2 (\|u\|_{3,\Omega_i}^2 + R_e^2 h_i^2 \|f\|_{0,\Omega_i}^2))^{\frac{1}{2}} \\
&\quad + C_{*2}(\sum_{i}^{N} h_i^2 (\|\partial_x u \triangle u\|_{0,\Omega_i}^2 + \|\partial_y u \triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}] \|u\|_3 \|\psi\|_3,
\end{aligned}$$

where we have used the following inequality

$$\|v\|_{1,\infty} \leq C_8 \|u\|_3, \quad \forall v \in H^3(\Omega).$$

For the term $II_3$, we have

$$\begin{aligned}
II_3 &= \frac{1}{R_e} a_h(\psi, \Pi_h e_h) - \frac{1}{R_e} E_h(\psi, \Pi_h e_h) \\
&= \frac{1}{R_e} a_h(\psi, \Pi_h u - u) + \frac{1}{R_e} a_h(\psi - \Pi_h \psi, e_h) \\
&\quad + \frac{1}{R_e} a_h(\Pi_h \psi, e_h) - \frac{1}{R_e} E_h(\psi, \Pi_h e_h) \\
&\hat{=} \sum_{i=1}^{5} J_i.
\end{aligned}$$

For $J_1$(cf. [13], [16]),

$$\begin{aligned}
|J_1| &= |-\frac{1}{R_e}(\nabla(\triangle\psi), \nabla(\Pi_h u - u))_h| + \frac{1}{R_e} E_h(\psi, \Pi_h u - u) \\
&\leq \frac{\sqrt{2} C_5}{R_e} (\sum_{i=1}^{N} h_i^4 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} \\
&\quad + \frac{C_5 C_7}{R_e} (\sum_{i=1}^{N} h_i^2 \|\psi\|_{3,\Omega_i}^2)^{\frac{1}{2}} \cdot (\sum_{i=1}^{N} h_i^2 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} \\
&\leq \frac{C_5(\sqrt{2} + C_7)}{R_e} h (\sum_{i=1}^{N} h_i^2 \|u\|_{3,\Omega_i}^2)^{\frac{1}{2}} \|\psi\|_3.
\end{aligned}$$

By the interpolate estimate and Theorem 4.1, we know

$$|J_2| \leq \frac{1+\sigma}{R_e} C_5 h |e_h|_{2,h} \|\psi\|_3.$$

For the term $J_3$, using a similar argument as in (4.5), we have

$$\begin{aligned} J_3 &= \frac{1}{R_e}(-\nabla(\triangle u), \nabla(\Pi_h\psi - \psi))_h + (f, \psi - \Pi_h\psi) + \frac{1}{R_e}E_h(u, \Pi_h\psi - \psi) \\ &\quad +[(\partial_x u\triangle u, \partial_y(\psi - \Pi_h\psi_h))_h - (\partial_y u\triangle u, \partial_x(\psi - \Pi_h\psi))_h] \\ &\quad +[\partial_x e_h\triangle u, \partial_y\Pi_h\psi)_h + (\partial_x u_h\triangle e_h, \partial_y\Pi_h\psi)_h \\ &\quad -(\partial_y e_h\triangle u, \partial_x\Pi_h\psi)_h - (\partial_y u_h\triangle e_h, \partial_x\Pi_h\psi)_h] \\ &\hat{=} \sum_{i=1}^{5} H_i, \end{aligned}$$

here we have used the fact

$$E_h(u, \psi) = 0.$$

It is easy to check that

$$\begin{aligned} |\sum_{i=1}^{4} H_i| &\leq (\frac{\sqrt{2}}{R_e} + \frac{C_7}{R_e} + 1)C_5 h(\sum_{i=1}^{N} h_i^2(\|u\|_{3,\Omega_i}^2 + R_e^2 h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\ &\quad +C_5 h(\sum_{i}^{N} h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \end{aligned}$$

Then

$$\begin{aligned} J_3 &\leq (\frac{\sqrt{2}}{R_e} + \frac{C_7}{R_e} + 1)C_5 h(\sum_{i=1}^{N} h_i^2(\|u\|_{3,\Omega_i}^2 + h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\ &\quad +C_5 h(\sum_{i}^{N} h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\ &\quad +[\partial_x e_h\triangle u, \partial_y\Pi_h\psi)_h + (\partial_x u_h\triangle e_h, \partial_y\Pi_h\psi)_h \\ &\quad -(\partial_y e_h\triangle u, \partial_x\Pi_h\psi)_h - (\partial_y u_h\triangle e_h, \partial_x\Pi_h\psi)_h]. \end{aligned}$$

For $J_4$,

$$J_4 \leq \frac{C_7 h}{R_e}\|\psi\|_3|\Pi_h e_h|_{2,h}.$$

So, for the term $II_3$, we get

$$\begin{aligned} |II_3| &\leq (\frac{2(\sqrt{2} + C_7)}{R_e} + 1)C_5 h(\sum_{i=1}^{N} h_i^2(\|u\|_{3,\Omega_i}^2 + h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\ &\quad +C_5 h(\sum_{i}^{N} h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\ &\quad +\frac{1+\sigma}{R_e}C_5 h|e_h|_{2,h}\|\psi\|_3 + \frac{C_7}{R_e}h|\Pi_h e_h|_{2,h}\|\psi\|_3 \\ &\quad +[\partial_x e_h\triangle u, \partial_y\Pi_h\psi)_h + (\partial_x u_h\triangle e_h, \partial_y\Pi_h\psi)_h \\ &\quad -(\partial_y e_h\triangle u, \partial_x\Pi_h\psi)_h - (\partial_y u_h\triangle e_h, \partial_x\Pi_h\psi)_h] \end{aligned}$$

For the term $II_4$, using Green's formula, we get

$$\begin{aligned}
II_4 &= (\triangle(\partial_x u\partial_y\psi) - \triangle(\partial_y u\partial_x\psi) + \partial_y(\triangle u)\partial_x\psi - \partial_x(\triangle u)\partial_y\psi, e_h)_h \\
&= (\triangle u, \partial_x(\partial_y\psi e_h))_h - (\triangle u, \partial_y(\partial_x\psi e_h))_h + (\partial_x u\partial_y\psi, \triangle e_h)_h - (\partial_y u\partial_x\psi, \triangle e_h)_h + \sum_{i=1}^{4} E_i \\
&= (\triangle u, \partial_x e_h\partial_y\psi - \partial_y e_h\partial_x\psi)_h + (\triangle e_h, \partial_x u\partial_y\psi - \partial_y u\partial_x\psi)_h + \sum_{i=1}^{4} E_i,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \sum_K \int_{\partial K} (\partial_x u\partial_y\psi + \partial_y u\partial_x\psi)\frac{\partial e_h}{\partial n} ds \\
E_2 &= -\sum_K \int_{\partial K} \frac{\partial}{\partial n}(\partial_x u\partial_y + \partial_y u\partial_x\psi) e_h ds \\
E_3 &= \sum_K \int_{\partial K} \triangle u\partial_x\psi e_h n_2 ds \\
E_4 &= \sum_K \int_{\partial K} \triangle u\partial_y\psi e_h n_1 ds.
\end{aligned}$$

So

$$\begin{aligned}
II_3 + II_4 &\le (\frac{2(\sqrt{2}+C_7)}{R_e} + 1)C_5 h(\sum_{i=1}^{N} h_i^2(\|u\|_{3,\Omega_i}^2 + h_i^2\|f\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\
&\quad + C_5 h(\sum_{i}^{N} h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2 + \|\partial_y u\triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}}\|\psi\|_3 \\
&\quad + \frac{1+\sigma}{R_e} C_5 h|e_h|_{2,h}\|\psi\|_3 + \frac{C_7}{R_e} h|\Pi_h e_h|_{2,h}\|\psi\|_3 \\
&\quad + (-\partial_y e_h\partial_x(\Pi_h\psi - \psi)_h + \partial_x e_h\partial_y(\Pi_h\psi - \psi), \triangle u)_h \\
&\quad + (\triangle e_h, \partial_x u_h\partial_y\Pi_h\psi - \partial_x u\partial_y\psi)_h + (\triangle e_h, \partial_y u\partial_x\psi - \partial_y u_h\partial_x\Pi_h\psi)_h + \sum_{i=1}^{4} E_i \\
&\hat{=} \sum_{i=1}^{7} K_i + \sum_{i=1}^{4} E_i.
\end{aligned}$$

By[13],[23], we know that

$$|\sum_{i=1}^{4} E_i| \le C_9 h\|u\|_3|e_h|_{2,h}\|\psi\|_3.$$

For the term $K_5$, we have

$$\begin{aligned}
|K_5| &\le \sqrt{2}C_4^2\|u\|_3|\psi - \Pi_h\psi|_{2,h}|e_h|_{2,h} \\
&\le \sqrt{2}C_4^2 C_5 h|e_h|_{2,h}\|u\|_3\|\psi\|_3.
\end{aligned}$$

For $K_6$, we can derive

$$\begin{aligned} |K_6| &= |(\triangle e_h, \partial_x u_h \partial_y \Pi_h \psi - \partial_x u \partial_y \psi)_h| \\ &= |(\triangle e_h, \partial_x u_h \partial_y \Pi_h \psi - \partial_x u \partial_y \Pi_h \psi + \partial_x u \partial_y \Pi_h \psi - \partial_x u \partial_y \psi)_h| \\ &= |(\triangle e_h, \partial_x e_h \partial_y \Pi_h \psi)_h + (\triangle e_h, \partial_x u \partial_y (\psi - \Pi_h \psi_h))_h| \\ &\leq \sqrt{2} C_4^2 |e_h|_{2,h}^2 |\Pi_h \psi|_{2,h} + \sqrt{2} C_4^2 |e_h|_{2,h} \|u\|_2 |\psi - \Pi_h \psi|_{2,h} \\ &\leq \sqrt{2} C_4^2 (1 + C_5 h) |e_h|_{2,h}^2 \|\psi\|_3 + \sqrt{2} C_4^2 C_5 M h \|u\|_2 |e_h|_{2,h} \|\psi\|_3. \end{aligned}$$

Similarly,

$$|K_7| \leq \sqrt{2} C_4^2 (1 + C_5 h) |e_h|_{2,h}^2 \|\psi\|_3 + \sqrt{2} C_4^2 C_5 h \|u\|_2 |e_h|_{2,h} \|\psi\|_3.$$

Then

$$\begin{aligned} II_3 + II_4 \leq\ & (\frac{2(\sqrt{2} + C_7)}{R_e} + 1) C_5 h (\sum_{i=1}^N h_i^2 (\|u\|_{3,\Omega_i}^2 + R_e^2 h_i^2 \|f\|_{0,\Omega_i}^2))^{\frac{1}{2}} \|\psi\|_3 \\ & + C_5 h (\sum_i^N h_i^2 (\|\partial_x u \triangle u\|_{0,\Omega_i}^2 + \|\partial_y u \triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}} \|\psi\|_3 \\ & + \frac{1+\sigma}{R_e} C_5 h |e_h|_{2,h} \|\psi\|_3 + \frac{C_7}{R_e} h |\Pi_h e_h|_{2,h} \|\psi\|_3 \\ & + C_9 h \|u\|_3 |e_h|_{2,h} \|\psi\|_3 + \sqrt{2} C_4^2 C_5 h |e_h|_{2,h} \|u\|_3 \|\psi\|_3 \\ & + 2\sqrt{2} C_4^2 (1 + C_5 h) |e_h|_{2,h}^2 \|\psi\|_3 + 2\sqrt{2} C_4^2 C_5 h \|u\|_2 |e_h|_{2,h} \|\psi\|_3. \end{aligned}$$

Finally, by Theorem 4.1 and a simple manipulation, we obtain

$$\begin{aligned} |\pi_h(\Pi_h e_h)|_1^2 \leq\ & \sum_{i=1}^4 II_i \leq C'_{*1} h (\|u\|_{3,\Omega_i}^2 + R_e^2 h_i^2 \|f\|_{0,\Omega_i}^2))^{\frac{1}{2}} \|\psi\|_3 \\ & + C'_{*2} h (\sum_i^N h_i^2 (\|\partial_x u \triangle u\|_{0,\Omega_i}^2 + \|\partial_y u \triangle u\|_{0,\Omega_i}^2))^{\frac{1}{2}} \|\psi\|_3 \\ & + C'_{*3} \sum_{i=1}^N h_i^2 (\|u\|_{3,\Omega_i}^2 + h_i^2 \|f\|_{0,\Omega_i}^2) \|\psi\|_3 \\ & + C'_{*2} \sum_i^N h_i^2 (\|\partial_x u \triangle u\|_{0,\Omega_i}^2 + \|\partial_y u \triangle u\|_{0,\Omega_i}^2) \|\psi\|_3 \\ \hat{=}\ & T(h, h_i) \|\psi\|_3, \end{aligned}$$

where

$$\begin{aligned} C'_{*1} &\hat{=} \frac{C_2}{R_e}(C_5+C_{*1})+8\sqrt{2}C_8C_2(C_5+C_{*1}+1)Mh^2 \\ &+(\frac{2(\sqrt{2}+C_7)}{R_e}+1)C_5+\frac{1+\sigma}{R_e}C_5C_{*1}+\frac{C_7}{R_e}(C_5+C_{*1}) \\ &+C_9C_{*1}\|u\|_3+\sqrt{2}C_4^2C_5C_{*1}\|u\|_3+2\sqrt{2}C_4^2C_5C_{*1}\|u\|_2; \\ C'_{*2} &\hat{=} \frac{C_2C_{*2}}{R_e}+8\sqrt{2}C_8C_2C_{*2}\|u\|_3Mh^2 \\ &+C_5+\frac{1+\sigma}{R_e}C_5C_{*2}+\frac{C_7}{R_e}C_{*2}+C_9C_{*2}\|u\|_3 \\ &+\sqrt{2}C_4^2C_5C_{*2}\|u\|_3+2\sqrt{2}C_4^2C_5C_{*2}; \\ C'_{*3} &\hat{=} 4\sqrt{2}C_4^2(1+C_5h)(C_{*1})^2; \\ C'_{*4} &\hat{=} 4\sqrt{2}C_4^2(1+C_5h)(C_{*2})^2. \end{aligned}$$

So, by (4.8), we get

$$|\pi_h(\Pi_h e_h)|_1 \le C'T(h,h_i).$$

Based on the above inequality and Lemma 3.1, we get

$$\begin{aligned} |\Pi_h e_h|_{1,h} &\le |\Pi_h e_h-\pi_h(\Pi_h e_h)|_{1,h}+|\pi_h(\Pi_h e_h)|_{1,h} \\ &\le C_2Mh|\Pi_h e_h|_{2,h}+C'T(h,h_i). \end{aligned}$$

Finally

$$\begin{aligned} |u-u_h|_{1,h} &\le |u-\Pi_h u|_{1,h}+|\Pi_h e_h|_{1,h} \\ &\le T_1(h,h_i), \end{aligned}$$

where

$$\begin{aligned} T_1(h,h_i) &= C_2(C_{*1}+2C_5)Mh(\sum_i h_i^2(\|u\|_{3,\Omega_i}^2+R_e^2h_i^2\|f\|_{0,\Omega_i}^2)^{\frac{1}{2}} \\ &+C_2C_{*2}Mh(\sum_i^N h_i^2(\|\partial_x u\triangle u\|_{0,\Omega_i}^2+\|\partial_y u\triangle u\|_{0,\Omega_i}^2)^{\frac{1}{2}}+C'T(h,h_i). \end{aligned}$$

## Appendix

In this appendix, we shall give the proofs of Lemmas 3.2, 3.3 and 4.2.

**The proof of Lemma 3.2.** For any $g\in H^{-1}(\Omega)$, we consider the following auxiliary problem

$$\begin{cases} -\triangle w=g & \text{in } \Omega, \\ w=0 & \text{on } \partial\Omega, \end{cases}$$

It is easy to check that the above equation has a unique solution $w\in H_0^1(\Omega)$ satisfies

$$\|w\|_1\le C'_1\|g\|_{-1}, \tag{1}$$

and

$$\int_\Gamma(\partial_n w-\partial_{n'}w)vds=0,\quad \forall v\in H_0^1(\Omega),\ \forall\Gamma\subset\Omega, \tag{2}$$

where $\Gamma$ is a broken line in $\Omega$, $n$ and $n'$ are two opposite normal direction of $\Gamma$.

By (2) and Green's formula, let $\pi_h$ denote the operator from $V_h$ to $H_0^1(\Omega)$ given in the section 3 of this paper, we have

$$\begin{aligned}(\pi_h v, g) &= (\pi_h v, -\triangle w)\\ &= -\sum_{K\in\Gamma_l}\int_K \partial_n w \pi_h v ds + (\nabla w, \nabla \pi_h v)\\ &= (\nabla w, \nabla \pi_h v) = (\nabla w, \nabla v)_h + (\nabla w, \nabla(\pi_h v - v))_h\\ &= \sum_{K\in\Gamma_h}\int_{\partial K} w\partial_n v ds - (w, \triangle v)_h + (\nabla w, \nabla(\pi_h v - v))_h.\end{aligned}$$

It follows from [19], [13]that

$$\begin{aligned}|\sum_{K\in\Gamma_h}\int_{\partial K} w\partial_n v ds| &\le C_2'(\sum_i h_i^2\|w\|_{1,\Omega_i}^2)^{\frac{1}{2}}|v|_{2,h}\\ &\le C_2' h\|w\|_1|v|_{2,h}.\end{aligned}$$

By (3.3) and the fact $h<1$, we have

$$\begin{aligned}&|-(w,\triangle v)_h + (\nabla w, \nabla(\pi_h v - v))_h|\\ &\le (\sqrt{2}\|w\|_0 + C_2 M h|w|_1)|v|_{2,h}\\ &\le \max\{\sqrt{2}, C_2 M\}\|w\|_1|v|_{2,h}.\end{aligned}$$

Combining above inequalities, and using the fact

$$|\pi_h v|_1 = \sup_{g\in H^{-1}(\Omega)}\frac{(\pi_h v, g)}{\|g\|_{-1}},$$

we get

$$|\pi_h v|_1 \le C_1'(C_2' + \max\{\sqrt{2}, C_2 M\})|v|_{2,h}.$$

Note that

$$\|\pi_h v\|_0 \le C_4'|\pi_h v|_1 \le C_4' C_1'(C_2' + \max\{\sqrt{2}, C_2 M\})|v|_{2,h}.$$

Finally, we can derive

$$\begin{aligned}\|v\|_0 + |v|_{1,h} &\le \|\pi_h v\|_0 + |\pi_h v|_1 + \|v - \pi_h v\|_0 + |v - \pi_h v|_{1,h}\\ &\le \|\pi_h v\|_0 + |\pi_h v|_1 + C_2 M h^2|v|_{2,h} + C_2 M h|v|_{2,h}\\ &\le C_3|v|_{2,h},\end{aligned}$$

where $C_3 = (C_4'+1)C_1'(C_2' + \max\{\sqrt{2}, C_2 M\}) + 2C_2 M$.

**The proof of Lemma 3.3.** We introduce an auxiliary mortar element space $S_h$. First, on each subdomains $\Omega_i$, define

$$\begin{aligned}\tilde{S}_{h,i} = \{v|v|_K \in P_1(K),\ \forall K\in\Gamma_{h,i},\ v \text{ is continuous}\\ \text{at midpoint } m \text{ of each edge of } K\text{ . Moreover}\\ v(m)=0,\ \text{if } m \text{ also belong to } \partial\Omega\}.\end{aligned}$$

Let

$$\tilde{S}_h = \prod_{i=1}^{N} \tilde{S}_{h,i}.$$

Next, define

$$S_h = \{v_h | v_h \in \tilde{S}_h, Q_{h,\delta_{m(j)}}(v_h|_{\delta_{m(j)}}) = Q_{h,\delta_{m(j)}}(v_h|_{\gamma_{m(i)}}), \text{ for } \forall \gamma_{m(i)} = \delta_{m(j)} \in \Gamma\},$$

where the operator $Q_{h,\delta_{m(j)}}$ is defined in (3.1).

Because $\partial_n v$, $\partial_\tau v$ are continuous at the midpoints of each edge of the element $K \in \Gamma_h$, $\partial_x v$, $\partial_y v \in S_{h,i}$. On the other hand, by the mortar condition, we have

$$\begin{aligned} Q_{h,\delta_{m(j)}}(\partial_x v_h|_{\delta_{m(j)}}) &= Q_{h,\delta_{m(j)}}(\partial_{n_\delta} v_h|_{\delta_{m(j)}}) cos(n_\delta, x) + Q_{h,\delta_{m(j)}}(\partial_{\tau_\delta} v_h|_{\delta_{m(j)}}) cos(\tau_\delta, x) \\ &= Q_{h,\delta_{m(j)}}(\partial_{n_\delta} v_h|_{\gamma_{m(i)}}) cos(n_\delta, x) + Q_{h,\delta_{m(j)}}(\partial_{\tau_\delta} v_h|_{\gamma_{m(i)}}) cos(\tau_\delta, x) \\ &= Q_{h,\delta_{m(j)}}(\partial_x v_h|_{\gamma_{m(i)}}), \end{aligned}$$

where $n_\delta$ is deifined in section 3, and $\tau_\delta$ denotes the unit tangent vector along $\gamma_m$. So $\partial_x v_h \in S_h$. Similarly $\partial_y v_h \in S_h$.

Based on the above observation, we only need to prove that for any $w \in S_h$ we have

$$\|w\|_{L^4} \leq C_1^* |w|_{1,h}.$$

Then Lemma 3.3 is valid.

First we introduce the following auxiliary problem

$$\begin{cases} -\triangle \xi = \theta & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases}$$

It is known that

$$\|\xi\|_1 \leq C_2^* \|\theta\|_{-1}, \quad \|\xi\|_2 \leq C_3^* \|\theta\|_0.$$

Using Green's formula, we get

$$\begin{aligned} (\theta, w) &= (-\triangle \xi, w) \\ &= \sum_{K \in \Gamma_h} (\nabla \xi, \nabla w)_K - \sum_{K \in \Gamma_h} \int_{\partial K} \partial_n \xi w ds. \end{aligned}$$

By [21], we know

$$|-\sum_{K \in \Gamma_h} \int_{\partial K} \partial_n \xi w ds| \leq C_4^* h \|\xi\|_2 |v|_{1,h} \leq C_4^* C_3^* h \|\theta\|_0 |v|_{1,h}.$$

So

$$(\theta, w) \leq (C_2^* \|\theta\|_{-1} + C_4^* C_3^* h \|\theta\|_0) |w|_{1,h}.$$

Taking $\theta = w^3$ in the above inequality, then

$$\|w\|_{L^4}^4 \leq (C_2^* \|w^3\|_{-1} + C_4^* C_3^* h \|w^3\|_0) |w|_{1,h}.$$

Using the inverse inequality, it is easy to check that

$$\|w^3\|_0^2 = (\int_\Omega w^6 dx)^{\frac{1}{2}} = \|w\|_{L^6}^3 \leq C_5^* \underline{h}^{-\frac{1}{4}} \|w\|_{L^4}^3.$$

On the other hand, for any $\xi \in H_0^1(\Omega)$,

$$\begin{aligned}(w^3,\xi) &= (\int_\Omega w^4 dx)^{\frac{3}{4}}(\int_\Omega \xi^4 dx)^{\frac{1}{4}} \\ &= \|w\|_{L^4}^3\|\xi\|_{L^4} \le C_6^*\|w\|_{L^4}^3|\xi|_1,\end{aligned}$$

where we have used the following Sobolev inequality

$$\|\xi\|_{L^4} \le C_6^*|\xi|_1.$$

Then

$$\|v^3\|_{-1} \le C_6^*\|v\|_{L^4}^3.$$

Finally, by condition (A), we have

$$\begin{aligned}\|w\|_{L^4} &\le (C_2^*C_6^* + C_4^*C_3^*C_5^*\underline{h}^{-\frac{1}{4}}h)|w|_{1,h} \\ &\le C_1^*|w|_{1,h},\end{aligned}$$

where $C_1^* = C_2^*C_6^* + C_4^*C_3^*C_5^*C_a$.

**The proof of Lemma 4.2.** First we consider the following biharmonic equation

$$\begin{cases} \frac{1}{R_e}\triangle^2\psi = Gv + g & \text{in } \Omega, \\ \psi = \partial_n\psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

It is known that the above equation has a unique solution for any $v \in H_0^2(\Omega)$. Thus, there exists a linear operator $T : H_0^2(\Omega) \to H^3(\Omega) \cap H_0^2(\Omega)$ such that

$$\psi = Tv. \tag{4}$$

Equation (3) can be written as

$$\psi = T\psi. \tag{5}$$

We now prove that $T$ is a compact operator. In fact, based on the regularity result in [7], we know that

$$\|Tv\|_3 = \|\psi\|_3 \le \bar{C}_2\|Gv + g\|_{-1} \le \bar{C}_2(\|Gv\|_{-1} + \|g\|_{-1}).$$

Now

$$\begin{aligned}\|Gv\|_{-1} &= \|\partial_x u\triangle(\partial_y v) - \partial_y u\triangle(\partial_x v) + 2\nabla(\partial_x u)\cdot\nabla(\partial_y v) - 2\nabla(\partial_y u)\cdot\nabla(\partial_x v)\|_{-1} \\ &\le \|\partial_x u\triangle(\partial_y v)\|_{-1} + \|\partial_y u\triangle(\partial_x v)\|_{-1} \\ &\quad +2\|\nabla(\partial_x u)\cdot\nabla(\partial_y v)\|_{-1} + 2\|\nabla(\partial_y u)\cdot\nabla(\partial_x v)\|_{-1} \\ &\hat{=} \sum_{i=1}^4 K_i.\end{aligned}$$

We estimate each term separately.

$$\begin{aligned}K_1 &= \sup_{\xi\in H_0^1(\Omega)} \frac{|(\partial_x u\triangle(\partial_y v),\xi)|}{|\xi|_1} \\ &= \sup_{\xi\in H_0^1(\Omega)} \frac{|(\triangle v, \partial_{xy}u\xi + \partial_y\xi\partial_x u)|}{|\xi|_1} \\ &\le \sup_{\xi\in H_0^1(\Omega)} \frac{C_0^2\|v\|_2\|u\|_3|\xi|_1 + \|v\|_2|\xi|_1\|u\|_{1,\infty}}{|\xi|_1} \\ &\le (C_0^2 + C_8)\|u\|_3\|v\|_2.\end{aligned} \tag{6}$$

Similarly,

$$K_2 \le (C_0^2 + C_8)\|u\|_3\|v\|_2,$$

and for $i = 3, 4$,

$$K_i \le 2C_0^2\|u\|_3\|v\|_2.$$

Finally, we obtain

$$\|Tv\|_3 \le \bar{C}_2[(6C_0^2 + 2C_8)\|u\|_3\|v\|_2 + \|g\|_{-1}]. \tag{7}$$

Since $H^3(\Omega)$ embeds into $H^2(\Omega)$ compactly, the operator $T$ is compact. In the following, we only need to prove that the solution of the following equation is bounded in $H_0^2(\Omega)$ :

$$\psi = tT\psi, \quad 0 < t \le 1,$$

that is,

$$\frac{1}{R_e}\triangle^2\psi = t(G\psi + g). \tag{8}$$

Using Green's formula, we get

$$\begin{aligned}
\frac{1}{R_e}\|\triangle\psi\|_0^2 &= t(\partial_x u\partial_y\psi - \partial_y u\partial_x\psi, \triangle\psi) + t(g, \psi) \\
&\le \|\nabla u\|_{L^4(\Omega)}\|\nabla\psi\|_{L^4(\Omega)}\|\triangle\psi\|_0 + \|g\|_{-1}|\psi|_1 \\
&\le C_0^2C_1R_e\|f\|_0\|\triangle\psi\|_0^2 + \bar{C}_3\|g\|_{-1}\|\triangle\psi\|_0,
\end{aligned}$$

where $\bar{C}_3$ satisfies the following Sobolev inequality

$$|\psi|_1 \le \bar{C}_3\|\triangle\psi\|_0, \quad \psi \in H_0^2(\Omega).$$

Since $1 - C_0^2C_1R_e^2\|f\|_0 > 0$ ,

$$\|\triangle\psi\|_0 \le \bar{C}_4\|g\|_{-1}, \tag{9}$$

where $\bar{C}_4 = \frac{\bar{C}_3R_e}{1-C_0^2C_1R_e^2\|f\|_0}$. By the Schauder fixed point theorem, we know that (3) has a unique solution.

Finally, we prove that the prior estimate is true.

$$\begin{aligned}
\|\psi\|_3 &\le \bar{C}_2(\|\partial_x u\triangle(\partial_y\psi) - \partial_y u\triangle(\partial_x\psi) + 2\nabla(\partial_x u)\cdot\nabla(\partial_y\psi) \\
&\quad -2\nabla(\partial_y u)\cdot\nabla(\partial_x\psi)\|_{-1}) + C_2\|g\|_{-1} \\
&\le \bar{C}_2(\|\partial_x u\triangle(\partial_y\psi)\|_{-1} + \|\partial_y u\triangle(\partial_x\psi)\|_{-1} \\
&\quad +2\|\nabla(\partial_x u)\cdot\nabla(\partial_y\psi)\|_{-1} + 2\|\nabla(\partial_y u)\cdot\nabla(\partial_x\psi)\|_{-1}) + C_2\|g\|_{-1} \\
&\hat{=} \bar{C}_2\sum_{i=1}^{4}K_i + \bar{C}_2\|g\|_{-1}.
\end{aligned}$$

Using the same argument as in (6), we can derive

$$K_i \le (C_0^2 + C_8)\|u\|_3\|\triangle\psi\|_0, \quad i = 1, 2, \tag{10}$$

and

$$K_i \le 2C_0^2\|u\|_3\|\triangle\psi\|_0, \quad i = 3, 4. \tag{11}$$

Then

$$\begin{aligned}
\|\psi\|_3 &\le \bar{C}_2(6C_0^2 + 2C_8)\|u\|_3\|\triangle\psi\|_0 + \bar{C}_2\|g\|_{-1} \\
&\le \bar{C}_2(6C_0^2 + 2C_8)\|u\|_3(\eta\|\psi\|_3 + \eta^{-1}|\psi|_1) + \bar{C}_2\|g\|_{-1}.
\end{aligned}$$

So if $\eta$ is sufficiently small, we get

$$\|\psi\|_3 \le C'\|g\|_{-1},$$

where $C' = \frac{\bar{C}_2(6C_0^2+2C_8)\eta^{-1}\bar{C}_3\bar{C}_4\|u\|_3+\bar{C}_2}{1-\eta\bar{C}_2(6C_0^2+2C_8)\|u\|_3}$.

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