Journal of Computational Mathematics, Vol.23, No.2, 2005, 177–184.

ON CONVERGENCE OF MULTIGRID METHOD FOR NONNEGATIVE DEFINITE SYSTEMS *1)

Qian-shun Chang
(*Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, The Chinese Academy of Sciences, Beijing 100080, China*)

Wei-wei Sun
(*Department of Mathematics, City University of Hong Kong, Hong Kong*)

Abstract

In this paper, we consider multigrid methods for solving symmetric nonnegative definite matrix equations. We present some interesting features of the multigrid method and prove that the method is convergent in L_2 space and the convergent solution is unique for such nonnegative definite system and given initial guess.

Mathematics subject classification: 65N55, 65F10, 65N30.
Key words: Multigrid, Singular Problem, Convergence.

1. Introduction

Multigrid (MG) methods have been successfully applied to many scientific computing problems. The main advantage of this method is its asymptotically optimal convergence, *i.e.*, the computational work required to achieve a fixed accuracy is proportional to the number of discrete unknowns. The convergence analysis of multigrid methods has been studied extensively by many papers (see [3, 6, 9, 11, 12, 16, 17, 18]). Recent effort for indefinite systems has been made in [5, 8, 21].

In this paper, we consider convergence of the multigrid method for linear systems with symmetric nonnegative definite matrices. Classical iterative algorithms, such as Jacobi iteration and Gauss-Seidel iteration, for solving such nonnegative definite systems have been well studied in many literatures (*e.g.*, see [1]). Some semiconvergent iterative methods were discussed in [7,13]. An incomplete factorization and an extrapolation technique were presented in [15] and [19], respectively. The convergence analysis of these classical iterations for the semidefinite problems can be obtained due to simple structures of algorithms. It has been proved theoretically and numerically that multigrid methods are usually more efficient than those classical iterations. Some numerical investigation of multigrid methods has been presented for solving certain singular systems arising from eigenproblems, second-order elliptic PDEs with Neumann boundary conditions, queuing networks, and image reconstruction ([3,5,10]). Theoretical analysis for the indefinite systems is less explored. The major difficulties lie on the fact that there exist infinite many solutions for a consistent singular system and the structure of multigrid methods is more complicated than those of classical iterations. The concept of classical convergence should be modified. In fact, for a singular system, one only expects to find an approximation to one of solutions. In this case, the main point for an iterative algorithm is as follows: when the iteration stops, the difference between the iterative solution and some exact solution is less than a given tolerance. In this paper, we shall prove that multigrid methods for symmetric nonnegative

* Received 14, 2003, final revised March 23, 2004.
1) The work of the second author was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CityU 1061/00P).

definite systems are convergent in a classical sense (L_2-norm). Some important features will be discussed. We present a convergence rate in a quotient space (in an energy norm) and an asymptotic convergence rate in the classical sense (L_2-norm). Thus, the multigrid method, similar to some classical iterations, is a semiconvergent method and can be applied directly to symmetric nonnegative systems of equations.

The paper is organized as follows. We present a multigrid algorithm and some new features for the singular case in section 2. The general convergence theory of multigrid methods for semidefinite problems are discussed in section 3.

2. Multigrid Algorithm

Let V^m, $m = 1, 2, ..., M$, be nested n_m dimensional Hilbert spaces with inner product $(\cdot, \cdot)$ and $\|v^m\|_{L_2} = (v^m, \quad v^m)^{1/2}$. Let $A^m \geq 0$ be an $n_m \times n_m$ symmetric nonnegative definite matrix on V^m with null space $N(A^m)$. Denote the quotient space $V^m/N(A^m)$ by H^m. Then $A^m > 0$ on H^m and $H^m = span\{v_1^m, v_2^m, ..., v_{l_m}^m\}$, where v_l^m, $1 \leq l \leq l_m$, are the eigenvectors corresponding to nonzero eigenvalues and l_m is the rank of the matrix A^m.Let $P_H^m : V^m \to H^m$ and $P_0^m : V^m \to N(A^m)$ be the orthogonal projection operators. For any $v \in V^m$, we have $v = v_H + v_0$, where $v_H = P_H^m v$ and $v_0 = P_0^m v$.

Consider the matrix problem

$$A^1 U^1 = R^1, \tag{2.1}$$

and assume that A^1 is irreducible and symmetric nonnegative definite and the right-hand side R^1 is given properly such that there exists at least a solution for the above problem (2.1), *i.e.*, R^1 is in the quotient space H^1.

Let $I_m^{m+1} : V^m \to V^{m+1}$ define a restriction and $I_{m+1}^m : V^{m+1} \to V^m$ an interpolation, $1 \leq m \leq M-1$. Let G^m: $V^m \times V^m \to V^m$ be smoothing operators and F^M represents an exact solver, in which case $F^M(U^M, R^M) = U^{M,*}$, where $A^M U^{M,*} = R^M$. The following defines a standard μ-cycle multigrid algorithm (called a V-cycle if $\mu = 1$ and a W-cycle if $\mu = 2$) for solving

$$A^m U^m = R^m, \quad 1 \leq m \leq M. \tag{2.2}$$

Standard MG Algorithm

(i) If $m = M$, then $U^M \leftarrow F^M(U^M, R^M)$.

(ii) If $m < M$, then

(1) $U^m \leftarrow G^m(U^m, R^m)$ (pre-smoothing step);

(2) perform μ iterations of Standard MG Algorithm on level $m+1$ (with fixed value of U^m) for the following correction problem, starting from zero initial value :

$$A^{m+1} U^{m+1} = I_m^{m+1}(R^m - A^m U^m), \quad U^{m+1} \in V^{m+1}; \tag{2.3}$$

(3) $U^m \leftarrow U^m + I_{m+1}^m U^{m+1}$ (correction);

(4) $U^m \leftarrow G^m(U^m, R^m)$ (post-smoothing step).

Here we assume that the interpolation I_{m+1}^m is full rank and

$$I_m^{m+1} = (I_{m+1}^m)^T \quad \text{and} \quad A^{m+1} = I_m^{m+1} A^m I_{m+1}^m. \tag{2.4}$$

It has been noted that the standard MG algorithm is given in a recurrence form. The matrix A^{m+1} is irreducible and symmetric nonnegative definite if A^m possesses these properties. More important features are given in the following lemma.

Lemma 2.1. *Assume that A^m is irreducible and symmetric nonnegative definite, and the interpolation operator I_{m+1}^m is full rank. Then, we have*

$$I_{m+1}^m : N(A^{m+1}) \to N(A^m),$$

and

$$I_m^{m+1} : H^m \to H^{m+1}.$$

Proof. For any $v^{m+1} \in N(A^{m+1})$,

$$A^{m+1}v^{m+1} = 0.$$

By (2.4) ,

$$I_m^{m+1} A^m I_{m+1}^m v^{m+1} = 0$$

and therefore,

$$(A^m I_{m+1}^m v^{m+1}, \quad I_{m+1}^m v^{m+1}) = 0,$$

which leads to

$$I_{m+1}^m v^{m+1} \in N(A^m). \tag{2.5}$$

Thus, we have from (2.5) that

$$(I_m^{m+1}u^m, \quad v^{m+1}) = (u^m, \quad I_{m+1}^m v^{m+1}) = 0, \qquad \forall u^m \in H^m, \quad v^{m+1} \in N(A^{m+1}), \tag{2.6}$$

i.e. ,

$$I_m^{m+1} : H^m \to H^{m+1}, \qquad I_{m+1}^m : N(A^{m+1}) \to N(A^m).$$

It follows from lemma 2.1 that the right-hand side R^m belongs to the quotient space H^m for each level, $1 \le m \le M$. The following corollary is obtained immediately.

Corollary. *If the system (2.1) is consistent, the system (2.3) is also consistent for all m.*

In level m+1, the correction U^{m+1} can be decomposed as

$$U^{m+1} = U_H^{m+1} + U_0^{m+1}.$$

It follows from lemma 2.1 that $I_{m+1}^m U_0^{m+1}$ must be in the null space $N(A^m)$ and

$$P_H^m I_{m+1}^m U^{m+1} = P_H^m I_{m+1}^m U_H^{m+1}. \tag{2.7}$$

It is obvious that the component of the correction $I_{m+1}^m U^{m+1}$ on the quotient space H^m is independent of U_0^{m+1}, *i.e.*, the correction does not bring any component in $N(A^{m+1})$ produced in the coarse grid into the component in H^m on the fine grid. The standard MG algorithm and the assumption (2.4) guarantee that such a property holds in all levels. On other side, the component in the null space can be controlled by suitably choosing the smoothing operator. The convergence of the standard MG algorithm will be discussed in the following section.

3. The Convergence Analysis for the Semidefinite Systems

In addition to the Euclidean inner product $(\cdot,\cdot)$, we use the energy inner product: $(u,v)_1 = (Au,v)$, along with its associated norm $\|\cdot\|_1 \equiv \|\cdot\|$. The energy norm is a norm in the quotient space H^m (seminorm in V^m). Note that

$$(A^m u, v) = (A^m u_H, v_H), \quad for \quad u, v \in V^m.$$

The following notations will be used in this paper:

$U^{m,s}$: approximation solution in level m produced by the the standard MG algorithm in the s-th V-cycle.

$U^{m,*}$: solution of (2.2), in which $U_H^{m,*}$ is unique and $P_0^m U^{m,*}$ may be some vector in the null space.

$U^{m,s(\alpha)}$, $\alpha = 0,1,2,3$: approximation solutions in level m and the s-th V-cycle before step (1) and after steps (1), (3) and (4) in the standard MG algorithm, respectively, where $U^{m,s} = U^{m,s(0)}$ and $U^{m,s+1} = U^{m,s(3)}$.

$e^{m,s} = U^{m,s} - U_H^{m,*}$ and $e^{m,s(\alpha)} = U^{m,s(\alpha)} - U_H^{m,*}$.

ϵ_H^m : convergence factor of the multigrid method in the quotient space H^m which is defined by $\|e^{m,s+1}\| \le \epsilon_H^m \|e^{m,s}\|$.

It should be noted that $e^{m,s}$ and $e^{m,s(\alpha)}$ do not represent error measurements. They do not converge to zero in general when $s \to \infty$. We will omit the superscript m if no confusion arises. Throughout this section c will denote a generic constant which is independent of s.

The convergence analysis in the vector space V^m is the key to an iterative algorithm. Since there are infinite many solutions in the singular problems (2.2), the sequence $U^{m,s}$, if convergent, will converge to one of solutions, which depends upon the initial guess. Clearly the general solution $U^{m,*}$ of (2.2) can be decomposed into $U^{m,*} = U_H^{m,*} + U_0^{m,*}$ and the component $U_H^{m,*}$ is unique in the quotient space. Then only the component in the null space is dependent upon the initial guess $U^{m,0}$.

Since the residual of $U^{m,s}$ is defined by

$$R^m - A^m U^{m,s} = A^m(U^{m,*} - U^{m,s}) = A^m(U_H^{m,*} - U_H^{m,s})$$

and

$$\lambda_{min}^m \|U_H^{m,*} - U_H^{m,s}\|_{L_2} \leq \|R^m - A^m U^{m,s}\|_{L_2} \leq \lambda_{max}^m \|U_H^{m,*} - U_H^{m,s}\|_{L_2},$$

where λ_{max}^m and λ_{min}^m denote maximal eigenvalue and minimal nonzero eigenvalue of the matrix A^m, respectively, the convergence in the quotient space is equivalent to one of residual. Therefore, the iteration can stop in practical computation when the residual $R^m - A^m U^{m,s}$ is less than a given tolerance in L_2 norm.

We consider the convergence of general multigrid (MG) method (including GMG and AMG) for the symmetric and nonnegative definite problems. We need to consider two problems for $A \geq 0$: (1) Components of sequence of approximation solutions produced by MG method are bounded in null spaces (Computation can not be completed if the components in the null space tend to infinite); (2) The sequence of approximation solutions produced by MG method is convergent in vector spaces V^m.

Let $A = A_1 - A_2$ be a splitting, where A_1 is nonsingular. A general iterative scheme for (2.2) is given by

$$U^{new} = \pounds U^{old} + \pounds_r R, \tag{3.1}$$

where $\pounds = (A_1)^{-1} A_2$, $\pounds_r = (A_1)^{-1}$.

A complete convergence analysis of the iterative scheme (3.1) were given in [1] by introducing the semiconvergence for matrix $A \geq 0$.

First, the boundness is considered.

Lemma 3.1. *If the conditions*
(i) A^m is symmetric nonnegative definite;
(ii) MG method is convergent with factor ϵ_H^m in the quotient space H^m ;
(iii) the smoothing operators G^m is of the form (3.1) with semiconvergent matrix $\pounds^m$,
are satisfied, then

$$\|U^{m+1,s(3)}\|_{L_2} \leq c\|e_H^{m,s(0)}\|_{L_2}.$$

Proof. We use mathematical induction to prove the theorem. Let λ_{max}^m and λ_{min}^m denote maximal eigenvalue and minimal nonzero eigenvalue of the matrix A^m, respectively.

First, two-level grids (1, 2) are considered.

The approximation after the pre-smoothing step in level 1 is given by

$$U^{1,s(1)} = U_H^{1,s(1)} + U_0^{1,s(1)}.$$

By (3.1) and noting the fact that the null space of A^1 is the same as the eigenspace of $\pounds^1$ corresponding to the eigenvalue 1, we have

$$e^{1,s(1)} = \pounds^1 e^{1,s(0)} = e_0^{1,s(0)} + \pounds^1 e_H^{1,s(0)},$$

which implies

$$\|e_H^{1,s(1)}\|_{L_2} \leq \|e_H^{1,s(0)}\|_{L_2},$$

where the semiconvergence is used. The defect $d^{1,s} = R^1 - A^1U^{1,s(1)}$ is estimated by

$$\|d^{1,s}\|_{L_2} \leq \lambda^1_{max}\|e_H^{1,s(1)}\|_{L_2} \leq c\|e_H^{1,s(0)}\|_{L_2},$$

since

$$d^{1,s} = d_H^{1,s} = -A^1e^{1,s(1)} = -A^1e_H^{1,s(1)}.$$

In level 2, we have

$$\|R^{2,s}\|_{L_2} \leq \|I_1^2\|_{L_2}\|d^{1,s}\|_{L_2} \leq c\|e_H^{1,s(0)}\|_{L_2},$$

since

$$R^{2,s} = I_1^2(R^1 - A^1U^{1,s(1)}) = I_1^2d^{1,s}.$$

In level 2 (the coarsest grid), we use the same iterative algorithm as in smoothing step until convergent. Then We see that

$$e^{2,s,i} = (\pounds^2)^ie^{2,s(0)} = -(\pounds^2)^iU_H^{2,*}$$

and

$$e^{2,s} = \lim_{i\to\infty} e^{2,s,i} = -(\lim_{i\to\infty}(\pounds^2)^i)U_H^{2,*},$$

where $e^{2,s,i}$ denotes the i-th iterative value for $e^{2,s}$ on the coarsest grid and $U^{2,s(0)} = 0$ is used. The definition of semiconvergence implies that

$$\|U^{2,s(3)}\|_{L_2} \equiv \|U^{2,s+1}\|_{L_2} \leq \|U_H^{2,*}\|_{L_2} + \|e^{2,s}\|_{L_2} \leq c\|U_H^{2,*}\|_{L_2} \leq \frac{c}{\lambda^2_{min}}\|R^{2,s}\|_{L_2} \leq c\|e_H^{1,s(0)}\|_{L_2},$$

where $A^2U_H^{2,*} = R^{2,s}$ is used.

Now, assume that

$$\|U^{m+2,s(3)}\|_{L_2} \leq c\|e_H^{m+1,s(0)}\|_{L_2} = c\|U_H^{m+1,*}\|_{L_2}, \tag{3.2}$$

since $U^{m+1,s(0)} = 0$.

We are to prove that in the finer grid

$$\|U^{m+1,s(3)}\|_{L_2} \leq c\|e_H^{m,s(0)}\|_{L_2}. \tag{3.3}$$

The approximation after the pre-smoothing step in level m+1 is given by

$$U^{m,s(1)} = U_H^{m,s(1)} + U_0^{m,s(1)}.$$

Then, we have in the similar argument to the previous one

$$\|e_H^{m,s(1)}\|_{L_2} \leq \|e_H^{m,s(0)}\|_{L_2},$$

and

$$\|d^{m,s}\|_{L_2} \leq \lambda^m_{max}\|e_H^{m,s(1)}\|_{L_2} \leq c\|e_H^{m,s(0)}\|_{L_2}.$$

In level $m+1$, we have from the standard MG algorithm that

$$\|R^{m+1,s}\|_{L_2} \leq \|I_m^{m+1}\|_{L_2}\|d^{m,s}\|_{L_2} \leq c\|e_H^{m,s(0)}\|_{L_2},$$

and

$$\|U^{m+1,s(1)}\|_{L_2} \leq \|\pounds_r^{m+1}\|_{L_2}\|R^{m+1,s}\|_{L_2} \leq c\|e_H^{m,s(0)}\|_{L_2}.$$

The corrected solution and new approximation after the post-smoothing step in the level $m+1$ are given by

$$U^{m+1,s(2)} = U^{m+1,s(1)} + I_{m+2}^{m+1}U^{m+2,s(3)}$$

and

$$U^{m+1,s(3)} = \pounds^{m+1}U^{m+1,s(2)} + \pounds_r^{m+1}R^{m+1,s},$$

respectively. Using the previous inequalities, assumption (3.2) and estimation

$$\|U_H^{m+1,*}\|_{L_2} \leq \frac{1}{\lambda_{min}^{m+1}} \|R^{m+1,s}\|_{L_2},$$

we have

$$\|U^{m+1,s(3)}\|_{L_2} \leq c \|e_H^{m,s(0)}\|_{L_2}.$$

The proof is completed.

Theorem 3.1. *If the conditions (i)-(iii) in lemma 3.1 are satisfied, then the sequence $U^{m,s}$ produced by the MG algorithm is convergent in the vector space V^m.*

Proof. We have the correct approximation and the approximation on the (s+1)-th V-cycle in level m

$$U^{m,s(2)} = U^{m,s(1)} + I_{m+1}^m U^{m+1,s(3)}$$

and

$$U^{m,s+1} \equiv U^{m,s(3)} = \pounds^m U^{m,s(2)} + \pounds_r^m R^m,$$

which imply that

$$e^{m,s+1} = \pounds^m e^{m,s(2)} = \pounds^m e^{m,s(1)} + \pounds^m I_{m+1}^m U^{m+1,s(3)} = (\pounds^m)^2 e^{m,s} + \pounds^m I_{m+1}^m U^{m+1,s(3)}.$$

In view of recurrence, we have

$$\begin{aligned} e^{m,s} = &\ (\pounds^m)^{2s} e^{m,0} + ((\pounds^m)^{2s-1} I_{m+1}^m U^{m+1,0(3)} \\ &+ (\pounds^m)^{2s-3} I_{m+1}^m U^{m+1,1(3)} + \cdots + \pounds^m I_{m+1}^m U^{m+1,s-1(3)}). \end{aligned}$$

Thus, we get for $i > j$, that

$$\begin{aligned} & e^{m,i} - e^{m,j} \\ &= (\pounds^m)^{2j}((\pounds^m)^{2(i-j)} - I) e^{m,0} + [(\pounds^m)^{2j-1}((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,0(3)} \\ &+ (\pounds^m)^{2j-3}((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,1(3)} + \cdots + \pounds^m((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,j-1(3)}] \\ &+ [\pounds^m(\pounds^m)^{2(i-j-1)} I_{m+1}^m U^{m+1,j(3)} + \pounds^m(\pounds^m)^{2(i-j-2)} I_{m+1}^m U^{m+1,j+1(3)} \\ &+ \cdots + \pounds^m I_{m+1}^m U^{m+1,i-1(3)}]. \end{aligned} \tag{3.4}$$

Since $\pounds^m$ is semiconvergent, there exists the decomposition (see [1])

$$\pounds^m = P \begin{bmatrix} I_1 & 0 \\ 0 & K \end{bmatrix} P^{-1}, \tag{3.5}$$

where I_1 is the identity matrix and $\rho(K) < 1$. Then,

$$\begin{aligned} & (\pounds^m)^{2j}((\pounds^m)^{2(i-j)} - I) \\ &= P \begin{bmatrix} I_1 & 0 \\ 0 & K^{2j} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & K^{2(i-j)} - I_2 \end{bmatrix} P^{-1} = P \begin{bmatrix} 0 & 0 \\ 0 & K^{2j}(K^{2(i-j)} - I_2) \end{bmatrix} P^{-1}. \end{aligned}$$

By some basic argument in linear algebra [1], for any $\xi > 0$, there exists a j_ξ such that $\|K^j\|_{L_2}^{1/j} \leq (\rho(K) + \xi)$, for $j > j_\xi$. We see that

$$\|(\pounds^m)^{2j}((\pounds^m)^{2(i-j)} - I)\|_{L_2} \leq c(\rho(K) + \xi)^{2j}.$$

The following estimate is obtained by using lemmas 3.1 and convergence in the quotient space

$$\begin{aligned} & \|(\pounds^m)^{2j-1}((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,0(3)} + (\pounds^m)^{2j-3}((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,1(3)} \\ & + \cdots + \pounds^m((\pounds^m)^{2(i-j)} - I) I_{m+1}^m U^{m+1,j-1(3)}\|_{L_2} \\ & \leq c((\rho(K) + \xi)^{2j-1} + (\rho(K) + \xi)^{2j-3} \epsilon_H^m + \cdots + (\rho(K) + \xi)(\epsilon_H^m)^{j-1}) \\ & \leq c((\epsilon^m)^{2j-1} + (\epsilon^m)^{2j-2} + \cdots + (\epsilon^m)^j) \\ & \leq c(\epsilon^m)^j, \end{aligned}$$

where

$$\epsilon^m = max(\epsilon_H^m, \rho(K) + \xi). \tag{3.6}$$

The last term in (3.4) is estimated as follows

$$\begin{aligned}&\|\pounds^m(\pounds^m)^{2(i-j-1)}I^m_{m+1}U^{m+1,j(3)}+\pounds^m(\pounds^m)^{2(i-j-2)}I^m_{m+1}U^{m+1,j+1(3)}\\&+\cdots+\pounds^m I^m_{m+1}U^{m+1,i-1(3)}\|_{L_2}\\&\le c(\|U^{m+1,j(3)}\|_{L_2}+\|U^{m+1,j+1(3)}\|_{L_2}+\cdots+\|U^{m+1,i-1(3)}\|_{L_2})\\&\le c(\epsilon^m_H)^j.\end{aligned}$$

Then, for any $\xi>0$ and $j>j_\xi$,

$$\|e^{m,i}-e^{m,j}\|_{L_2}\le c(\epsilon^m)^j. \tag{3.7}$$

$e^{m,s}=U^{m,s}-U^{m,*}_H$ is convergent since $\epsilon^m<1$, $\rho(K)<1$ and ξ is arbitrary. The convergence of $U^{m,s}$ follows immediately.

Theorem 3.2. *If the conditions (i)-(iii) in lemma 3.1 are satisfied, then for $\xi>0$, there exists an s_ξ such that*

$$\|U^{m,s}-U^{m,*}\|_{L_2}\le c(\epsilon^m)^s,\quad for\quad s>s_\xi,$$

where ϵ^m is given in (3.6) and $U^{m,*}$ is a solution of (2.2).

Proof. Since $e^{m,i}=U^{m,i}-U^{m,*}_H$,

$$e^{m,i}-e^{m,j}=U^{m,i}-U^{m,j}\,.$$

It follows from (3.7) that

$$\|U^{m,i}-U^{m,j}\|_{L_2}\le c(\epsilon^m)^j. \tag{3.8}$$

By theorem 3.1, $U^{m,s}$ is convergent in the vector space. In view of the assumption, $U^{m,s}_H$ of the sequence produced by the standard MG algorithm is convergent to $U^{m,*}_H$ in the quotient space. It is clear that sequence $U^{m,s}_0$ converges to a vector of null space. Therefore, $U^{m,s}=U^{m,s}_H+U^{m,s}_0\to U^{m,*}$ where $U^{m,*}$ is a solution of (2.2). Let $i\to\infty$ in (3.8). Then for any $\xi>0$ and $j>j_\xi$,

$$\|U^{m,*}-U^{m,j}\|_{L_2}\le c(\epsilon^m)^j,$$

The proof is completed.

Remark 1. There are some interesting features in our convergence analysis. The asymptotic convergence factor in L_2-norm given in theorem 3.2 depends upon the two factors, ϵ^m_H and $\rho(K)$, which is larger. The former is the convergence rate in the quotient space H^m and the latter represents the asymptotic convergence rate of the iterative scheme (3.1), which usually is some classical iteration. Let

$$U^{m,s}=U^{m,s}_H+U^{m,s}_0$$

and $U^{m,s}\to U^{m,*}$. Some features are summarized below.

- In the classical sense of convergence (L_2-norm in the vector space V^m), the standard multigrid algorithm for solving the singular problems has the same convergence rate as those classical iterations. It is reasonable since the interpolation is required to reduce only the component in the quotient space H^m.
- In the sense of convergence in the quotient space H^m, general multigrid algorithms for solving the symmetric nonnegative definite systems have the same convergence rate as the algorithms for solving symmetric positive definite systems.
- Since one only expects to find the approximation to some solution of problem, the convergence in the quotient space seems to be more realistic in practical computations. When the computation stops after S V-cycles, the difference between the approximation and some exact solution in L_2-norm is

$$\|U^{m,S}-(U^{m,*}_H+U^{m,S}_0)\|_{L_2}=\|U^{m,S}_H-U^{m,*}_H\|_{L_2}\le c(\epsilon^m_H)^S,$$

which gives the same convergence rate as in the quotient space.

Remark 2. The Gauss-Seidel and damped Jacobi iterations are semiconvergent (see [1]). Therefore, theorems 3.1 and 3.2 demonstrate that the convergence of the MG method in the quotient space implies one in the vector space in L_2 norm if the Gauss-Seidel or damped Jacobi iterations is chosen as the smoothing operator.

Remark 3. In [12], J. Mandel, S. McCormick and J. Ruge proved convergence of the multigrid method for the symmetric positive definite systems of linear equations, i.e., convergence in the quotient space for the symmetric nonnegative definite systems. The convergence of two-level grids of the AMG methods for the symmetric positive definite systems is proved in [4,6,16]. Under certain conditions, some properties of matrices , for example, symmetry, diagonally dominance and positive definite property, can be kept in all level. Hence, we can have convergence of the AMG methods in a fixed number of levels (the convergence factor is dependent on the number of levels). Therefore, convergence of the MG methods for the symmetric nonnegative definite systems can be obtained from these results and our theorems under certain conditions

References

[1] A. Berman and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, SIAM, Philadephia, 1994.
[2] J.M. Borwein and W. Sun, The stability of dynamic SPECT models, *Numerische Mathematik*, **77** (1997), 283–298.
[3] Z. Cai, J. Mandel and S. McCormick, Multigrid methods for nearly singular linear equations and eigenvalue problems, *SIAM J. Numer. Anal.*, **34** (1997), 178–200.
[4] Q. Chang, Yaushu Wong and H. Fu, On algebraic multigrid method, *J. Comput. Phys.*, **125** (1996), 279-292.
[5] Q. Chang, S. Ma and G. Lei, Algebraic multigrid method for queuing networks, *Intern. J. of Computer Math.*, **70** (1999), 539–552.
[6] Q. Chang and Z. Huang, Efficient algebraic multigrid algorithms and their convergence, *SIAM J. Sci. Comput.*, **24** (2002), 597-618 (electric version).
[7] M. Eiermann, I. Marek and W. Niethammer, On the solution of singular linear systems of algebraic equations by semi-iterative methods, *Numer. Math.*, **53** (1988), 265–283.
[8] M. Griebel, Multigrid algorithms considered as iterative methods on semidefinite systems, *SIAM J. Sci. Comput.*, **15** (1994), 547–565.
[9] W. Hackbusch, Multi-grid Methods and Applications, Springer-Verlag, Berlin, 1985.
[10] V. Henson, M. Limber, S. McCormick and B. Robinson, Multilevel image reconstruction with natural pixels, *SIAM J. Sci. Comput.*, **17** (1996), 193–216.
[11] Z.H. Huang and Q.S. Chang, Guass-Seidel-Type multigrid methods, *J. Comput. Math.,* **21** (2003), 421-434.
[12] J. Mandel, S. McCormick and J. Ruge, An algebraic theory for multigrid methods for variational problems, *SIAM J. Numer. Anal.*, **25** (1988), 91–110.
[13] C.D. Meyer and R. J. Plemmons, Convergent powers of a matrix with applications to iterative methods for singular linear systems, *SIAM J. Numer. Anal.*, **14** (1977), 699–705.
[14] A. Neumaier, Solving ill-conditioned and singular linear systems: a tutorial on regularization, *SIAM Rev.*, **40** (1998), 636–666.
[15] Y. Notay, Imcomplete factorization of singular linear systems, *BIT*, **29** (1989), 682–702.
[16] J. Ruge and K. Stüben, Algebraic Multigrid, in Multigrid Methods, (S. F. McCormick, ed.) 4, *SIAM*, Philadephia, (1987), 73–130.
[17] Z.C. Shi and X.J. Xu, A V-cycle multigrid for quadrilateral rotated Q_1 element with numercial integration, *J. Comput. Math.,* **21** (2003), 545-554.
[18] X.J. Xu and J.R. Chen, Multigrid for the Mortar finite element for parabolic problem, *J. Comput. Math.,* **21** (2003), 411-420.
[19] Y. Song, Semiconvergence of extrapolated iterative methods for singular linear systems, *J. Comput. Appl. Math.*, **106** (1999), 117–129.
[20] K. Stüben and U. Tottenberg, Multigrid Methods: Fundamental Algorithms, Model Problem Analysis an Application, In Lecture Note in Mathematics, Vol. 962, Springer-Verlag, Berlin, 1982.
[21] J.P. Wang, Convergence analysis of multigrid algorithms for nonselfadjoint and indefinite elliptic problems, *SIAM J. Numer. Anal.*, **30** (1993), 275–285.