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SUPERCONVERGENCE OF TETRAHEDRAL QUADRATIC FINITE ELEMENTS *1)

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Abstract

For a model elliptic boundary value problem we will prove that on strongly regular families of uniform tetrahedral partitions of a pohyhedral domain, the gradient of the quadratic finite element approximation is superclose to the gradient of the quadratic Lagrange interpolant of the exact solution. This supercloseness will be used to construct a post-processing that increases the order of approximation to the gradient in the global L^2-norm.

Mathematics subject classification: 65N30.
Key words: Tetrahedron, Superconvergence, Supercloseness, Post-processing, Gauss points.

1. Introduction

The topic of this paper is supercloseness and superconvergence of a finite element that is frequently used in practical applications: the tetrahedral quadratic element. As a matter of fact, in the engineering society, it is even more popular than the linear element, in spite of the fact that the latter has been studied in much more detail. Before we discuss our results in Section 1.2, we will introduce the term superconvergence and put it in its historical context in Section 1.1. In particular, we comment on work by other authors having direct links to our results.

1.1. Overview

Superconvergence of standard and mixed finite elements is a well-known and practically useful topic in finite element analysis. Usually, a finite element method is called superconvergent, if at special points (or on special lines) the rate of convergence is higher than what is globally possible (cf. [11, 13, 23, 27, 28]). Oganesjan and Ruhovets [24] proved that for linear triangular elements on uniform partitions the gradient of the finite element approximation is a higher order perturbation of the gradient of a local interpolant of the exact solution. This property, which lies also at the basis of the papers (cf. [4, 9, 10, 12, 19, 22, 29]), is often called supercloseness. In both cases, one can usually construct, without too much additional computational effort, approximations that are globally better than the original one. This procedure is called post-processing. The difference between the original and the post-processed approximation may then be used as an asymptotically exact error estimation. For some interesting papers and an abundance of references, we refer to [21].

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Already in 1981, Zhu [30] proved superconvergence of the gradient of quadratic triangular elements on uniform triangulations, so, in the two-dimensional case. In [31], he discusses superconvergence at nodal points for this setting. Later, similar results were obtained by Andreev in [1] and Andreev and Lazarov in [2], who proved that the tangential component of the gradient is superconvergent at the two Gauss points at each edge of each triangle (see also Goodsell and Whiteman [16, 17]). Recently, Brandts rederived some of these results in [5] as a by-product of a superconvergence proof for one-but-lowest order Raviart-Thomas mixed finite elements.

Superconvergence results for three-dimensional problems are relatively scarce, since not all techniques for the two-dimensional case can be generalized. Typical difficulties with superconvergence in $\mathbb{R}^3$ are surveyed in [6]. As far as we know, the Chinese were the first to prove superconvergence for the gradient in a three-dimensional setting. In 1980, Chen considered linear elements on tetrahedra in [7], which was followed in the second half of the eighties by Kantchev and Lazarov with the paper [18]. A short note by Pehlivanov [25] reflects on the quadratic case, but unfortunately without any (reference to a) proof. In 1994, Goodsell derived, for the gradient, pointwise superconvergence results for linear tetrahedral elements in [15]. Finally, results in [26, 27] imply superconvergence at nodal values for quadratic three-dimensional elements on locally point-symmetric meshes. In fact, we will need a result from [26] in our proofs. In the two-dimensional case [5], this was not necessary because of favorable properties of a mixed finite element Fortin interpolation, which do not generalize to the three-dimensional case (see [6], p. 29).

1.2. Outline

The Poisson equation with homogeneous Dirichlet boundary conditions will be our model elliptic problem. We employ regular family of uniform tetrahedral partitions of the domain. The gradient of the standard quadratic finite element approximations will then be proved superclose to the gradient of the nodal quadratic Lagrange interpolant of the exact solution. Once more we stress that the Fortin-like interpolant that was used in the two-dimensional setting, has no special advantages in 3D as in 2D. See [5] and [6] for details.

The outline of this paper is as follows. Section 2 contains some preliminaries. In Section 3, we derive auxiliary results for so-called quadratic bubble functions, which will be used in the proofs of our main results in Section 4. There we prove the supercloseness between the gradients of the finite element approximation and the quadratic Lagrange interpolant of the exact solution. A numerical test is presented for illustration. In Section 5, we discuss the generalization of the results to other elliptic problems with varying coefficients, and the extension of a post-processing scheme by Andreev and Lazarov [2], which will lead to a higher order approximation of the gradient.

2. Preliminaries

Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with Lipschitz boundary. Denote by $H^k(\Omega)$ the usual Sobolev spaces of functions having generalized partial derivatives up to order k in $L^2(\Omega)$ and their usual norm and seminorm by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. The subspace of functions from $H^1(\Omega)$ with vanishing traces on $\partial\Omega$ we denote by $H^1_0(\Omega)$. Before turning to the discrete spaces, we will elaborate on uniform partitions of the domain.

2.1. Uniform partitions of a domain into tetrahedra

A triangulation of a planar domain is called *uniform* if the union of any two triangles sharing an entire edge forms a parallelogram. The feature of interest is, that a parallelogram is a set that is invariant under reflection in its center of gravity. For brevity, we will refer to such sets as "point-symmetric sets". Since two tetrahedra having a face in common never form a point-symmetric set, the generalization of the concept of uniform partition to three dimensions needs to be different. It turns out that "edge sharing" is the property of interest. Hence, we will call a partition into tetrahedra uniform if it satisfies the following condition:

(U) For each internal edge $e \not\subset \partial\Omega$, the patch P_e of tetrahedra sharing e is a point-symmetric set with respect to the midpoint M_e of e, by which we mean that $x \in P_e \Leftrightarrow 2M_e - x \in P_e$ for all $x \in P_e$.

There exist indeed partitions into tetrahedra satisfying (U). They are based on the so-called *Kuhn partition* of the unit cube. The study of such partitions in arbitrary dimension, however, traces back to Freudenthal [14].

Kuhn's partition is the partition of the cube $[0,1]^3$ into six tetrahedra $T_\sigma (\sigma \in \Sigma)$, where Σ is the group of permutations of the numbers $1, 2$, and 3. Each T_σ is given by

$$T_\sigma = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq x_{\sigma(3)} \leq 1\}. \quad (1)$$

One way to visualize this partition is by first cutting the cube into two prisms. Then, each prism is cut into a pyramid and a tetrahedron. Finally, both pyramids are cut into two tetrahedra. See Figure 1 below.

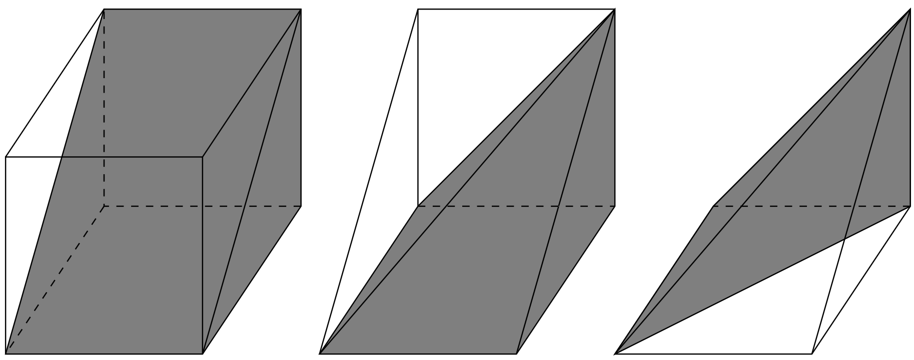

Figure 1. Kuhn's partition of the unit cube

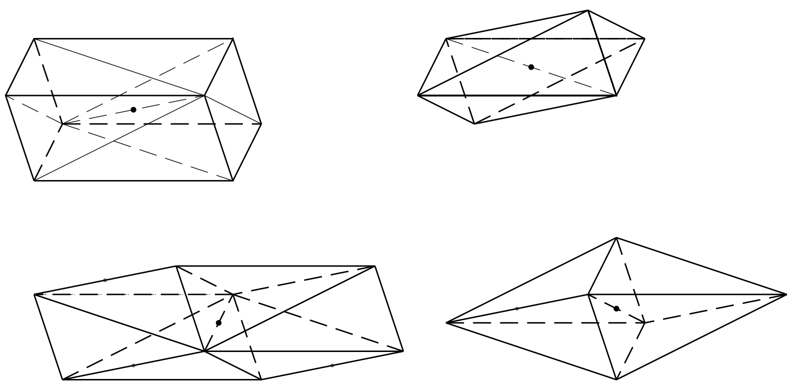

Figure 2. Examples of patches around edges

Let $\mathcal{T}$ be the partition of $\mathbb{R}^3$ into tetrahedra defined by translation over all $z \in \mathbb{Z}^3$ of the Kuhn-partitioned unit cube. Then each pair (B, w), where B is a non-singular 3×3 matrix and $w \in \mathbb{R}^3$, yields a uniform partition $\mathcal{T}(B, w) = \{BT + w \mid T \in \mathcal{T}\}$ of $\mathbb{R}^3$, since it satisfies (U). If B and w are such that $T \subset \overline{\Omega}$ or $T \subset \mathbb{R}^3 \setminus \Omega$ for all $T \in \mathcal{T}(B, w)$, then

$$\mathcal{T}_\Omega(B, w) = \{T \in \mathcal{T}(B, w) \mid T \subset \overline{\Omega}\} \tag{2}$$

is a uniform partition of $\overline{\Omega}$. This is motivated by the obvious fact that each patch of tetrahedra that share an edge is point-symmetric, which is illustrated in Figure 2. Notice that in Figure 2, the only two topologically different types of patches are depicted: two consisting of six tetrahedra forming a parallelepiped, and two consisting of four tetrahedra (the union of two pyramids).

We also remind the reader of the following terminology. A family of partitions into tetrahedra is called *regular* if there exists a $C > 0$ such that for all elements from all partitions, $\operatorname{Vol} T \geq C h_T^3$, where h_T is the diameter of T. We will denote families of uniform partitions of $\overline{\Omega}$ by $\mathcal{T}_\Omega(B_j, w_j)_j$. Their regularity clearly only depends on the matrices B_j.

2.2. Discrete spaces and interpolation operators

Relative to a partition of the domain, let V_h^k be the space of continuous piecewise polynomials of degree k, and set $V_{0h}^k = V_h^k \cap H_0^1(\Omega)$. Let $L_h : H^2(\Omega) \to V_h^1$ be the linear Lagrange interpolant on the vertices of the tetrahedra, and $Q_h : H^2(\Omega) \to V_h^2$ the quadratic Lagrange interpolant on the vertices and midpoints of edges of the tetrahedra. It is well-known that the following a priori bound holds.

Proposition 2.1. *Let $\mathcal{T}_\Omega(B_j, w_j)_j$ be a regular family of uniform partitions of $\overline{\Omega}$. Then there exists a constant $C > 0$ such that for all elements T from all partitions and all $v \in H^3(\Omega)$ we have*

$$|v - Q_h v|_{1,T} \leq C h^2 |v|_{3,T}. \tag{3}$$

In Section 3, we will study the subspace $B_{0h}^2 \subset V_{0h}^2$ of so-called quadratic bubble functions, defined by

$$B_{0h}^2 = \{(I - L_h) v_h \mid v_h \in V_{0h}^2\}. \tag{4}$$

This definition induces the following space-decomposition,

$$V_{0h}^2 = V_{0h}^1 \oplus B_{0h}^2, \tag{5}$$

which expresses that each $v_h \in V_{0h}^2$ can be uniquely written as $\ell_h + b_h$ with $\ell_h \in V_{0h}^1$ and $b_h \in B_{0h}^2$. This decomposition will be frequently used in our main results.

Clearly, B_{0h}^2 is spanned by the basis $\psi_i, (i = 1, \ldots, M)$, where each $\psi_i \in V_{0h}^2$ has value one at the midpoint of the internal edge e_i, and vanishes at all other edges.

Remark 2.2. The support S_i of a bubble basis function ψ_i equals the patch of elements that share the edge e_i. Recall that if the partition of the domain is uniform, each S_i is point-symmetric (see also Figure 2).

2.3. The tetrahedral quadratic finite element method

The tetrahedral quadratic finite element formulation for the Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions results from discretizing the associated weak formulation that aims to find $u \in H_0^1(\Omega)$ such that $(\nabla u, \nabla v) = (f, v)$ for all $v \in H_0^1(\Omega)$. The discrete problem consists of finding $u_h \in V_{0h}^2$ such that $(\nabla u_h, \nabla v_h) = (f, v_h)$ for all $v_h \in V_{0h}^2$. Note that subtraction gives the Galerkin orthogonality relation

$$\forall v_h \in V_{0h}^2 : \quad (\nabla(u - u_h), \nabla v_h) = 0. \tag{6}$$

In the Section 4, we will study the difference between the Galerkin approximation u_h of u and the interpolant $Q_h u$ of u.

Remark 2.3. It may seem restrictive to concentrate on the Poisson equation only, but all arguments can be extended to the more general equation $-\text{div}\,(A\nabla u) = f$ as long as the matrix A is uniformly positive definite with entries having derivatives in $L^\infty(\Omega)$. We refer to Section 5.1 for an outline.

3. Quadratic Bubble Functions

We will now derive some results that are needed in the proofs of our main theorems in Section 4. The lemmas are generalizations of those in Section 3 of [5] to the three-dimensional setting, but with simplified proofs based on similar results from [3].

Lemma 3.1. *Let $\mathcal{T}_\Omega(B_j, w_j)_j$ be a regular family of uniform partitions of $\overline{\Omega}$. Then there exists a constant $C > 0$ such that for all j and all $b_h = \sum_i \alpha_i \psi_i \in B^2_{0h}$*

$$\sqrt{\sum_i \alpha_i^2 |\psi_i|_1^2} \leq C|b_h|_1. \tag{7}$$

Proof. Let j be given and $b_h = \sum_i \alpha_i\psi_i \in B^2_{0h}$. Let $T \in \mathcal{T}_\Omega(B_j, w_j)$. Then there exists a $\sigma \in \Sigma$ and a $z_T \in \mathbb{Z}^3$ such that $F_j : \hat{x} \mapsto B_j(\hat{x} + z_T) + w_j$ maps T_σ onto some reference tetrahedron T_σ from (1). In particular, F_j gives rise to a one-to-one correspondence between the bubble functions ψ_i on T and the bubble functions $\hat{\psi}_i$ on T_σ, i.e., $\hat{\psi}_i = F_j^{-1}(\psi_i)$. The usual transformation rules between elements and reference elements (cf. [20], p. 40) tell us that there exists a constant C independent of the mesh parameters such that

$$|\hat{\psi}|_{1,T_\sigma} \leq C\|B_j\|| \det B_j|^{-\frac{1}{2}} |\psi|_{1,T} \quad \text{and} \quad |\psi|_{1,T} \leq C\|B_j^{-1}\|| \det B_j|^{\frac{1}{2}} |\hat{\psi}|_{1,T_\sigma}. \tag{8}$$

Using the rightmost rule from (8), we find that

$$\sqrt{\sum_{i=1}^{6} \alpha_i^2 |\psi_i|_1^2} \leq C\|B_j^{-1}\|| \det B_j|^{\frac{1}{2}} \sqrt{\sum_{i=1}^{6} \alpha_i^2 |\hat{\psi}_i|_1^2}. \tag{9}$$

Now, there also exists a constant C, depending only on the reference tetrahedron T_σ, such that

$$\sum_{i=1}^{6} \alpha_i^2 |\hat{\psi}_i|^2_{1,T_\sigma} \leq C|\sum_{i=1}^{6} \alpha_i \hat{\psi}_i|^2_{1,T_\sigma} \tag{10}$$

simply because the semi-norm on the right-hand side does not vanish unless all α_i vanish, in which case such C trivially exists. Using the left transformation rule from (8), we find that

$$|\sum_{i=1}^{6} \alpha_i \hat{\psi}_i|_{1,T_\sigma} = |F_j^{-1}(b_h)|_{1,T_\sigma} \leq C\|B_j\|| \det B_j|^{-\frac{1}{2}} |b_h|_{1,T}. \tag{11}$$

Combining (9), (10) and (11), we get

$$\sqrt{\sum_{i=1}^{6} \alpha_i^2 |\psi_i|_1^2} \leq C\|B_j\|\|B_j^{-1}\||b_h|_{1,T}, \tag{12}$$

By the regularity of the family of uniform partitions, there exists a constant $C > 0$ such that for all j we have $\|B_j^{-1}\| \leq Ch^{-1}$ and $\|B_j\| \leq Ch$. Finally, (7) is proved by summing over all T in the partition.

Another result about the basis function ψ_i for the bubble functions in B^2_{0h} is the following. In the proof for supercloseness in Section 4, it will enable the application of a Bramble-Hilbert type argument to gain an additional factor h.

Lemma 3.2. *Let $\mathcal{T}_\Omega(B, w)$ be a uniform partition, e_i an internal edge, and $\psi_i \in B^2_{0h}$ its corresponding basis function. Then for all cubic polynomials p on the support S_i of ψ_i we have,*

$$(\nabla(p - Q_h p), \nabla\psi_i)_{S_i} = 0. \tag{13}$$

Proof. Without loss of generality we may assume that S_i is centered around the origin. Since Q_h maps quadratic functions onto themselves, we only need to check the identity for the ten cubic monomials. All these monomials are odd functions about the origin, and so are their interpolants. Hence, $\nabla(p - Q_h p)$ is even. Similarly, one sees that $\nabla\psi_i$ is odd. Hence, their product has zero mean on S_i.

Lemma 3.3. *There exists a constant $C > 0$ such that for all j, $T \in \mathcal{T}_\Omega(B_j, w_j)$ and quadratic polynomials p,*

$$|(I - L_h)p|_{1,T} \leq C|p|_{1,T}. \tag{14}$$

Consequently, each $v_h \in V_{0h}^2$ having decomposition $v_h = \ell_h + b_h$ with $\ell_h \in V_{0h}^1$ and $b_h \in B_{0h}^2$ satisfies

$$|b_h|_1 \leq C|v_h|_1 \tag{15}$$

with the same constant as in (14).

Proof. Let $\sigma \in \Sigma$ be arbitrary and write L for linear Lagrange interpolation on T_σ. Then there exists a constant $C > 0$ such that $|(I-L)p|_{1,T_\sigma} \leq C|p|_{1,T_\sigma}$ for all quadratic polynomials p. Indeed, if p is linear, the left-hand side vanishes, and if p is not linear, both terms are positive. Using the transformation rules (8), the first statement follows for arbitrary $T \in \mathcal{T}_\Omega(B, v)$ similar as in Lemma 3.1. The second statement follows from the first one by summing over all elements in the partition and realizing that $b_h = (I - L_h)v_h$.

4. Supercloseness of u_h and $Q_h u$

We will now prove an H^1-bound for $\omega_h = u_h - Q_h u$ based on the decomposition

$$\omega_h = \lambda_h + \beta_h, \quad \text{with } \lambda_h \in V_{0h}^1 \text{ and } \beta_h \in B_{0h}^2. \tag{16}$$

Galerkin orthogonality (6) and the Cauchy-Schwarz inequality give

$$|\omega_h|_1^2 = (\nabla\omega_h, \nabla\lambda_h) + (\nabla(u_h - Q_h u), \nabla\beta_h) \leq |\omega_h|_1|\lambda_h|_1 + |(\nabla(I - Q_h)u, \nabla\beta_h)|. \tag{17}$$

First consider the product of norms on the right-hand side of (17). To bound the right factor, we make use of the fact that u_h is superconvergent at the vertices, which follows from [27].

Theorem 4.1. *Suppose $u \in H^4(\Omega)$ and that $\mathcal{T}_\Omega(B_j, w_j)_j$ is a regular family of uniform tetrahedral partitions. Then*

$$|\lambda_h|_1 \leq C(u)h^3. \tag{18}$$

Proof. Since the family of uniform partitions is regular, we can use the discrete inverse inequality [8], and then switch to the L^∞ norm. This gives

$$|\lambda_h|_1 \leq Ch^{-1}|\lambda_h|_0 \leq Ch^{-1}|\lambda_h|_\infty. \tag{19}$$

Since λ_h is linear, its maximum value is attained at some vertex of the partition. Notice also that $\lambda_h(N) = (u - u_h)(N)$ at all vertices N. Since by Schatz [26], p. 245 (see also [27]), at each vertex N we have

$$|(u - u_h)(N)| \leq C(u)h^4, \tag{20}$$

the statement is proved.

Theorem 4.2. *Suppose $u \in H^4(\Omega)$ and that $\mathcal{T}_\Omega(B_j, w_j)_j$ is a regular family of uniform tetrahedral partitions. Then for all $b_h \in B_{0h}^2$,*

$$|(\nabla(I - Q_h)u, \nabla b_h)| \leq Ch^3|u|_4|b_h|_1. \tag{21}$$

Proof. Let u be given and write $b_h = \sum \alpha_i\psi_i$. Application of the triangle inequality, followed by the Cauchy-Schwarz inequality and Lemma 3.1 results in

$$|(\nabla(I - Q_h)u, \nabla b_h)| \leq \sum_i |\alpha_i||\psi_i|_1|(\nabla(I - Q_h)u, \nabla\psi_i|)||\psi_i|_1^{-1}$$

$$\leq C|b_h|_1 \sqrt{\sum_i |(\nabla(I-Q_h)u, \nabla\psi_i)_{S_i}|^2 |\psi_i|_1^{-2}}, \tag{22}$$

where S_i is the support of ψ_i. By Lemma 3.2, the Cauchy-Schwarz inequality, the a priori estimate (3) for Q_h, we have that for all $p \in \mathcal{P}^3(S_i)$,

$$|(\nabla(I-Q_h)u, \nabla\psi_i)_{S_i}| = |(\nabla(I-Q_h)(u-p), \nabla\psi_i)_{S_i}| \leq Ch^2 |u-p|_{3,S_i} |\psi_i|_1. \tag{23}$$

Taking the infimum over all $p \in \mathcal{P}^3(S_i)$ gives, using interpolation theory in Sobolev spaces (cf. [8]),

$$|(\nabla(I-Q_h)u, \nabla\psi_i)_{S_i}| \leq Ch^3 |u|_{4,S_i} |\psi_i|_1. \tag{24}$$

Substituting this into (22) together with the fact that at most six tetrahedra in $\mathcal{T}_h$ share a given edge, results in the statement.

Corollary 4.3. *Under the same assumptions as in Theorem 4.2 we have that*

$$|u_h - Q_h u|_1 \leq C(u) h^3. \tag{25}$$

Proof. According to (17) and Theorems 4.2 and 4.1 we have

$$|\omega_h|_1^2 \leq |\lambda_h|_1 |\omega_h|_1 + Ch^3 |u|_4 |\beta_h|_1.$$

The rest follows from (15).

A numerical experiment

The supercloseness result (25) was numerically verified on the unit cube $\Omega = (0,1)^3$ and with right-hand side $f(x_1,x_2,x_3) = 3\pi^2 \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$, hence, $u = -f/3\pi^2$. The finite element approximation was computed on uniform grids with mesh sizes $h = 2^{-j}$, $j = 0, \ldots, 4$, meaning that $\overline{\Omega}$ was first subdivided into 2^{3j} congruent subcubes, after which each subcube was partitioned into six tetrahedra according to Section 2.2. To evaluate the right-hand side vector, f was replaced by $Q_h f$, so in fact we solved

$$(\nabla \hat{u}_h, \nabla v_h) = (Q_h f, v_h) \quad \text{instead of} \quad (\nabla u_h, \nabla v_h) = (f, v_h). \tag{26}$$

Subtracting these two equations and applying the Cauchy-Schwarz inequality and the a priori estimate (3) for Q_h gives

$$|u_h - \hat{u}_h|_1^2 = (f - Q_h f, u_h - \hat{u}_h) \leq Ch^3 |f|_3 |u_h - \hat{u}_h|_0 \leq Ch^3 |f|_3 |u_h - \hat{u}_h|_1, \tag{27}$$

where the latter inequality is Friedrichs'. This shows that if $\hat{u}_h$ is superclose to $Q_h u$ in the H^1-seminorm, then so is u_h. In the tabular below we present the relevant numbers. The reduction factors of $|\hat{u}_h - Q_h u|_1$ in the third column seem to confirm the predicted $\mathcal{O}(h^3)$ behavior.

$1/h$	$\lvert\hat{u}_h - Q_h u\rvert_1$	reduction	nr. of dofs
1	$5.278e-1$	–	1
2	$2.654e-1$	1.99	27
4	$5.376e-2$	4.93	343
8	$8.380e-3$	6.40	3375
16	$1.149e-3$	7.29	29791

(28)

5. Further Issues

In this section, we will discuss how to generalize the results of Section 4 to elliptic equations with varying coefficient, and how to post-process the finite element solution. Since both topics do not essentially differ from the two-dimensional case, we will only give an outline and refer the reader who needs a more detailed description to relevant papers.

5.1. Elliptic equations with varying coefficients

Here we will sketch what happens when we consider the more general second order elliptic problem

$$-\mathrm{div}\,(A\nabla u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{29}$$

in which A has Lipschitz continuous entries. By coercivity and continuity of the corresponding bilinear form, there exist positive constants γ, M such that

$$\gamma|\omega_h|_1^2 \le |(A\nabla\omega_h, \nabla(\lambda_h + \beta_h))| \le M|\omega_h|_1|\lambda_h|_1 + |(A\nabla(I - Q_h)u, \nabla\beta_h)|,$$

which is the analogue of (17). The first term on the right-hand side is exactly the same as before apart from the factor M. The second term needs a slightly different treatment than in Theorem 4.2. The main idea is that with m_i the midpoint of the edge e_i and $A_i = A(m_i)$ we have for the bubble basis function ψ_i that

$$\begin{aligned}|(A\nabla(I - Q_h)u, \nabla\psi_i)| &\le |((A - A_i)\nabla(I - Q_h)u, \nabla\psi_i)| + |(A_i\nabla(I - Q_h)u, \nabla\psi_i)|\\ &\le Ch|A|_{1,\infty}Ch^2|u|_{3,S_i}|\psi_i|_1 + |(A_i\nabla(I - Q_h)u, \nabla\psi_i)|.\end{aligned}$$

Since (13) is also valid with a constant matrix in front of the left gradient, we skip the rest of the adapted proof of the following theorem. See [4, 24], where similar techniques are applied in more detail.

Theorem 5.1. *The quadratic finite element approximation u_h of (29) on a regular family of uniform tetrahedral partitions satisfies*

$$|u_h - Q_h u|_1 \le C(u)h^3. \tag{30}$$

5.2. Post-processing of the gradient approximation

Post-processing of ∇u_h will be done in two steps. First, we develop a post-processor for $\nabla Q_h u$ with suitable properties. Then, using either supercloseness bound (25) or (30), we prove that application of the same post-processor to ∇u_h yields a higher order approximation than ∇u_h. The two-dimensional version of the procedure described here originates from [2] and was also described in detail in [16, 5]. The three-dimensional case is not more difficult, hence, we refer also to [2, 16, 5] for all proofs of the statements below.

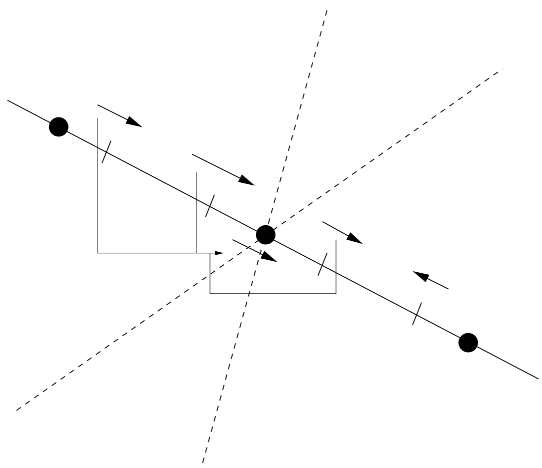

Figure 3. Sampling of three exact tangential derivatives at Gauss points to reconstruct the exact tangential derivative of a cubic polynomial at a nodal point. Three linearly independent reconstructed tangential derivatives at this node result in the exact gradient

Let N be an internal vertex in a uniform partition, and P the union of tetrahedra for which N is a vertex. Let e and e' be edges in P that meet at N and lie in the same direction, and τ a unit vector along $e \cup e'$. Denote differentiation in the direction τ by D_τ. Define G_1, G_2 on e as the unique (Gauss) points at which $D_\tau(p - Q_h p)$ vanishes for all cubic p, and similarly, define G'_1 and G'_2 on e'. Hence, supposing that we are given $Q_h p$ on P, we can *reconstruct* the quadratic function $D_\tau p$ on $e \cup e'$, and in particular at N by interpolation. This procedure is illustrated in Figure 3 below. Since the reconstruction can be done at N for three linearly independent directions, we are able to find $\nabla p(N)$.

Suppose we are given $Q_h u$ on P for some $u \in H_0^1(\Omega) \cap H^4(\Omega)$. Naturally, it will be impossible to reconstruct $\nabla u(N)$ from these data, but by using the same sampling scheme and interpolation as above, we may hope to reconstruct its "quadratic part". Applying the scheme to all vertices N, and then using, for instance, interpolation on those vertices to define values also on the midpoints of edges, it is possible to define a vector quadratic approximation $K_h \nabla Q_h u$ of ∇u globally on Ω, for which

$$|\nabla u - K_h \nabla Q_h u|_0 \le C h^3 |u|_4. \tag{31}$$

Since K_h satisfies $|K_h \nabla v_h|_0 \le C |\nabla v_h|_0$ for all $v_h \in V_{0h}^2$, using the supercloseness (25), a simple derivation based on the triangle inequality gives

$$|\nabla u - K_h \nabla u_h|_0 \le |\nabla u - K_h \nabla Q_h u|_0 + |K_h \nabla (Q_h u - u_h)|_0 \le C(u) h^3. \tag{32}$$

If, in the context of the more general elliptic problem, (30) is used, then the dependence on u in $C(u)$ in the right-hand side of (32) changes accordingly.

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References

[1] A.B. Andreev. Error estimate of type superconvergence of the gradient for quadratic triangular elements. *C. R. Acad. Bulgare Sci.*, **36** (1984), 1179–1182.

[2] A.B. Andreev and R.D. Lazarov, Superconvergence of the gradient for quadratic triangular finite elements, *Numer. Methods Partial Differential Equations*, **4** (1988), 15–32.

[3] F.A. Bornemann, B. Erdmann, and R. Kornhuber, A posteriori error estimates for elliptic problems in two and three space dimensions, *SIAM J. Numer. Anal.*, **33**:3 (1996), 1188-1204.

[4] J.H. Brandts, Superconvergence and a posteriori error estimation for triangular mixed finite elements, *Numer. Math.*, **68**:3 (1994), 311–324.

[5] J.H. Brandts, Superconvergence for triangular order $k = 1$ Raviart-Thomas mixed finite elements and for triangular standard quadratic finite element methods. *Appl. Numer. Anal.*, **34** (2000), 39–58.

[6] J.H. Brandts and M. Křížek, History and future of superconvergence in three-dimensional finite element methods. *Proc. Conf. Finite Element Methods: Three-dimensional Problems*, GAKUTO Internat. Ser. Math. Sci. Appl., 15:24–35, Gakkotosho, Tokyo, 2001.

[7] C.M. Chen, Optimal points of stresses for tetrahedron linear element (in Chinese). *Natur. Sci. J. Xiangtan Univ.*, **3** (1980), 16–24.

[8] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.

[9] J. Douglas, T. Dupont, and M.F. Wheeler, An l^∞ estimate and a superconvergence result for a Galerkin method for elliptical equations based on tensor products of piecewise polynomials, *RAIRO Anal. Numér.*, **8** (1974), 61–66.

[10] J. Douglas and J.E. Roberts, Global estimates for mixed methods for second order elliptic problems, *Math. Comp.*, **44**:169 (1985), 39–52.

[11] J. Douglas and J. Wang, Superconvergence of mixed finite element methods on rectangular domains, *Calcolo*, **26** (1989), 121–133.

[12] R. Durán, Superconvergence for rectangular mixed finite elements. *Numer. Math.*, **58** (1990), 287–298.

[13] R.E. Ewing, R.D. Lazarov, and J. Wang., Superconvergence of the velocity along the Gauss lines in mixed finite element methods, *SIAM J. Numer. Anal.*, **28** (1991), 1015–1029.

[14] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, *Ann. Math.*, **43** (1942), 580–582.

[15] G. Goodsell, Pointwise superconvergence of the gradient for the linear tetrahedral element *Numer. Methods Partial Differential Equations*, **10** (1994), 651–666.

[16] G. Goodsell and J.R. Whiteman, Superconvergence of recovered gradients of piecewise quadratic finite element approximations, Part I, L^2-error estimates. *Numer. Methods Partial Differential Equations*, **7** (1991), 61–83.

[17] G. Goodsell and J.R. Whiteman, Superconvergence of recovered gradients of piecewise quadratic finite element approximations, Part 2, L^∞-error estimates. *Numer. Methods Partial Differential Equations*, **7** (1991), 85–99.

[18] V. Kantchev and R.D. Lazarov, Superconvergence of the gradient of linear finite elements for 3D Poisson equation. Proc. Conf. Optimal Algorithms (B. Sendov ed.), Publ. Bulg. Acad. Sci., Sofia, 172:182, 1986.

[19] M. Křížek and P. Neittaanmäki, On a global superconvergence of the gradient of linear triangular elements, *J. Comput. Appl. Math.*, **18** (1987), 221-233.

[20] M. Křížek and P. Neittaanmäki, Finite element approximation of variational problems and applications, Longman, Harlow, 1990.

[21] M. Křížek, P. Neittaanmäki, and R. Stenberg (Eds.), Finite element methods: superconvergence, post-processing and a posteriori estimates, Proc. Conf. Univ. of Jyväskylä, 1996, Vol. 196 of: Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1998.

[22] N. Levine, Superconvergent recovery of the gradient from piecewise linear finite element approximations, *IMA J. Numer. Anal.*, **5** (1985), 407-427.

[23] Q. Lin and J.C. Xu, Linear finite elements with high accuracy, *J. Comp. Math.*, **3** (1985), 115–133.

[24] L.A. Oganesjan and L.A. Ruhovets, Study of the rate of convergence of variational difference schemes for second-order elliptic equations in a two-dimensional field with a smooth boundary, *Ž. Vyčisl. Mat. i Mat. Fiz.*, **9** (1969), 1102-1120.

[25] A. Pehlivanov, Superconvergence of the gradient for quadratic 3D simplex finite elements, *Proc. Conf. Numer. Methods and Appl., Publ. Bulgar. Acad. Sci., Sofia,* (1989), 362–366.

[26] A.H. Schatz, Pointwise error estimates, superconvergence and extrapolation. Proc. Conf. Finite Element Methods: Superconvergence, Post-processing, and A Posteriori Estimates, Marcel Dekker, New York, (1998), 237–247.

[27] A.H. Schatz, I.H. Sloan, and L.B. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, *SIAM J. Numer. Anal.*, **33** (1996), 505-521.

[28] J. Wang, Superconvergence and extrapolation for mixed finite element methods on rectangular domains, *Math. Comp.,* **56** (1991), 477-503.

[29] M.F. Wheeler and J.R. Whiteman, Superconvergent recovery of gradients on subdomains from piecewise linear finite-element approximations, *Numer. Methods Partial Differential Equations*, **3** (1987), 357–374.

[30] Q.D. Zhu, The derivative good points for the finite element method with 2-degree triangular element (in Chinese), *Natur. Sci. J. Xiangtan Univ.*, **4** (1981), 36–45.

[31] Q.D. Zhu, *Natural inner superconvergence for the finite element method,* In: Proc. China-France Sympos. on the FEM, Bejing, (1982), 935–960.