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# STABILITY OF THEORETICAL SOLUTION AND NUMERICAL SOLUTION OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PIECEWISE DELAYS *1)

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**Abstract**

This paper is concerned with the stability of theoretical solution and numerical solution of a class of nonlinear differential equations with piecewise delays. At first, a sufficient condition for the stability of theoretical solution of these problems is given, then numerical stability and asymptotical stability are discussed for a class of multistep methods when applied to these problems.

*Mathematics subject classification:* 65L06, 65L20.
*Key words*: Stability, Delay differential equations, Linear multistep methods.

## 1. Introduction

In recent years, many authors discussed the stability of numerical methods for the solution of delay differential equations(DDEs) (see, e.g., [1, 2, 3, 4, 10] and their references) with constant delay. Recently, H.Tian [9] has given the exponential asymptotic stability of singularly perturbed delay differential equations with a bounded lag, and this type stability can be applied to general delay differential equations with a variable lag. However, the stability results of numerical methods for differential equations with variable delays are much less. In 1997, Zennaro[7] first investigated asymptotical stability of nonlinear delay differential equations(DDEs) with a variable delay, and gave the stability result of Runge-Kutta methods applied to this systems. In 1984, Cooke et al [6] described the existence, asymptotic behavior, periodic and oscillating solutions of the differential equations with piecewise constant delays. More results can also be found in [8] about differential equations with piecewise continuously variable arguments. In [5], the stability of $\theta$-methods has been studied by Zhang Changhai et al, which is based on the linear problem

$$\begin{cases} y'(t) = ay(t) + by([t]), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

where a, b denote real constants and $[\cdot]$ denotes the greatest integer function. In this paper, we further investigate the stability of the theoretical solution and numerical solution of a class of initial value problems in nonlinear differential equations with piecewise delays. In section 2, we fix our attention on the stability of the theoretical solution of the problems. In section 3, we analyze the stability and asymptotical stability of a class of linear multistep methods when applied to the problems. Our results are further verified by the numerical experiment in section 4.

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## 2. Test Problems

Let $\langle\cdot,\cdot\rangle$ be an inner product in $C^N$ and $\|\cdot\|$ the corresponding norm. Consider the following initial value problem in nonlinear differential equations with piecewise delay:

$$\begin{cases} y'(t) = f(t, y(t), y([t])), \quad t \geq 0, \\ \qquad y(0) = y_0, \end{cases} \tag{2.1}$$

where $[\cdot]$ is the largest-integer function, and $f : [0, +\infty) \times C^N \times C^N \longrightarrow C^N$ is a given continuous mapping. Assume that there exist continuous bounded functions $\alpha(t)$ and $\beta(t)$ on the interval $[0, +\infty)$, which satisfies the following conditions:

$$\alpha(t) \leq 0, \quad \alpha(t) + \beta(t) \leq 0 \quad \forall t \geq 0, \tag{2.2}$$

such that

$$\begin{cases} Re < u_1 - u_2, f(t, u_1, v) - f(t, u_2, v) > \leq \alpha(t)\|u_1 - u_2\|^2, \forall t \geq 0, u_1, u_2, v \in C^N & (2.3a) \\ \|f(t, u, v_1) - f(t, u, v_2)\| \leq \beta(t)\|v_1 - v_2\|, \ \forall t \geq 0, u, v_1, v_2 \in C^N, & (2.3b) \end{cases}$$

and that the problem (2.1) has a unique true solution $y(t)$ on the interval $[0, +\infty)$.

In order to discuss the contractivity and asymptotic stability of (2.1), we introduce the perturbed problem

$$\begin{cases} z'(t) = f(t, z(t), z([t])), \quad t \geq 0, \\ \qquad z(0) = z_0, \end{cases} \tag{2.4}$$

and assume that the problem (2.4) has a unique true solution $z(t)$.

**Theorem 2.1.** *If the mapping $f$ satisfies the condition* (2.3) *with (2.2), then we have*

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \qquad \forall \ \ t \in [0, +\infty) \tag{2.5}$$

*Proof.* Define $Y(t) := \|y(t) - z(t)\|^2 =< y(t) - z(t), y(t) - z(t) >$. Noting the conditions (2.2) and (2.3), and Cauchy-Schwartz inequality, we have

$$\begin{aligned} Y'(t) &= 2Re < y(t) - z(t), y'(t) - z'(t) > \\ &= 2Re < y(t) - z(t), f(t, y(t), y([t])) - f(t, z(t), y([t])) > \\ &+ 2Re < y(t) - z(t), f(t, z(t), y([t])) - f(t, z(t), z([t])) > \\ &\leq 2\alpha(t)Y(t) + 2\beta(t)\|y(t) - z(t)\|\|y([t]) - z([t])\| \\ &\leq 2\alpha(t)Y(t) + \beta(t)(Y(t) + Y([t])) \\ &= \alpha(t)Y(t) + (\alpha(t) + \beta(t))Y(t) + \beta(t)Y([t]) \\ &\leq \alpha(t)Y(t) + \beta(t)Y([t]). \end{aligned}$$

Let $A(x) := \int_0^x \alpha(t)dt$, for every $t_0 \geq 0, t \geq t_0$, we have

$$\int_{t_0}^{t} (e^{-A(x)}Y(x))'dx \leq \int_{t_0}^{t} \beta(x)e^{-A(x)}Y([x])dx. \tag{2.6}$$

Hence

$$Y(t) \leq Y(t_0)e^{A(t)-A(t_0)} - e^{A(t)} \int_{t_0}^{t} \alpha(x)e^{-A(x)}Y([x])dx.$$

For the case $m \leq t \leq m + 1$ with integer $m \geq 0$. Let $t_0 = m$, we have

$$\begin{aligned} Y(t) &\leq Y(m)[e^{A(t)-A(m)} - e^{A(t)} \int_{m}^{t} \alpha(x)e^{-A(x)}dx] \\ &\leq Y(m)[e^{A(t)-A(m)} + 1 - e^{A(t)-A(m)}] \leq Y(m). \end{aligned} \tag{2.7}$$

By iterating, the (2.5) is true.

Modifying the conditions of theorem 2.1 further, we can obtain the following conclusion.

**Theorem 2.2.** *Assume the mapping $f$ satisfies the condition* (2.3) *with*

$$\alpha(t) < 0,\ 0 \le \gamma(t) < 1,\ -\gamma(t)\alpha(t) = \beta(t),\ \forall t \ge 0, \tag{2.8}$$

*where continuous functions $\alpha(t), \gamma(t)$ satisfy*

$$\bar{\alpha} = \sup_{t\ge 0} \alpha(t) < 0, \quad \bar{\gamma} = \sup_{t\ge 0} \gamma(t) < 1. \tag{2.9}$$

*Then we have*

$$\lim_{t\to+\infty} \|y(t) - z(t)\| = 0. \tag{2.10}$$

*Proof.* According to the proof of Theorem 2.1, and noting (2.6) and (2.8), we have

$$Y(t) \le Y(t_0)e^{A(t)-A(t_0)} - \bar{\gamma}e^{A(t)} \int_{t_0}^{t} \alpha(x)e^{-A(x)}Y([x])dx.$$

For the case $m \le t \le m+1$ with integer $m \ge 0$, let $t_0 = m$, we have

$$Y(t) \le Y(m)[\bar{\gamma} + (1-\bar{\gamma})e^{A(t)-A(m)}] \le Y(m)[\bar{\gamma} + (1-\bar{\gamma})e^{\bar{\alpha}(t-m)}]. \tag{2.11}$$

Put $q = \bar{\gamma} + (1-\bar{\gamma})e^{\bar{\alpha}}$. Because $\bar{\alpha} < 0, \bar{\gamma} < 1$, so $0 < q < 1$. Then we have

$$Y(m) \le qY(m-1) \le q^m Y(0).$$

Therefore we can obtain (2.10) easily.

## 3. Stability Analysis of Linear Multistep Methods

In this paper, we consider a specific class of linear multistep methods(LMM's) for solving ODE's with the generating polynomials $\rho(\xi) = \sum_{j=0}^{k} a_j\xi^j, \sigma(\xi) = b\xi^k$, which are assumed to have real coefficient, no common divisor. We also assume(cf.[3])

$$\rho(1) = 0, \rho'(1) = \sigma(1), a_k = 1, a_j \le 0, j = 0, 1, \cdots, k-1, \tag{3.1}$$

consequently $b > 0$.

Applying the k-step LMM $(\rho, \sigma)$ to (2.1) and (2.4) respectively, we obtain

$$\sum_{j=0}^{k} a_j y_{n+j} = hbf(t_{n+k}, y_{n+k}, \bar{y}_{n+k}) \tag{3.2}$$

and

$$\sum_{j=0}^{k} a_j z_{n+j} = hbf(t_{n+k}, z_{n+k}, \bar{z}_{n+k}), \tag{3.3}$$

where $h > 0$ is the fixed stepsize, $y_n$ and $z_n$ are approximations to the exact solutions $y(t)$ and $z(t)$, and $\bar{y}_n, \bar{z}_n$ to $y([t_n]), z([t_n])$ respectively, with $t_n = nh$.

**Definition 3.1.** *A LMM $(\rho, \sigma)$ for solving (2.1) is said to be stable, if the numerical approximations $y_n$ and $z_n$ to the solutions of any given problems (2.1) and (2.4) respectively, satisfy the condition*

$$\|y_n - z_n\| \le \max_{0\le j\le k-1} \|y_j - z_j\|, \forall n \ge k. \tag{3.3}$$

**Definition 3.2.** *A LMM $(\rho, \sigma)$ for solving (2.1) is said to be asymptotically stable, if the numerical approximations $y_n$ and $z_n$ to the solutions of any given problems (2.1) and (2.4) respectively, satisfy the condition*

$$\lim_{n\to+\infty} \|y_n - z_n\| = 0. \tag{3.4}$$

*So as to reduce the error, we let* $h = \frac{1}{m}$*, where* $m$ *is a positive integer. Then we have*

$$\bar{y}_n = y_{m[\frac{n}{m}]}, \bar{z}_n = z_{m[\frac{n}{m}]}.$$

**Theorem 3.1.** *Assume the mapping* $f$ *satisfies the condition* (2.3) *with (2.2), and the step-size satisfies* $h = \frac{1}{m}$*, with a positive integer* $m$*, the coefficients of the method* $(\rho, \sigma)$ *satisfy (3.1). Then the method* $(\rho, \sigma)$ *for solving (2.1) is stable.*

*Proof.* Let $\omega_n = y_n - z_n, \Delta f_n = f(t_n, y_n, \bar{y}_n) - f(t_n, z_n, \bar{z}_n), \bar{\omega}_n = \bar{y}_n - \bar{z}_n, n = 0, 1, \cdots$. It follows from (3.2) – (3.3) that

$$\sum_{j=0}^{k} a_j \omega_{n+j} = hb\Delta f_{n+k}. \tag{3.5}$$

We take the inner products in (3.5) with $\omega_{n+k}$ and obtain

$$\begin{aligned} \|\omega_{n+k}\|^2 &= -\sum_{j=0}^{k-1} a_j Re < \omega_{n+k}, \omega_{n+j} > + hbRe < \omega_{n+k}, \Delta f_{n+k} > \\ &\leq -\sum_{j=0}^{k-1} a_j \|\omega_{n+j}\| \|\omega_{n+k}\| + hbRe < \omega_{n+k}, \Delta f_{n+k} > . \end{aligned} \tag{3.6}$$

It follows from the conditions (2.3) and (2.4) and Cauchy inequality that

$$\begin{aligned} Re < \omega_{n+k}, \Delta f_{n+k} > &= Re < y_{n+k} - z_{n+k}, f(t_{n+k}, y_{n+k}, \bar{y}_{n+k}) - f(t_{n+k}, z_{n+k}, \bar{y}_{n+k}) > \\ &\quad + Re < y_{n+k} - z_{n+k}, f(t_{n+k}, z_{n+k}, \bar{y}_{n+k}) - f(t_{n+k}, z_{n+k}, \bar{z}_{n+k}) > \\ &\leq \alpha(t_{n+k}) \|\omega_{n+k}\|^2 + \beta(t_{n+k}) \|\omega_{n+k}\| \|\bar{\omega}_{n+k}\|. \end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$(1 - hb\alpha(t_{n+k}))\|\omega_{n+k}\| \leq -\sum_{j=0}^{k-1} a_j \|\omega_{n+j}\| + hb\beta(t_{n+k})\|\bar{\omega}_{n+k}\|, \tag{3.8}$$

where $\bar{\omega}_n = y_{m[\frac{n}{m}]} - z_{m[\frac{n}{m}]}$.

For the case m=1, (3.8) leads to

$$(1 - hb\alpha(t_{n+k}) - hb\beta(t_{n+k}))\|\omega_{n+k}\| \leq -\sum_{j=0}^{k-1} a_j \|\omega_{n+j}\|. \tag{3.9}$$

By the condition (2.2) and (3.1), and noting (3.9), we have

$$\|\omega_{n+k}\| \leq \frac{1}{1 - hb(\alpha(t_{n+k}) + \beta(t_{n+k}))} \max_{0 \leq j \leq k-1} \|\omega_{n+j}\| \leq \max_{0 \leq j \leq k-1} \|\omega_{n+j}\|.$$

Therefor,

$$\max_{0 \leq j \leq k-1} \|\omega_{pk+j}\| \leq \max_{0 \leq j \leq k-1} \|\omega_{(p-1)k+j}\| \qquad p = 1, 2, \cdots . \tag{3.10}$$

For the case $m > 1$, we consider two cases:

i) $m > k$. For any integer $p \geq 0$ we have

$$\begin{aligned} \|\omega_{mp+k}\| &\leq \frac{\sum_{j=0}^{k-1} a_j \|\omega_{mp+j}\| + hb\beta(t_{mp+k})\|\omega_{mp}\|}{1 - hb\alpha(t_{mp+k})} \\ &\leq \frac{1 + hb\beta(t_{mp+k})}{1 - hb\alpha(t_{mp+k})} \max_{0 \leq j \leq k-1} \|\omega_{mp+j}\| \leq \max_{0 \leq j \leq k-1} \|\omega_{mp+j}\|, \end{aligned}$$

we also can obtain that

$$\|\omega_{mp+i}\| \leq \max_{0 \leq j \leq k-1} \|\omega_{mp+j}\|, \qquad i = k, k+1, \cdots, m-1.$$

Moreover

$$\|\omega_{mp+m}\| \leq \frac{-\sum_{j=0}^{k-1} a_j \|\omega_{mp+m-j}\|}{1 - hb(\alpha(t_{(p+1)m}) + \beta(t_{(p+1)m}))} \leq \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|.$$

Clearly,

$$\max_{0\leq j\leq m} \|\omega_{mp+j}\| \leq \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|.$$

On the other hand

$$\begin{aligned}\|\omega_{m(p+1)+i}\| &\leq \frac{-\sum_{j=0}^{k-1} a_j \|\omega_{m(p+1)-k+i+j}\| + hb\beta(t_{m(p+1)+i})\|\omega_{m(p+1)}\|}{1 - hb\alpha(t_{m(p+1)+i})} \\ &\leq \max_{0\leq j\leq k-1} \|\omega_{m(p+1)-j}\| \leq \max_{0\leq j\leq m} \|\omega_{mp+j}\| \qquad i = 0, 1, 2, \cdots, k-1.\end{aligned}$$

Consequently, we have

$$\max_{0\leq j\leq m} \|\omega_{m(p+1)+j}\| \leq \max_{0\leq j\leq m} \|\omega_{mp+j}\| \leq \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|. \tag{3.11}$$

ii) $1 < m \leq k$. For any integer $n \geq 0$, if $[\frac{n+k}{m}] = \frac{n+k}{m}$, then from (3.8) we obtain

$$\|\omega_{n+k}\| \leq \frac{-\sum_{j=0}^{k-1} a_j \|\omega_{n+j}\|}{1 - hb(\alpha(t_{n+k}) + \beta(t_{n+k}))} \leq \max_{0\leq j\leq k-1} \|\omega_{n+j}\|,$$

while $[\frac{n+k}{m}] \neq \frac{n+k}{m}$, then we obtain with (3.8)

$$\|\omega_{n+k}\| \leq \frac{1 + hb\beta(t_{n+k})}{1 - hb\alpha(t_{n+k})} \max_{0\leq j\leq k-1} \|\omega_{n+j}\| \leq \max_{0\leq j\leq k-1} \|\omega_{n+j}\|.$$

Consequently, we have

$$\max_{0\leq j\leq k-1} \|\omega_{k(p+1)+j}\| \leq \max_{0\leq j\leq k-1} \|\omega_{kp+j}\| \quad p = 0, 1, 2, \cdots. \tag{3.12}$$

Together with (3.10), (3.11) and (3.12), we complete the proof of Theorem 3.1.

**Theorem 3.2.** *Assume the mapping $f$ satisfies the conditions of Theorem 2.2, and the step-size satisfies $h = \frac{1}{m}$, with a positive integer $m$, the coefficients of the method $(\rho, \sigma)$ satisfy (3.1). Then the method $(\rho, \sigma)$ for solving (2.1) is asymptotically stable.*

*Proof.* By the proof procedure of Theorem 3.1, from (3.8) and (2.8) we have

$$(1 - hb\alpha(t_{n+k}))\|\omega_{n+k}\| \leq -\sum_{j=0}^{k-1} a_j \|\omega_{n+j}\| - hb\gamma(t_{n+k})\alpha(t_{n+k})\|\bar{\omega}_{n+k}\|. \tag{3.13}$$

When $m = 1$, (3.13) leads to

$$\begin{aligned}\|\omega_{n+k}\| &\leq \frac{1}{1 - hb\alpha(t_{n+k})(1 - \gamma(t_{n+k}))} \max_{0\leq j\leq k-1} \|\omega_{n+j}\| \\ &\leq \Gamma \max_{0\leq j\leq k-1} \|\omega_{n+j}\| \qquad n = 0, 1, \cdots,\end{aligned}$$

where $\Gamma = \frac{1}{1-hb\bar{\alpha}(1-\bar{\gamma})}$. From (2.9), it is obvious that $0 < \Gamma < 1$. Hence

$$\begin{aligned}\max_{0\leq j\leq k-1} \|\omega_{k(p+1)+j}\| &\leq \Gamma \max_{0\leq j\leq k-1} \|\omega_{kp+j}\| \\ &\leq \Gamma^p \max_{0\leq j\leq k-1} \|\omega_j\| \qquad p = 0, 1, 2, \cdots. \end{aligned} \tag{3.14}$$

When $m > 1$, we consider two cases

i) $m > k$. From (3.13), for any integer $p \geq 0$, we have

$$\|\omega_{mp+k}\| \leq \frac{-\sum_{j=0}^{k-1} a_j \|\omega_{mp+j}\| - hb\gamma(t_{mp+k})\alpha(t_{mp+k})\|\omega_{mp}\|}{1 - hb\alpha(t_{mp+k})} \leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|,$$

where $\Gamma_1 = \frac{1-hb\bar{\gamma}\bar{\alpha}}{1-hb\bar{\alpha}}$, and $0 < \Gamma_1 < 1$ obviously. Similarly

$$\|\omega_{mp+i}\| \leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{mp+j}\| \quad fori = k, k+1, \cdots, m-1.$$

Moreover

$$\|\omega_{mp+m}\| \leq \frac{-\sum_{j=0}^{k-1} a_j \|\omega_{mp+m-j}\|}{1 - hb\alpha(t_{(p+1)m})(1 - \gamma(t_{(p+1)m}))} \leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|.$$

Then

$$\max_{k\leq i\leq m} \|\omega_{mp+i}\| \leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|$$

and

$$\max_{0\leq j\leq m} \|\omega_{mp+j}\| = \max_{0\leq j\leq k-1} \|\omega_{mp+j}\|.$$

On the other hand

$$\begin{aligned}\|\omega_{mp+1}\| &\leq \frac{-\sum_{j=1}^{k} a_j \|\omega_{mp-j+1}\| - hb\gamma(t_{mp+1})\alpha(t_{mp+1})\|\omega_{mp}\|}{1 - hb\alpha(t_{mp+1})} \\ &\leq \frac{\max_{1\leq j\leq k} \|\omega_{mp+1-j}\| - hb\gamma(t_{mp+1})\alpha(t_{mp+1})\Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{m(p-1)+j}\|}{1 - hb\alpha(t_{mp+1})} \\ &\leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{m(p-1)+j}\|.\end{aligned}$$

Likewise,

$$\max_{0\leq j\leq k-1} \|\omega_{mp+j}\| \leq \Gamma_1 \max_{0\leq j\leq k-1} \|\omega_{m(p-1)+j}\|.$$

Hence,

$$\max_{0\leq j\leq m} \|\omega_{mp+j}\| \leq \Gamma_1 \max_{0\leq j\leq m} \|\omega_{m(p-1)+j}\| \leq \Gamma_1^p \max_{0\leq j\leq k-1} \|\omega_j\|. \tag{3.15}$$

ii) $0 < m < k$, by similar discussion as above, we obtain that

$$\max_{0\leq j\leq k-1} \|\omega_{k(p+1)+j}\| \leq \Gamma_1^p \max_{0\leq j\leq k-1} \|\omega_{kp+j}\| \quad for \quad p = 0, 1, 2, \cdots. \tag{3.16}$$

From (3.14), (3.15) and (3.16), we complete the proof of Theorem 3.2.

## 4. Numerical Experiment

Consider the parabolic problem with piecewise delay term

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \exp(-t)u(x,[t]) + F(x,t), & 0 < x < 1, t > 0 \\ u(x,0) = x - x^2, & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & t > 0, \end{cases} \tag{4.1}$$

where $F(x,t) = (2 - x(1-x)(exp([t]) + 1))exp(-t)$. This problem has a unique true solution $u(x,t) = (x - x^2)\exp(-t)$. Therefore, without regard to truncation error we can replace the

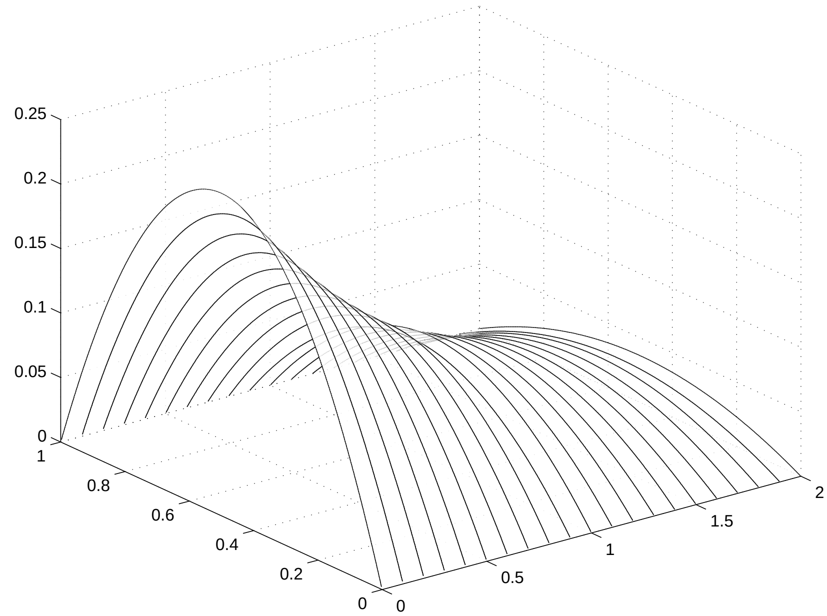


Figure 1: Numerical solution of problem (4.1) by method (4.4) with $N = 500, a = -0.5, h = 0.1, 0 < t \leq 2$

second spatial derivatives by finite differences on a grid of points $x_i = i/N, i = 1(1)N-1$, with $N$ an arbitrarily given natural number. Write

$$u_i(t) = u(x_i, t), F_i(t) = (2 - i\triangle x(1 - i\triangle x)(exp([t]) + 1))exp(-t), \triangle x = 1/N,$$

we thus obtain

$$\begin{cases} \frac{du_i(t)}{dt} = \frac{1}{\triangle x^2}(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + \exp(-t)u_i([t]) + F_i(t), \quad t > 0 \\ u_0(t) \equiv u_N(t) \equiv 0, u_i(0) = i\triangle x(1 - i\triangle x), \quad i = 1, 2, \cdots, N-1. \end{cases} \tag{4.2}$$

Refer to Section 2, the problem (4.2) can be regarded as a special case of the initial value problem (2.1) in space $\mathbf{R}^{N-1}$, we can easily verify conditions (2.2) and (2.3) are satisfied with $\alpha(t) = -4N^2 \sin^2 \frac{\pi}{2N} \simeq -\pi^2$ for large $N$, $\beta(t) = exp(-t)$ and

$$f(t, u, v) = [f_1(t, u, v), f_2(t, u, v), \cdots, f_{N-1}(t, u, v)]^T$$

with

$$f_i(t, u, v) = \frac{1}{\triangle x^2}(u_{i-1} - 2u_i + u_{i+1}) + \exp(-t)v_i + F_i(t)$$

$$\forall t > 0, u, v \in \mathbf{R}^{N-1}. \tag{4.3}$$

Let $\gamma(t) = \frac{exp(-t)}{4N^2 \sin^2 \frac{\pi}{2N}}$. It is easy to see that (2.8) and (2.9) are satisfied.

We apply the two-step one-order method

$$ay_n - (1 + a)y_{n+1} + y_{n+2} = h(a + 2)f_{n+2} \tag{4.4}$$

to problem (4.2) and perturbed problem

$$\begin{cases} \frac{dv_i(t)}{dt} = \frac{1}{\triangle x^2}(v_{i-1}(t) - 2v_i(t) + v_{i+1}(t)) + \exp(-t)v_i([t]) + F_i(t), \ t > 0 \\ v_0(t) \equiv v_N(t) \equiv 0, v_i(0) = i\triangle x(1 - i\triangle x) + 0.5, \quad i = 1, 2, \cdots, N-1, \end{cases} \tag{4.5}$$

where $-1 \leq a \leq 0$ and the step-size satisfies $h = \frac{1}{m}$, with a positive integer $m$, and obtain the numerical approximation solutions $y_n$ and $z_n$ respectively, where $y_n \approx [u_1(t_n), u_2(t_n), \cdots, u_{N-1}(t_n)]^T$

Table 1: Maximum global errors $E(T)$ of method (3.2)-(3.21) applied to Problem (4.2).

| | | T=2 | T=5 | T=10 | T=15 |
|---|---|---|---|---|---|
| $a=-0.5$ | h=1 | $3.22850\times10^{-1}$ | $6.58560\times10^{-4}$ | $4.31390\times10^{-7}$ | $9.13491\times10^{-11}$ |
| | h=0.5 | $4.50465\times10^{-2}$ | $2.28225\times10^{-5}$ | $2.33157\times10^{-10}$ | $3.11673\times10^{-15}$ |
| | h=0.1 | $2.11418\times10^{-2}$ | $6.17889\times10^{-4}$ | $8.61215\times10^{-7}$ | $3.30468\times10^{-10}$ |
| $a=-1$ | h=1 | $9.38517\times10^{-1}$ | $1.71286\times10^{-3}$ | $6.73092\times10^{-5}$ | $1.12901\times10^{-11}$ |
| | h=0.5 | $3.05128\times10^{-1}$ | $1.53902\times10^{-3}$ | $2.30905\times10^{-7}$ | $3.25682\times10^{-11}$ |
| | h=0.1 | $5.60568\times10^{-2}$ | $1.73493\times10^{-3}$ | $2.53492\times10^{-6}$ | $3.97812\times10^{-9}$ |

and $z_n \approx [v_1(t_n), v_2(t_n), \cdots, v_{N-1}(t_n)]^T$. According to Theorem 3.2, the method (4.4) for solving (4.2) is asymptotically stable. Now we let $N=500$, $a=-0.5$ and $a=-1$ respectively, for $h=1, 0.5, 0.1$, let $E(T)$ denote the global error of $y_n$ and $z_n$ at $t=T$, i.e. $E(T)=\|y_n - z_n\|$ $(nh=T)$. We list the values of $E(T)$ in Table 1 for $T=2,5,10,15$ respectively. Moreover in Figure 1, we describe the numerical solutions of problem (4.1) by method (4.4) when $N=500, a=-0.5, h=0.1, 0<t\le 2$. It is clear that the results given by Table 1 and Figure 1 confirm our results in Section 3.

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