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SMOOTHING BY CONVEX QUADRATIC PROGRAMMING *1)

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Abstract

In this paper, we study the relaxed smoothing problems with general closed convex constraints. It is pointed out that such problems can be converted to a convex quadratic minimization problem for which there are good programs in software libraries.

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1. Introduction

Let

$$x_1 < x_2 < \cdots < x_n < x_{n+1}$$

and

$$y_1, y_2, \ldots, y_n, y_{n+1} = y_1.$$

The mathematical form of the problems considered in this paper is to find a twice continuous differentiable periodic function $g(x)$ with $g(x_{n+i}) = g(x_i)$, such that $g(x)$ is the optimal solution of the following problem:

$$\min \quad \int_{x_1}^{x_{n+1}} |g''(x)|^2 dx \tag{1.1}$$

$$\text{s. t} \quad u \in \Omega \tag{1.2}$$

where

$$u = (u_1, u_2, \ldots, u_n)^T, \qquad u_i = \frac{g(x_i) - y_i}{\delta y_i}, \tag{1.3}$$

$\delta y_i, i = 1, \ldots, n$ are given positive numbers and Ω is a closed convex set in R^n. We refer the problem to *relaxed smoothing problem* whenever $\Omega \neq \{0\}$. For $\Omega = \{v \in R^n \mid \|v\|_2 \leq r\}$, the problem was investigated by Reinsch [2] and it was converted to a smooth convex unconstrained optimization. Problem (1.1) with general closed convex constraints have more applications, for example, $\Omega = \{v \in R^n \mid \|v\|_\infty \leq r\}$ is also interesting in real problems.

It is well known that the solution of the non-relaxed problem of (1.1) is a spline function. We will prove that the solution of the relaxed smoothing problem with general closed convex constraints is the spline function $g(x) \in C^2$ of the following form:

$$g(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3, \quad x \in [x_i, x_{i+1}). \tag{1.4}$$

Then the task of solving problem (1.1)-(1.2) is to find $a_i, b_i, c_i, d_i, i = 1, \ldots, n$.

In next section, we summarize some notations and the basic relations of the spline function. Section 3 illustrates that the coefficients of the spline function can be obtained by solving a

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convex quadratic programming. Finally, in Section 4, we prove that the obtained spline function is the solution of the original problem and give our conclusions.

2. Notations and the Basic Relations

For analysis convenience, we need the following notations. Let $h_i := x_{i+1} - x_i$,

$$D = \begin{pmatrix} \delta y_1 & & & \\ & \delta y_2 & & \\ & & \ddots & \\ & & & \delta y_n \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_n \end{pmatrix}$$

be diagonal matrices in $R^{n\times n}$. Denote

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}.$$

Note that a, b, c, d are unknown vectors. Since $g(x_i) = a_i$, using these notations, the relation (1.3) can be written as

$$u = D^{-1}(a - y). \tag{2.1}$$

In addition, we needs the following permutation matrix

$$P := \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}.$$

For this matrix P we have $P^T P = I$,

$$Pa = \begin{pmatrix} a_2 \\ \vdots \\ a_n \\ a_1 \end{pmatrix} \qquad \text{and} \qquad P^T a = \begin{pmatrix} a_n \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}.$$

Now, let us list the basic properties of the periodic spline function $g(x) \in C^2$. First, since $g(x_{i+1}^-) = g(x_{i+1}^+)$, we have $a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = a_{i+1}$ and thus

$$a + Hb + H^2 c + H^3 d = Pa. \tag{2.2}$$

In addition, because $g'(x_{i+1}^-) = g'(x_{i+1}^+)$, we have $b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}$ and

$$b + 2Hc + 3H^2 d = Pb. \tag{2.3}$$

Finally, since $g''(x_{i+1}^-) = g''(x_{i+1}^+)$, we have $c_i + 3d_i h_i = c_{i+1}$ and thus

$$c + 3Hd = Pc. \tag{2.4}$$

3. The Convex Quadratic Programming

If the solution of Problem (1.1)-(1.2) is a spline function of form (1.4), the objective function can be written as

$$\begin{aligned}\int_{x_1}^{x_{n+1}} |g''(x)|^2 dx &= \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} |2c_i + 6d_i(x - x_i)|^2 dx \\ &= \sum_{i=1}^{n} (4h_i c_i^2 + 12c_i d_i h_i^2 + 12d_i^2 h_i^3) \\ &= 4c^T Hc + 6c^T H^2 d + 6d^T H^2 c + 12d^T H^3 d. \qquad (3.1)\end{aligned}$$

Substituting $Hd = \frac{1}{3}(P - I)c$ (see (2.4)) in (3.1) and by a manipulation we get

$$\int_{x_1}^{x_{n+1}} |g''(x)|^2 dx = \frac{2}{3} c^T Mc, \qquad (3.2)$$

where

$$M = 2H + 2P^T HP + HP + P^T H. \qquad (3.3)$$

Note that

$$HP = \begin{pmatrix} 0 & h_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & h_{n-1} \\ h_n & & & 0 \end{pmatrix}, \qquad P^T HP = \begin{pmatrix} h_n & & & \\ & h_1 & & \\ & & \ddots & \\ & & & h_{n-1} \end{pmatrix}$$

and thus

$$M = \begin{pmatrix} 2(h_1 + h_n) & h_1 & & & h_n \\ h_1 & 2(h_2 + h_1) & h_2 & & \\ & h_2 & \ddots & \ddots & \\ & & \ddots & \ddots & h_{n-1} \\ h_n & & & h_{n-1} & 2(h_n + h_{n-1}) \end{pmatrix}$$

is a positive definite matrix (since it is diagonal dominate).

It follows from (2.2) that

$$H^{-1}(P - I)a = b + Hc + H^2 d \qquad (3.4)$$

and

$$-P^T H^{-1}(P - I)a = -P^T b - P^T Hc - P^T H^2 d. \qquad (3.5)$$

From (2.4) we have

$$H^2 d = \frac{1}{3} H(P - I)c. \qquad (3.6)$$

Adding (3.4) and (3.5) and using (3.6), we get

$$Qa = b - P^T b + Hc - P^T Hc + \frac{1}{3}(I - P^T)H(P - I)c, \qquad (3.7)$$

where

$$Q = (I - P^T)H^{-1}(P - I). \qquad (3.8)$$

It follows from (2.3) that

$$\begin{aligned} b - P^T b &= 2P^T Hc + 3P^T H^2 d \\ &\overset{(3.6)}{=} 2P^T Hc + P^T H(P-I)c \\ &= P^T Hc + P^T HPc. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7), we obtain (see (3.3))

$$Qa = \frac{1}{3}(2H + 2P^T HP + HP + P^T H)c = \frac{1}{3}Mc. \tag{3.10}$$

According to (3.10), the objective function (3.2) can be rewritten as

$$6a^T Q^T M^{-1} Qa. \tag{3.11}$$

By using $u = D^{-1}(a-y)$, we convert the original problem to the following convex quadratic minimization problem:

$$\begin{array}{ll} \min & \frac{1}{2}u^T D^T Q^T M^{-1} QDu + y^T Q^T M^{-1} QDu \\ \text{s. t} & u \in \Omega. \end{array} \tag{3.12}$$

After getting the solution of (3.12), we can get the solution of the vectors a, b, c and d by

$$\begin{aligned} a &\overset{(2.1)}{=} Du + y, \\ c &\overset{(3.10)}{=} 3M^{-1}Qa, \\ d &\overset{(2.4)}{=} \frac{1}{3}H^{-1}(Pc - c), \\ b &\overset{(2.2)}{=} H^{-1}(Pa - a) - H(c + Hd). \end{aligned}$$

4. Optimality

The purpose of this section is to prove that the spline function (1.4) with a, b, c, d obtained from the last section is the solution of Problem (1.1)-(1.2). First, we prove the following lemma.
Lemma 1. *Let u be a solution of (3.12). Then we have*

$$(u' - u)^T DQc \geq 0, \quad \forall u' \in \Omega. \tag{4.1}$$

Proof. Denote the objective function of (3.12) by $\theta(u)$. Since Ω is closed convex and u is a solution of (3.12), it follows that $u \in \Omega$ and

$$(u' - u)^T \nabla\theta(u) \geq 0, \quad \forall u' \in \Omega.$$

Note that

$$\nabla\theta(u) = D^T Q^T M^{-1} QDu + D^T Q^T M^{-1} Qy.$$

Since D and Q are symmetric, it follows that

$$\begin{aligned} \nabla\theta(u) &\overset{(2.1)}{=} DQM^{-1}Q(a-y) + DQM^{-1}Qy \\ &\overset{(3.10)}{=} \frac{1}{3}DQc. \end{aligned}$$

The assertion of this lemma is proved.

Now, we are in the stage to prove the optimality theorem.

Theorem 1. *Let $f(x)$ be a twice continuous differentiable periodic function, $f(x_i) = \tilde{a}_i$, $f(x_i) = f(x_{n+i})$ and $\tilde{u} = D^{-1}(\tilde{a} - y) \in \Omega$. Then we have*

$$\int_{x_1}^{x_{n+1}} |g''(x)|^2 dx \leq \int_{x_1}^{x_{n+1}} |f''(x)|^2 dx.$$

Proof. Since

$$\begin{aligned}\int_{x_1}^{x_{n+1}} |f''(x)|^2 dx &= \int_{x_1}^{x_{n+1}} |g''(x)|^2 dx + \int_{x_1}^{x_{n+1}} |f''(x) - g''(x)|^2 dx \\ &\quad +2\int_{x_1}^{x_{n+1}} [g''(x)(f''(x) - g''(x))]dx,\end{aligned}$$

we only need to show that

$$\int_{x_1}^{x_{n+1}} g''(x)(f''(x) - g''(x))dx \geq 0.$$

Using $f, g \in C^2$ and by a manipulation (integration by parts), we get

$$\begin{aligned}&\int_{x_1}^{x_{n+1}} g''(x)(f''(x) - g''(x))dx \\ &= \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} g''(x)(f''(x) - g''(x))dx \\ &= \sum_{i=1}^{n} (f'(x) - g'(x))g''(x)\Big|_{x_i}^{x_{i+1}} - \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (f'(x) - g'(x))g'''(x)dx \\ &= -\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (f'(x) - g'(x))g'''(x)dx. \qquad (4.2)\end{aligned}$$

The last equation of (4.2) is followed from the periodicity of g. Integrate the function again and use $g^{(4)} = 0$, we obtain

$$\begin{aligned}&\int_{x_1}^{x_{n+1}} g''(x)(f''(x) - g''(x))dx \\ &= -\sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (f'(x) - g'(x))g'''(x)dx \\ &= -\sum_{i=1}^{n} (f(x) - g(x))g'''(x)\Big|_{x_i}^{x_{i+1}} + \sum_{i=1}^{n} \int_{x_i}^{x_{i+1}} (f(x) - g(x))g^{(4)}(x)dx \\ &= -\sum_{i=1}^{n} (f(x) - g(x))g'''(x)\Big|_{x_i}^{x_{i+1}} \qquad \text{(since } g^{(4)} = 0) \\ &= 6\Big((f(x_1) - a_1)(d_1 - d_n) + \sum_{i=2}^{n} (f(x_i) - a_i)(d_i - d_{i-1})\Big) \\ &= 6(\tilde{a} - a)^T (d - P^T d). \qquad (4.3)\end{aligned}$$

Using (3.6), we obtain

$$d - P^T d = -\frac{1}{3}(P^T - I)H^{-1}(P - I)c \stackrel{(3.8)}{=} \frac{1}{3}Qc.$$

Substituting it into (4.2) and using the assertion of Lemma 1, we get

$$\begin{aligned}\int_{x_1}^{x_{n+1}} g''(x)(f''(x) - g''(x))dx &= 2(\tilde{a} - a)^T Qc \\ &= 2(D^{-1}(\tilde{a} - y) - D^{-1}(a - y))^T DQc \\ &= 2(\tilde{u} - u)^T DQc \\ &\geq 0.\end{aligned}$$

The proof is complete.

Conclusions remark. This paper pointed out that the relaxed smoothing problem with general closed convex constraints is equivalent to a convex quadratic programming (CQP) (3.12). For such CQP, if Ω is a box or a polytope, many excellent numerical methods have been designed in the literature [1, 3]. Hence, it is meaningful to derive Problem (1.1)-(1.2) to a convex quadratic programming of form (3.12) for which there are good programs in software libraries.

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