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CASCADIC MULTIGRID METHOD FOR THE MORTAR ELEMENT METHOD FOR P1 NONCONFORMING ELEMENT *1)

Chun-jia Bi
(*Department of Mathematics, Yantai University, Shandong 264005, China*)

Dan-hui Hong
(*Institute of Mathematics, Fudan University, Shanghai 200433, China*)

Abstract

In this paper, we consider the cascadic multigrid method for the mortar P_1 nonconforming element which is used to solve the Poisson equation and prove that the cascadic conjugate gradient method is accurate with optimal complexity.

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Key words: Mortar P_1 nonconforming element, Cascadic multigrid method.

1. Introduction

The mortar finite element method was first introduced by Bernardi, Maday and Patera in [3]. From then on, this method as a special nonconforming domain decomposition technique has aroused many researchers' attention because different types of discretizations can be employed in different parts of the computational domain. We refer to [3] for the general presentation of the mortar element method and [1], [2], [4], [10], [12], [13], [16], [17] and [25] for details.

In the mortar element methods, the computational domain is first decomposed into a polygonal partition. The meshes on different subdomains need not match across subdomain interfaces. The basic idea of this method is to replace the strong continuity condition on the interfaces between different subdomains by a weaker one, i.e., the so called mortar condition. The mortar condition guarantees optimal discretization schemes, this is, the global discretization error is bounded by the sum of the optimal approximation errors on different subdomains.

On the other hand, Bornemann and Deuflhard [6] [7] have proposed the cascadic multigrid method. Compared with usual multigrid methods, this method requires no coarse grid corrections at all that may be viewed as a "one way" multigrid method. Another distinctive feature of this method is performing more iterations on coarser levels so as to obtain less iterations on finer level. Numerical experiments [7] show that this method is very effective. A first candidate of such a cascadic multigrid method was the cascadic conjugate gradient method, in short CCG method, which used the conjugate gradient method as basic iteration method on each level. For the second-order elliptic problem in 2D discretized by the P_1 conforming element, Bornemann and Deuflhard [6] have proved that the CCG method is accurate with optimal computational complexity. The general framework to analyze the cascadic multigrid method has been established by Shi and Xu in [21]. The cascadic multigrid method also has been applied to the elliptic problems in domain with curved boundary by Bi and Li in [5], to the Stokes problems by Braess and Dahmen in [8], to the elliptic problems in domain with re-entrant corners by Shaidurov and Tobiska in [19], to the parabolic problems by Shi and Xu in [22], to the elliptic problems for finite volume methods by Shi, Xu and Man [23], and to the semilinear elliptic problems by Timmermann in [24].

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Recently, Braess, Deuflhard and Lipnikov [9] have proposed and analyzed a subspace cascadic multigrid method for the elliptic problems with strong material jumps in the framework of the mortar mixed method. In the mortar mixed method in [9], the finite element spaces associated with the subdomain grids and the interface between the subdomains are the P_1 conforming finite element space and the piecewise constant functions space.

Marcinkowski [17] has considered the mortar element method for P_1 nonconforming element, obtained the optimal order error estimate in H^1-norm, and proposed an additive Schwarz method to solve the system of linear equations. Xu and Chen [25] have considered the multigrid algorithm for the mortar element method for P_1 nonconforming element and proved that the W-cycle multigrid is optimal, i.e., the convergence rate is independent of the mesh size and mesh level, and constructed a variable V-cycle multigrid preconditioner which results in a preconditioned system with uniformly bounded condition number.

In this paper, for the Poisson problem, we consider the cascadic multigrid method for the mortar P_1 nonconforming element and show that the CCG method is accurate with optimal computational complexity.

The remainder of this paper is organized as follows. In Section 2, we introduce notation and the mortar P_1 nonconforming element. In Section 3, we obtain the optimal order error estimate in L^2-norm. The cascadic multigrid method is considered in Section 4. In Section 5, we present numerical experiments showing the optimality of our theoretical results.

In this paper, C denotes the positive constant independent of the meshsize and the number of the levels which will be stated below and may be different at different occurrence.

2. Mortar P_1 Nonconforming Element

In this section, we provide some notation and preliminaries. We consider the following Poisson problem:

$$-\triangle u = f, \quad \text{in} \quad \Omega, \tag{2.1}$$

$$u = 0, \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

where Ω is a bounded convex polygonal domain and $f \in L^2(\Omega)$.

The variational form of the problem (2.1)-(2.2) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \quad (f,v) = \int_\Omega f v \mathrm{d}x, \quad \forall u,v \in H_0^1(\Omega). \tag{2.4}$$

In this paper, we will need to assume the H^2-regularity on the problem (2.1)-(2.2), i.e., for any $f \in L^2(\Omega)$, the problem (2.1)-(2.2) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $\|u\|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$. The definitions of integral and fractional Sobolev spaces and associated norms are the same as those in [14].

In this paper, we consider a geometrically conforming version of the mortar element method, i.e., Ω is divided into non-overlapping polygonal subdomains Ω_i, $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$, where $\overline{\Omega}_i \cap \overline{\Omega}_j$ is an empty set or an edge or a vertex for $i \neq j$.

Each subdomain Ω_i is triangulated to produce a regular mesh $\mathcal{T}_h^i$ with mesh parameter h_i, where h_i is the largest diameter of the elements in $\mathcal{T}_h^i$. The triangulations of subdomains generally do not align at the subdomain interfaces. Let Γ_{ij} denote the open straight line segment which is common to $\overline{\Omega}_i$ and $\overline{\Omega}_j$ and Γ denote the union of all interfaces between the subdomains, i.e., $\Gamma = \cup\partial\Omega_i \backslash \partial\Omega$. We assume that the endpoints of each interface segment in Γ are vertices of $\mathcal{T}_h^i$ and $\mathcal{T}_h^j$. Let $\mathcal{T}_h$ denote the global mesh $\cup_i \mathcal{T}_h^i$ with $h = \max_{1\le i\le N} h_i$.

Since the triangulations on two adjacent subdomains are independent, the interface $\overline{\Gamma}_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ is provided with two different and independent 1-D meshes, which are denoted by $\mathcal{T}_h^i(\Gamma_{ij})$ and $\mathcal{T}_h^j(\Gamma_{ij})$, respectively. We define one of the sides of Γ_{ij} as a mortar one, the other as a nonmortar one, denoted by γ_i and δ_j, respectively. Let $\Omega_{M(\Gamma_{ij})}$ denote the mortar domain of Γ_{ij} and $\Omega_{NM(\Gamma_{ij})}$ the non-mortar domain of Γ_{ij}. Define $u|_{\gamma_i}$ and $u|_{\delta_j}$ to be the traces of $u|_{\Omega_{M(\Gamma_{ij})}}$ and $u|_{\Omega_{NM(\Gamma_{ij})}}$ on Γ_{ij} respectively. Define CR nodal points as the midpoints of the edges of elements in $\mathcal{T}_h$. The sets of CR nodal points belonging to $\overline{\Omega}_i$, $\partial\Omega_i$, $\partial\Omega$, γ_i and δ_j are denoted by $\Omega_{i,h}^{CR}$, $\partial\Omega_{i,h}^{CR}$, $\partial\Omega_h^{CR}$, γ_i^{CR} and δ_j^{CR} respectively. Let $\Omega_{i,h}$ and $\partial\Omega_{i,h}$ denote the sets of vertices of the triangulations $\mathcal{T}_h^i$ that in $\overline{\Omega}_i$ and $\partial\Omega_i$, respectively.

In order to define the mortar P_1 nonconforming finite element space, we first introduce the P_1 nonconforming finite element space over each subdomain Ω_i :

$$\widetilde{V}_i^h = \widetilde{V}_i^h(\Omega_i) = \{v : v|_K \text{ is linear for all } K \in \mathcal{T}_h^i, v \text{ is continuous at } \Omega_{i,h}^{CR} \backslash \partial\Omega_{i,h}^{CR} \text{ and } v = 0 \text{ at } \partial\Omega_{i,h}^{CR} \cap \partial\Omega_h^{CR}\},$$

with the broken norm $||v||_{1,h,\Omega_i} = (\sum_{K\in\mathcal{T}_h^i} ||v||^2_{H^1(K)})^{1/2}$ and the broken semi-norm $|v|_{1,h,\Omega_i} = (\sum_{K\in\mathcal{T}_h^i} |v|^2_{H^1(K)})^{1/2}$.

We can now introduce the global space $\widetilde{V}^h$:

$$\widetilde{V}^h = \prod_{i=1}^{N} \widetilde{V}_i^h(\Omega_i),$$

with the broken norm $||v||_{1,h} = (\sum_{i=1}^N ||v||^2_{1,h,\Omega_i})^{\frac{1}{2}}$ and the broken semi-norm $|v|_{1,h} = (\sum_{i=1}^N |v|^2_{1,h,\Omega_i})^{\frac{1}{2}}$.

Let $M(\delta_j)$ be the subspace of the space $L^2(\Gamma_{ij})$:

$$M(\delta_j) = \{v : v \in L^2(\Gamma_{ij}), v \text{ is piecewise constant on } \mathcal{T}_h^j(\delta_j)\}.$$

For each nonmortar side $\delta_j = \Gamma_{ij} \in \Gamma$, we introduce the L^2 orthogonal projection $Q^{\delta_j} : L^2(\Gamma_{ij}) \to M(\delta_j)$ defined by

$$(Q^{\delta_j}u, \psi)_{0,\delta_j} = (u, \psi)_{0,\delta_j}, \quad \forall \psi \in M(\delta_j).$$

Here and hereafter $(\cdot,\cdot)_{0,\delta_j}$ denotes the usual L^2 inner product over the space $L^2(\delta_j)$.

Lemma 2.1. (see [17]) *If $u \in H^s(\delta_j)$, then we have*

$$\left\|u - Q^{\delta_j}u\right\|_{0,\delta_j} \le Ch_j^s \left|u\right|_{H^s(\delta_j)}, \quad s \in \{0, \frac{1}{2}, 1\}. \tag{2.5}$$

We define the mortar P_1 nonconforming finite element space V^h as:

$$V^h = \{v \in \widetilde{V}^h, \quad Q^{\delta_j}(v|_{\gamma_i}) = Q^{\delta_j}(v|_{\delta_j}), \quad \forall \gamma_i = \delta_j \in \Gamma\}.$$

The condition on Γ is called the mortar condition.

This mortar condition is constructed in [17]. We note that this mortar condition is not only dependent on the degrees of freedom on the interface but also the degrees of freedom near the interface, see [17] for details.

The discrete problem of (2.3) is to find $u_h \in V^h$ such that

$$a_h(u_h, v) = (f, v), \quad \forall v \in V^h, \tag{2.6}$$

where

$$a_h(u,v)=\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla u\cdot\nabla v\mathrm{d}x,\quad (f,v)=\sum_{i=1}^{N}\int_{\Omega_i} fv\mathrm{d}x.$$

From [17], we know that the bilinear form $a_h(\cdot,\cdot)$ is elliptic on the discrete space V^h with constant independent of h and the number of subdomains. Then, the discrete problem (2.6) has a unique solution $u_h\in V^h$.

Let $\widetilde{W}^{h/2}(\Omega_i)$ be the conforming finite element space of piecewise linear continuous functions on the triangulation $\mathcal{T}_{h/2}^i$ which is constructed by joining the midpoints of the edges of the element of $\mathcal{T}_h^i$.

As in [17] [18], we introduce a local equivalence map $M_i:\widetilde{V}_i^h\to\widetilde{W}_i^{h/2}$.

Definition 2.1. *Given $u\in\widetilde{V}_i^h$, we define $M_iu\in\widetilde{W}_i^{h/2}$ by the values of M_iu at the vertices of the triangulation $\mathcal{T}_{h/2}^i$. The vertices are divided into three sets of points:*

(1). If $p\in\Omega_{i,h}^{CR}$, then

$$M_iu(p)=u(p).$$

(2). If $p\in\Omega_{i,h}\backslash\partial\Omega_{i,h}$ and p is a vertex of an element of $\mathcal{T}_h^i$, then

$$M_iu(p)=\frac{1}{N(p)}\sum_{\tau_j^h}u|_{\tau_j^h}(p),$$

where the sum is taken over all triangle τ_j^h with the common vertex p and $N(p)$ is the number of these triangles.

(3). If $q\in\partial\Omega_{i,h}$, then

$$M_iu(q)=\frac{|q_lq|}{|q_lq_r|}u(q_l)+\frac{|qq_r|}{|q_lq_r|}u(q_r),$$

where q_l,q_r are the left and right neighboring CR nodal points of q.

Lemma 2.2. (see [17]) *Let M_iu be defined as above, then for any $u\in\widetilde{V}_i^h(\Omega_i)$, we have*

$$C_1|u|_{1,h,\Omega_i}\le|M_iu|_{1,\Omega_i}\le C_2|u|_{1,h,\Omega_i},$$
$$\|u-M_iu\|_{L^2(\varepsilon)}\le Ch_i^{1/2}|u|_{1,h,\Omega_i}.$$

Here ε is an edge of Ω_i.

3. Error Estimation in L^2-norm

In this section, by means of the duality argument, we will obtain the error estimate in L^2-norm which will be used in the analysis of the cascadic multigrid method. For this purpose, we consider the following auxiliary problem: Find $\varphi\in H_0^1(\Omega)$ such that

$$\begin{cases} -\triangle\varphi=g, & \text{in}\quad \Omega,\\ \varphi=0, & \text{on}\quad \partial\Omega,\end{cases}\tag{3.1}$$

where $g\in L^2(\Omega)$. Obviously, the problem (3.1) also has the corresponding H^2-regularity.

In [17], for the Poisson problem, Marcinkowski proved that the error estimate in H^1-norm is of the same optimal order as in the standard P_1 nonconforming finite element method. For simplicity, we rewrite this result as follows.

Lemma 3.1. *Let u,u_h be the solutions of (2.3) and (2.6), respectively. Then we have*

$$\|u-u_h\|_{1,h}\le Ch\,\|u\|_{2,\Omega}\,.\tag{3.2}$$

The following Lemma 3.2 is proved in [11]. We formulate it here which will be used in our convergence proof.

Lemma 3.2. *There exists a constant C independent of h_K such that $v|_K \in H^1(K)$ for every $K \in \mathcal{T}_h$*

$$\int_{\partial K} v^2 \mathrm{d}s \le C(h_K^{-1}||v||_{0,K}^2 + h_K|v|_{1,K}^2), \quad \forall K \in \mathcal{T}_h. \tag{3.3}$$

In order to get the error estimate in L^2-norm, we first prove the following Lemmas 3.3 and 3.4.

Lemma 3.3. *Assume that u, u_h and φ are the solutions of (2.3), (2.6) and (3.1), respectively. Then we have*

$$\left|\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_{\partial K}(u-u_h)\frac{\partial\varphi}{\partial n}\mathrm{d}s\right| \le Ch^2\left\|u\right\|_{2,\Omega}\left\|g\right\|_{0,\Omega}, \tag{3.4}$$

where n is the unit outerward normal vector along ∂K.

Proof. Given a triangle $K \in \mathcal{T}_h^i$, we denote the set of the sides of K by $E(K)$ and $E_{h,i} = \cup_{K\in\mathcal{T}_h^i}E(K)$. Let $E_{h,i}^{\mathrm{in}}$ be the set of the interior sides of the triangulation $\mathcal{T}_h^i$.

We rewrite $\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_{\partial K}(u-u_h)\frac{\partial\varphi}{\partial n}\mathrm{d}s$ as the sum of three terms:

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_{\partial K}(u-u_h)\frac{\partial\varphi}{\partial n}\mathrm{d}s\\
=\;&\sum_{i=1}^{N}\sum_{e\in E_{h,i}^{\mathrm{in}}}\int_e[u-u_h]\frac{\partial\varphi}{\partial n}\mathrm{d}s+\sum_{e\in\partial\Omega}\int_e(u-u_h)\frac{\partial\varphi}{\partial n}\mathrm{d}s+\sum_{\delta_j\in\Gamma}\int_{\delta_j}[u-u_h]\frac{\partial\varphi}{\partial n}\mathrm{d}s\\
=\;&S_1+S_2+S_3,
\end{aligned}\tag{3.5}$$

where $[\cdot]$ denotes the jump of a function across e or δ_j.

First, we estimate $|S_1| + |S_2|$. Given $e \in E_{h,i}^{\mathrm{in}}$, there exist two triangles $K_1, K_2 \in \mathcal{T}_h^i$ that have e as a common side. Let $\widetilde{W}^h(\Omega_i)$ be the conforming finite element space of piecewise linear continuous functions on the triangulation $\mathcal{T}_h^i$ and π_h^i be the standard linear interpolation operator: $H^2(\Omega_i) \to \widetilde{W}^h(\Omega_i)$. Then we have

$$\begin{aligned}
\int_e[u-u_h]\frac{\partial\varphi}{\partial n}\mathrm{d}s &= \int_e[\pi_h^i u-u_h]\frac{\partial\varphi}{\partial n}\mathrm{d}s\\
&= \int_e[\pi_h^i u-u_h](\frac{\partial\varphi}{\partial n}-Q^e\frac{\partial\varphi}{\partial n})\mathrm{d}s\\
&= \int_e[(\pi_h^i u-u_h)-Q^e(\pi_h^i u-u_h)](\frac{\partial\varphi}{\partial n}-Q^e\frac{\partial\varphi}{\partial n})\mathrm{d}s,
\end{aligned}\tag{3.6}$$

where $Q^e : L^2(e) \to R$ is orthogonal projection onto one dimensional space of constant functions on e.

From the property of the orthogonal projection Q^e and Lemma 3.2, we have

$$||\frac{\partial\varphi}{\partial n}-Q^e\frac{\partial\varphi}{\partial n}||_{0,e} \le ||\frac{\partial\varphi}{\partial n}-Q^{K_1}\frac{\partial\varphi}{\partial n}||_{0,e} \le Ch_i^{1/2}|\varphi|_{2,K_1}, \tag{3.7}$$

where $Q^{K_1} : L^2(K_1) \to R$ is orthogonal projection onto one dimensional space of constant functions on K_1.
Similar to (3.7), we get

$$|||[(\pi_h^i u - u_h) - Q^e(\pi_h^i u - u_h)]|||_{0,e} \quad \leq \quad C h_i^{1/2} |\pi_h^i u - u_h|_{1,K_1}. \tag{3.8}$$

From (3.6)-(3.8), we obtain

$$|\int_e [u - u_h] \frac{\partial \varphi}{\partial n} \mathrm{d}s| \leq C h_i |\varphi|_{2,K_1 \cup K_2} \left(|\pi_h^i u - u_h|_{1,K_1} + |\pi_h^i u - u_h|_{1,K_2} \right). \tag{3.9}$$

For $e \in \partial\Omega \cap E(K)$, using the same method as above, we have

$$\begin{aligned} \int_e (u - u_h) \frac{\partial \varphi}{\partial n} \mathrm{d}s &= \int_e (\pi_h^i u - u_h) \frac{\partial \varphi}{\partial n} \mathrm{d}s \\ &= \int_e (\pi_h^i u - u_h - Q^e(\pi_h^i u - u_h))(\frac{\partial \varphi}{\partial n} - Q^e \frac{\partial \varphi}{\partial n}) \mathrm{d}s, \end{aligned}$$

therefore,

$$|\int_e (u - u_h) \frac{\partial \varphi}{\partial n} \mathrm{d}s| \leq C h_i |\varphi|_{2,K} |\pi_h^i u - u_h|_{1,K}. \tag{3.10}$$

From the triangle inequality, the standard interpolation theory, Lemma 3.1 and the H^2-regularity assumption, we get

$$|S_1| + |S_2| \leq C \sum_{i=1}^N h_i |\varphi|_{2,\Omega_i} \left| \pi_h^i u - u_h \right|_{1,h,\Omega_i} \leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega}. \tag{3.11}$$

Next, we estimate $|S_3|$. Let δ_j be the common side of Ω_i and Ω_j. From the mortar condition and the definition of the operator Q^{δ_j}, we get

$$\int_{\delta_j} \frac{\partial \varphi}{\partial n} [u_h] \mathrm{d}s = \int_{\delta_j} (\frac{\partial \varphi}{\partial n} - Q^{\delta_j} \frac{\partial \varphi}{\partial n}) [u_h] \mathrm{d}s,$$

$$\int_{\delta_j} (\frac{\partial \varphi}{\partial n} - Q^{\delta_j} \frac{\partial \varphi}{\partial n})(-Q^{\delta_j} M_i \pi_h^i u + Q^{\delta_j} M_i u_h^i + Q^{\delta_j} \pi_h^j u - Q^{\delta_j} u_h^j) \mathrm{d}s = 0,$$

where u_h^i and u_h^j are the restrictions of u_h to Ω_i and Ω_j, respectively.

Therefore, we have

$$\begin{aligned} & \left| \int_{\delta_j} [u - u_h] \frac{\partial \varphi}{\partial n} \mathrm{d}s \right| \\ = & \left| \int_{\delta_j} (\frac{\partial \varphi}{\partial n} - Q^{\delta_j} \frac{\partial \varphi}{\partial n}) \left(u - \pi_h^i u + \pi_h^i u - u_h^i - M_i(\pi_h^i u - u_h^i) \right. \right. \\ & \quad + M_i(\pi_h^i u - u_h^i) - Q^{\delta_j} M_i(\pi_h^i u - u_h^i) \\ & \quad \left. \left. + Q^{\delta_j}(\pi_h^j u - u_h^j) + u_h^j - \pi_h^j u + \pi_h^j u - u \right) \mathrm{d}s \right| \\ \leq & \ \| \frac{\partial \varphi}{\partial n} - Q^{\delta_j} \frac{\partial \varphi}{\partial n} \|_{0,\delta_j} \left(\|u - \pi_h^i u\|_{0,\delta_j} + \|\pi_h^i u - u_h^i - M_i(\pi_h^i u - u_h^i)\|_{0,\delta_j} \right. \\ & \quad + \|M_i(\pi_h^i u - u_h^i) - Q^{\delta_j} M_i(\pi_h^i u - u_h^i)\|_{0,\delta_j} \\ & \quad \left. + \|\pi_h^j u - u_h^j - Q^{\delta_j}(\pi_h^j u - u_h^j)\|_{0,\delta_j} + \|u - \pi_h^j u\|_{0,\delta_j} \right). \end{aligned} \tag{3.12}$$

By Lemma 2.1 and the trace theorem, we obtain

$$\|\frac{\partial\varphi}{\partial n} - Q^{\delta_j}\frac{\partial\varphi}{\partial n}\|_{0,\delta_j} \le Ch_j^{1/2}\left|\frac{\partial\varphi}{\partial n}\right|_{H^{1/2}(\delta_j)} \le Ch_j^{1/2}\|\varphi\|_{2,\Omega_j}. \tag{3.13}$$

By Lemma 2.2, we obtain

$$\|\pi_h^i u - u_h^i - M_i(\pi_h^i u - u_h^i)\|_{0,\delta_j} \le Ch_i^{1/2}\left|\pi_h^i u - u_h^i\right|_{1,h,\Omega_i}. \tag{3.14}$$

By Lemma 2.1 and Lemma 2.2, we obtain

$$\|M_i(\pi_h^i u - u_h^i) - Q^{\delta_j}M_i(\pi_h^i u - u_h^i)\|_{0,\delta_j} \le Ch_j^{1/2}\left|\pi_h^i u - u_h^i\right|_{1,h,\Omega_i}. \tag{3.15}$$

Similarly, we have

$$\|\pi_h^j u - u_h^j - Q^{\delta_j}(\pi_h^j u - u_h^j)\|_{0,\delta_j} \le Ch_j^{1/2}|\pi_h^j u - u_h|_{1,h,\Omega_j}. \tag{3.16}$$

By means of the standard interpolation theory and the trace theorem, we obtain

$$\|u - \pi_h^i u\|_{0,\delta_j} \le Ch_i^{3/2}\|u\|_{H^{3/2}(\delta_j)} \le Ch_i^{3/2}\|u\|_{2,\Omega_i}, \tag{3.17}$$

$$\|u - \pi_h^j u\|_{0,\delta_j} \le Ch_j^{3/2}\|u\|_{2,\Omega_j}. \tag{3.18}$$

From (3.12)-(3.18), Lemma 3.1 and the regularity assumption, we get

$$|S_3| = \left|\sum_{\delta_j\in\Gamma}\int_{\delta_j}[u-u_h]\frac{\partial\varphi}{\partial n}\mathrm{d}s\right| \le Ch^2\|u\|_{2,\Omega}\|g\|_{0,\Omega}. \tag{3.19}$$

From (3.5), (3.11) and (3.19), we get the desired result (3.4).

Lemma 3.4. *Assume that u, u_h and φ are the solutions of (2.3), (2.6) and (3.1) respectively. Then we have*

$$\left|\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla(u-u_h)\cdot\nabla\varphi\mathrm{d}x\right| \le Ch^2\|u\|_{2,\Omega}\|g\|_{0,\Omega}. \tag{3.20}$$

Proof. Let $\varphi_h \in V^h$ be the mortar P_1 nonconforming element approximation of (3.1). We have

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla(u-u_h)\nabla\varphi\mathrm{d}x\\
=\;&\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla(u-u_h)\nabla(\varphi-\varphi_h)\mathrm{d}x + \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla(u-u_h)\nabla\varphi_h\mathrm{d}x\\
=\;&E_1+E_2,
\end{aligned} \tag{3.21}$$

The estimation for E_1 is easy,

$$\begin{aligned} |E_1| &= \left|\sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla(u-u_h)\nabla(\varphi-\varphi_h)\mathrm{d}x\right| \\ &\le \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\|u-u_h\|_{1,K}\|\varphi-\varphi_h\|_{1,K} \le Ch^2\|u\|_{2,\Omega}\|g\|_{0,\Omega}, \end{aligned} \tag{3.22}$$

where Lemma 3.1 and the regularity assumption are used.

Using Green's formula, we get

$$\begin{aligned} E_2 &= \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla u\nabla\varphi_h\mathrm{d}x - \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \nabla u_h\nabla\varphi_{\mathrm{h}}\mathrm{d}x \\ &= \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_{\partial K}\frac{\partial u}{\partial n}\varphi_h\mathrm{d}s - \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K \triangle u\varphi_h\mathrm{d}x - \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_K f\varphi_h\mathrm{d}x \\ &= \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\int_{\partial K}\frac{\partial u}{\partial n}[\varphi_h-\varphi]\mathrm{d}s. \end{aligned}$$

Using a similar argument as Lemma 3.3, we get

$$|E_2| \le Ch^2||u||_{2,\Omega}\|g\|_{0,\Omega}. \tag{3.23}$$

Combining (3.21), (3.22) with (3.23) yields the desired result (3.20).

Theorem 3.5. *Let u, u_h be the solutions of (2.3) and (2.6) respectively. Then*

$$\|u-u_h\|_{0,\Omega} \le Ch^2\,\|u\|_{2,\Omega}\,. \tag{3.24}$$

Proof. By the definition of L^2-norm, we obtain

$$\|u-u_h\|_{0,\Omega} = \sup_{0\ne g\in L^2(\Omega)}\frac{|(u-u_h,g)|}{\|g\|_{0,\Omega}} = \sup_{0\ne g\in L^2(\Omega)}\frac{|(u-u_h,-\triangle\varphi)|}{\|g\|_{0,\Omega}}. \tag{3.25}$$

Using Green's formula, we have

$$(u-u_h,\triangle\varphi) = \sum_{i=1}^{N}\sum_{K\in\mathcal{T}_h^i}\left(-\int_K \nabla(u-u_h)\nabla\varphi\mathrm{d}x + \int_{\partial K}(u-u_h)\frac{\partial\varphi}{\partial n}\mathrm{d}s\right).$$

By Lemma 3.3 and Lemma 3.4, we obtain the desired result (3.24).

4. Cascadic Multigrid Method

Let $\mathcal{T}_1^i$ be a coarsest triangulation of Ω_i with the mesh size h_1^i. The triangulation generally does not align at the subdomain interface. Denote the global mesh $\cup_i\mathcal{T}_1^i$ by $\mathcal{T}_1$ and $h_1 = \max_{1\le i\le N} h_1^i$. We refine the triangulation $\mathcal{T}_1$ to produce $\mathcal{T}_2$ by connecting the midpoints of the edges of the triangles in $\mathcal{T}_1$. Obviously, the mesh size h_2 in $\mathcal{T}_2$ is $h_2 = h_1/2$. Repeating this process, we get k-level triangulation $\mathcal{T}_k$ with the mesh size $h_k = h_1 2^{-k}(k = 1, ..., L)$. Let V_k be the mortar P_1 nonconforming finite element space over the triangulation $\mathcal{T}_k$.

The discrete problem of (2.3) on V_k is to find $u_k \in V_k$ such that

$$a_k(u_k, v) = (f, v), \quad \forall v \in V_k, \tag{4.1}$$

where

$$a_k(u, v) = \sum_{i=1}^{N} \sum_{K \in \mathcal{T}_k^i} \int_K \nabla u \cdot \nabla v \mathrm{d}x, \quad (f, v) = \sum_{i=1}^{N} \int_{\Omega_i} f \cdot v \mathrm{d}x, \quad \forall u, v \in V_k.$$

From [17], we know that the bilinear form $a_k(\cdot, \cdot)$ is elliptic on the discrete space V_k. Then we can define the energy norm:

$$||v||_{1,k} = (a_k(v, v))^{\frac{1}{2}}, \quad \forall v \in V_k.$$

Before giving a basis of V_k, we first define an operator $\mathcal{E}_{\delta_j} : \widetilde{V}_k \to \widetilde{V}_k$ by

$$\mathcal{E}_{\delta_j} \tilde{v}(m) = \begin{cases} Q^{\delta_j}(\tilde{v}|_{\gamma_i} - \tilde{v}|_{\delta_j})(m), & m \in \delta_{k,j}^{CR}, \\ 0, & \text{otherwise}. \end{cases}$$

Then for any $\tilde{v} \in \widetilde{V}_k$, let

$$v = \tilde{v} + \sum_{\delta_j \in \Gamma} \mathcal{E}_{\delta_j} \tilde{v}.$$

We can check that $v \in V_k$ (see [25] for details).

Let $\{\tilde{\varphi}_k^l | l = 1, ..., \widetilde{N}_k\}$ be the nodal basis of the global nonconforming finite element space on k-level $\widetilde{V}_k(\Omega)$. The basis of V_k consists of the functions with the form

$$\varphi_k^l = \tilde{\varphi}_k^l + \sum_{\delta_j \in \Gamma} \mathcal{E}_{\delta_j} \tilde{\varphi}_k^l, \tag{4.2}$$

Apart from φ_k^l corresponding to those nodes on nonmortar side, it is not difficult to check these φ_k^l defined by (4.2) form a basis of V_k.

From above definition, we can see that there exist two kinds of basis function of space V_k : (a). φ_k^l and $\tilde{\varphi}_k^l$ at all nodes which are not in the interior of Γ are same. Denote the set of this kind basis function by $\Phi_0 = \{\varphi_k^l\}$; (b). φ_k^l at all nodes which are in the interior of each mortar edge $\gamma_i \in \Gamma$ are defined by (4.2). Denote the set of this kind basis function by $\Phi_\Gamma = \{\varphi_k^l\}$.

Let $< .,. >$ be the Euclidean scalar product of the basis in the finite element space V_k, the induced norm will be denoted by

$$|v|^2 =< v, v >, \quad \forall v \in V_k.$$

We define the operator $A_k : V_k \longrightarrow V_k$ by

$$< A_k u, v >= a_k(u, v), \quad \forall u, v \in V_k, \tag{4.3}$$

which is represented in the basis by the stiffness matrix.

Following [8] and [21], we introduce a projection operator $P_k : V_{k-1} + V_k \to V_k$ defined by

$$a_k(P_k u, v) = a_k(u, v), \quad \forall v \in V_k.$$

From the definition, it is easily seen that

$$||P_k v||_{1,k} \le ||v||_{1,k-1}, \quad \forall v \in V_{k-1}. \tag{4.4}$$

In this paper, we apply the theory framework developed by Shi and Xu in [21] to analyze the cascadic multigrid algorithm. Before giving the cascadic multigrid algorithm, we must define a suitable intergrid transfer operator for the nonnested mesh space V_k.

Following [25], we first define an operator $J_k^i : \widetilde{V}_{k-1}(\Omega_i) \to \widetilde{W}_k(\Omega_i)$ as follows:

(1). If $m \in \Omega_{k-1,i}^{CR}$, then $(J_k^i v)(m) = v(m)$,

(2). If $m \in \Omega_{k,i}^{N} \backslash \Omega_{k-1,i}^{CR}$ and $m \notin \partial\Omega$, then $(J_k^i v)(m) = \frac{1}{N(m)} \sum\limits_{K_i} v|_{K_i}(m)$, where $\Omega_{k,i}^N$ is the set of the vertices of the triangulation $\mathcal{T}_k^i$ that are in $\overline{\Omega}_i$ and the sum is taken over all triangles $K_i \in \mathcal{T}_k^i$ with the common vertex m and $N(m)$ is the number of those triangles.

(3). If $m \in \partial\Omega \cap \partial\Omega_{k,i}^N$, then $(J_k^i v)(m) = 0$, where $\partial\Omega_{k,i}^N$ is the set of the vertices of the triangulation $\mathcal{T}_k^i$ that are in $\partial\Omega_i$.

Based on the operator J_k^i, we define an intergrid operator $J_k : \widetilde{V}_{k-1} \to \widetilde{V}_k$ as follows: for any $v = (v_1, ..., v_N) \in \widetilde{V}_{k-1}$,

$$J_k v = (J_k^1 v_1, ..., J_k^N v_N) \in \widetilde{V}_k.$$

After above preparation, we can define an intergrid transfer operator $I_k : \widetilde{V}_{k-1} \to V_k$. For any $v \in \widetilde{V}_{k-1}$,

$$I_k v = J_k v + \sum_{\delta_j \in \Gamma} \mathcal{E}_{\delta_j}(J_k v). \tag{4.5}$$

We use the operator $C_k^{m_k} : V_k \to V_k$ to denote m_k steps of the CG iterative procedure on the level k.

The cascadic multigrid method can be written as follows:

Cascadic Multigrid Algorithm

(1). Set $u_1^0 = u_1^* \hat{=} u_1$, where u_1 is the solution of (4.1) on coarse initial triangulation $\mathcal{T}_1$. Let $u_k^0 = I_k u_{k-1}^*$,

(2). For $k = 2, \cdots, L$, $u_k^{m_k} = C_k^{m_k} u_k^0$,

(3). Set $u_k^* \hat{=} u_k^{m_k}$.

Following [6], we call a cascadic multigrid method optimal in the energy norm on the level L, if we obtain both the accuracy

$$||u_L - u_L^*||_{1,L} \approx ||u - u_L||_{1,L},$$

which means that the iterative error is comparable to the approximation error, and the multigrid complexity *amount of work* $= O(n_L), n_L = \dim V_L$.

Shi and Xu [21] gave three hypothesis to guarantee the convergence of the cascadic multigrid method. In this paper, we will prove three hypothesis hold for CCG method for the mortar P_1 nonconforming element space.

H1. For the intergrid transfer operator I_k, we assume that

$$(1). \quad \|v - I_k v\|_{0,k} \leq C h_k ||v||_{1,k-1}, \quad \forall v \in V_{k-1},$$

$$(2). \quad \|u_k - I_k u_{k-1}\|_{0,k} \leq C h_k^2 \|f\|_{0,\Omega},$$

where u_k is the mortar-type finite element solution of (4.1) on V_k.

H2. Assume that there exists a linear operator $\mathcal{T}_k^{m_k} : V_k \to V_k$ such that

$$\begin{aligned}
\left|u_k - C_k^{m_k} u_k^0\right| &\leq \left|\mathcal{T}_k^{m_k}(u_k - u_k^0)\right|, \\
\|\mathcal{T}_k^{m_k} v\|_{1,k} &\leq C \frac{h_k^{-1}}{m_k} ||v||_{0,k}, \quad \forall v \in V_k, \\
\|\mathcal{T}_k^{m_k} v\|_{1,k} &\leq \|v\|_{1,k}, \quad \forall v \in V_k,
\end{aligned}$$

where $|\cdot|$ denotes the Euclidean norm.

H3. For the operator P_k, we assume that

$$||u - P_k u||_{0,k} \le Ch_k ||u||_{1,k-1}, \quad \forall u \in V_k.$$

Lemma 4.1. *H1 holds for the mortar P_1 nonconforming finite element space.*

Proof. H1-(1) has been obtained by Xu and Chen in [25]. We only need to prove H1-(2) is also valid for the mortar P_1 nonconforming finite element space.

From the triangle inequality, we get

$$\|u_k - I_k u_{k-1}\|_{0,k} \le \|u_k - J_k u_{k-1}\|_{0,k} + \sum_{\delta_j \in \Gamma} \left\|\mathcal{E}_{\delta_j}(J_k u_{k-1})\right\|_{0,k}. \tag{4.6}$$

Using the similar argument in Lemma 3.3 in Shi and Xu [20] and Theorem 3.1, the first term can be estimated:

$$\|u_k - J_k u_{k-1}\|_{0,k} \le Ch_k^2 \|f\|_{0,\Omega}. \tag{4.7}$$

By means of the scaling argument and the definition of the operator $\mathcal{E}_{\delta_j}$, we can derive

$$\begin{aligned} ||\mathcal{E}_{\delta_j}(J_k u_{k-1})||_{0,k}^2 \le& Ch_k ||Q^{\delta_j}(J_k u_{k-1}|_{\gamma_i} - J_k u_{k-1}|_{\delta_j})||_{0,\delta_j}^2 \\ \le& Ch_k \Big(||Q^{\delta_j}(J_k u_{k-1}|_{\gamma_i} - u_k|_{\gamma_i})||_{0,\gamma_i}^2 + ||Q^{\delta_j}(u_k|_{\gamma_i} - u_k|_{\delta_j})||_{0,\delta_j}^2 \\ &+ ||Q^{\delta_j}(u_k|_{\delta_j} - J_k u_{k-1}|_{\delta_j})||_{0,\delta_j}^2 \Big). \end{aligned} \tag{4.8}$$

From the stability of the operator Q^{δ_j}, Lemma 3.2 and the inverse inequality, we get

$$\begin{aligned} \left\|Q^{\delta_j}(J_k u_{k-1}|_{\gamma_i} - u_k|_{\gamma_i})\right\|_{0,\gamma_i}^2 &\le ||J_k u_{k-1}|_{\gamma_i} - u_k|_{\gamma_i}||_{0,\gamma_i}^2 \\ &\le Ch_{k,i}^{-1} \|J_k u_{k-1} - u_k\|_{0,k,\Omega_i}^2, \end{aligned} \tag{4.9}$$

$$\left\|Q^{\delta_j}(u_{k-1}|_{\delta_j} - J_k u_{k-1}|_{\delta_j})\right\|_{0,\delta_j}^2 \le Ch_{k,j}^{-1} \|J_k u_{k-1} - u_k\|_{0,k,\Omega_j}^2. \tag{4.10}$$

Since $u_k \in V_k$, from the mortar condition, we have

$$\left\|Q^{\delta_j}(u_k|_{\gamma_i} - u_k|_{\delta_j})\right\|_{0,\delta_j}^2 = 0. \tag{4.11}$$

From (4.6), (4.7), (4.8), (4.9), (4.10) and (4.11), we obtain

$$\|u_k - I_k u_{k-1}\|_{0,k} \le Ch_k^2 \|f\|_{0,\Omega}.$$

Lemma 4.2. *H2 holds for the mortar P_1 nonconforming finite element space.*

Proof. From [6], for the CCG method, we have

$$\begin{aligned} \left|u_k - C_k^{m_k} u_k^0\right| &\le \left|\mathcal{T}_k^{m_k}(u_k - u_k^0)\right|, \\ \|\mathcal{T}_k^{m_k} v\|_{1,k} &\le \frac{\sqrt{\lambda_k^*}}{2m_k + 1} |v|, \\ \|\mathcal{T}_k^{m_k} v\|_{1,k} &\le \|v\|_{1,k}, \end{aligned}$$

where λ_k^* is the largest eigenvalue of A_k.

In order to complete the proof, we only need to prove the following results are valid.

$$\lambda_k^* \leq C, \quad |v| \leq Ch_k^{-1} \|v\|_{0,k}, \quad \forall v \in V_k. \tag{4.12}$$

Each $v \in V_k$ can be expressed by

$$v = v_0 + v_\Gamma = \sum_{\varphi_k^l \in \Phi_0} \mu_l \varphi_k^l + \sum_{\varphi_k^l \in \Phi_\Gamma} \mu_l \varphi_k^l.$$

Then, Using the Cauchy-Schwarz inequality, we get we have

$$\begin{aligned} < A_k v, v > &= a_k(v,v) \\ &\leq 2(a_k(v_0,v_0) + a_k(v_\Gamma, v_\Gamma)) \\ &\leq C \sum_{\varphi_k^l \in \Phi_0} \mu_l^2 a_k(\varphi_k^l, \varphi_k^l) + C \sum_{\varphi_k^l \in \Phi_\Gamma} \mu_l^2 a_k(\varphi_k^l, \varphi_k^l) \\ &= C \sum_{\varphi_k^l \in \Phi_0} \mu_l^2 ||\varphi_k^l||_{1,k} + C \sum_{\varphi_k^l \in \Phi_\Gamma} \mu_l^2 ||\varphi_k^l||_{1,k}. \end{aligned} \tag{4.13}$$

Obviously, the basis functions in Φ_0 and Φ_Γ have $O(h_k^2)$-supports. For each $\varphi_k^l \in \Phi_0$, we can calculate directly $\left\|\varphi_k^l\right\|_{1,k} = \left\|\widetilde{\varphi}_k^l\right\|_{1,k} \leq C$.

For each $\varphi_k^l \in \Phi_\Gamma$, we have

$$\left\|\varphi_k^l\right\|_{1,k} \leq \left\|\widetilde{\varphi}_k^l\right\|_{1,k} + \sum_{\delta_j \in \Gamma} ||\mathcal{E}_{\delta_j} \widetilde{\varphi}_k^l||_{1,k}. \tag{4.14}$$

Using the inverse inequality, the definition of $\mathcal{E}_{\delta_j}$, the property of the operator Q^{δ_j} and Lemma 3.2, we obtain

$$\begin{aligned} ||\mathcal{E}_{\delta_j} \widetilde{\varphi}_k^l||_{1,k} &\leq Ch_{k,j}^{-1} ||\mathcal{E}_{\delta_j} \widetilde{\varphi}_k^l||_{0,k} \leq Ch_{k,j}^{-1/2} ||Q^{\delta_j} \widetilde{\varphi}_k^l||_{0,\delta_j} \\ &\leq Ch_{k,j}^{-1/2} ||\widetilde{\varphi}_k^l||_{0,\delta_j} \leq Ch_{k,j}^{-1} ||\widetilde{\varphi}_k^l||_{0,k,\Omega_i} \leq C. \end{aligned} \tag{4.15}$$

From (4.14) and (4.15), we have

$$||\varphi_k^l||_{1,k} \leq C, \quad \varphi_k^l \in \Phi_\Gamma. \tag{4.16}$$

Then

$$< A_k v, v > \leq C \sum_{\varphi_k^l \in \Phi_0 \cup \Phi_\Gamma} \mu_l^2 = C < v, v >, \quad \lambda_k^* \leq C.$$

Since v is a linear function on element K, we have

$$\|v\|_{0,k}^2 = \sum_{K \in \mathcal{T}_k} \left(\frac{1}{3} |\text{meas}(K)| \sum_{t=1}^{3} v(m_t)^2 \right) \geq Ch_k^2 |v|^2,$$

where $m_t (1 \leq t \leq 3)$ are the midpoints of the edges of the element K.

In the following, we will use the duality argument [8] [21] to prove H3 holds for the mortar P_1 nonconforming finite element space. For this purpose, we consider the following auxiliary problem: for a given $v \in V_{k-1}$, find $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{cases} -\triangle \psi = v - P_k v, & \text{in} \quad \Omega, \\ \psi = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.17}$$

Lemma 4.3. *H3 holds for the mortar P_1 nonconforming finite element space.*

Proof. Let ψ_k be the mortar nonconforming approximation of the problem (4.17) in the discrete space V_k. From the definition of the operator P_k and Green's formula, we have

$$\begin{aligned}\|v-P_kv\|_{0,k}^2 &= \sum_{K\in\mathcal{T}_k}\int_K (v-P_kv)(-\triangle\psi)\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_k}\int_K \nabla(v-P_kv)\nabla(\psi-\psi_k)\mathrm{d}x \\ &\quad -\sum_{K\in\mathcal{T}_{k-1}}\int_{\partial K}\frac{\partial\psi}{\partial n}v\mathrm{d}s+\sum_{K\in\mathcal{T}_{k-1}}\int_{\partial K}\frac{\partial\psi}{\partial n}P_kv\mathrm{d}s \\ &= R_1+R_2+R_3, \end{aligned}\tag{4.18}$$

For the first term at the right side of (4.18), we estimate directly as follows

$$\begin{aligned}|R_1| &= |a_k(v-P_kv,\psi-\psi_k)|\le ||v-P_kv||_{1,k}||\psi-\psi_k||_{1,k} \\ &\le Ch_k||v-P_kv||_{1,k}\,\|\psi\|_{2,\Omega} \\ &\le Ch_k||v||_{1,k-1}\,\|v-P_kv\|_{0,k}\,, \end{aligned}\tag{4.19}$$

where the H^2-regularity assumption is used.

Following Lemma 3.7 in [17], we get

$$|R_2|\le Ch_k\,\|v-P_kv\|_{0,k}\,||v||_{1,k-1}.$$

$$|R_3|\le Ch_k\,\|v-P_kv\|_{0,k}\,||P_kv||_{1,k}\le Ch_k\,\|v-P_kv\|_{0,k}\,||v||_{1,k-1}.$$

Then we obtain

$$\|v-P_kv\|_{0,k}\le Ch_k||v||_{1,k-1}.$$

Let $m_k, 1\le k\le L$, be the smallest integer satisfying $m_k\ge\beta^{L-k}m_L$ for some fixed $\beta\ge 1$, where m_L is the number of iteration of the finest level L.

From Lemmas 4.1-4.3 and the framework given in [21], we have the following results.

Theorem 4.4. *Suppose that (H1), (H2) and (H3) hold. Then we have*

$$\|u_L-u_L^*\|_{1,L}\le C\sum_{k=1}^{L}\frac{h_k}{m_k}\,\|f\|_{0,\Omega}\,.$$

Theorem 4.5. *Suppose that (H1), (H2) and (H3) hold. Then the accuracy of the cascadic multigrid method is*

$$\|u_L-u_L^*\|_{1,L}\le\begin{cases} C\frac{1}{1-(\frac{2}{\beta})}\frac{h_L}{m_L}\,\|f\|_{0,\Omega}\,, & \beta>2. \\ CL\dfrac{h_L}{m_L}\,\|f\|_{0,\Omega}\,, & \beta=2. \end{cases}$$

Theorem 4.6. *Suppose that (H1), (H2) and (H3) hold. Then the computational cost of the cascadic multigrid is proportional to*

$$\sum_{l=0}^{L}m_kn_k\le\begin{cases} C\frac{1}{1-\beta/4}m_Ln_L, & \beta<4. \\ CLm_Ln_L, & \beta=4. \end{cases}$$

From Theorem 4.5 and Theorem 4.6, we know that

Theorem 4.7. *Suppose (H1), (H2) and (H3)hold, then the cascadic multigrid method is optimal for $2<\beta<4$.*

5. Numerical Results

In this section, we present the results of some numerical experiments which show that the CCG method is optimal with respect to the H^1-norm. The domain Ω is divided into two adjacent subdomains. Each subdomain is divided into a grid of smaller triangles. The meshes do not match on the interface. We assign the below side of the interface as the mortar side, the other as the nonmortar side, see Fig.1 below. Assume that the exact solution of the Poisson problem is $u(x,y) = x(1-x-y)(1-x+y)$ which satisfies the homogeneous Dirichlet boundary condition.

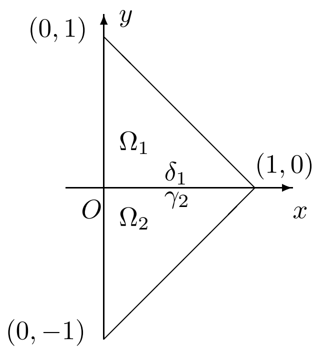

Fig.1 The domain Ω.

Table 5.1: $m_L = 1, \beta = 3, L = 5$

Nonmortar meshsize	Mortar meshsize	Nodes	H^1 error
0.03125	0.020833333	4920	0.037116383
0.015625	0.010416667	19824	0.028322420
0.0078125	0.005208333	79584	0.009157758
0.00390625	0.002604167	318912	0.004414613

Table 5.2: $m_L = 4, \beta = 3, L = 5$

Nonmortar meshsize	Mortar meshsize	Nodes	H^1 error
0.03125	0.020833333	4920	0.022102732
0.015625	0.010416667	19824	0.012487675
0.0078125	0.005208333	79584	0.004778748
0.00390625	0.002604167	318912	0.002805251

Table 5.3: $m_L = 8, \beta = 3, L = 5$

Nonmortar meshsize	Mortar meshsize	Nodes	H^1 error
0.03125	0.020833333	4920	0.021386326
0.015625	0.010416667	19824	0.009206107
0.0078125	0.005208333	79584	0.004450679
0.00390625	0.002604167	318912	0.002252639

Table 5.4: $m_L = 12, \beta = 3, L = 5$

Nonmortar meshsize	Mortar meshsize	Nodes	H^1 error
0.03125	0.020833333	4920	0.018765744
0.015625	0.010416667	19824	0.009112266
0.0078125	0.005208333	79584	0.004431573
0.00390625	0.002604167	318912	0.002238156

From the above results, we see that, for the mortar P_1 nonconforming element, the CCG method is optimal in H^1-norm for the Poisson problem.

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