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ON SOLUTIONS OF MATRIX EQUATION $AXA^T + BYB^T = C$ *1)

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Abstract

By making use of the quotient singular value decomposition (QSVD) of a matrix pair, this paper establishes the necessary and sufficient conditions for the existence of and the expressions for the general solutions of the linear matrix equation $AXA^T + BYB^T = C$ with the unknown X and Y, which may be both symmetric, skew-symmetric, nonnegative definite , positive definite or some cross combinations respectively. Also, the solutions of some optimal problems are derived.

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1. Introduction

It has been of interest for many authors to solve the linear matrix equations under constrained conditions. For the cases of one unknown matrix, such as $AX = B$ or $AXB = C$, the discussions can be seen in literatures [6,11,12,14,16] and [17]. The authors in [2,3,5,13] and [15] considered the solutions of the following linear matrix equation with two unknown matrices

$$AXB + CYD = E, \tag{1.1}$$

which originates from the applications to output feedback pole assignment problems in control theory and from an inverse scattering problem. As special cases, Jameson and Kreindler (1973), Jameson, Kreindler and Lancaster(1992), and Dobovisek (2001) developed the consistent conditions and representations of the solutions of homogeneous equations

$$AX \pm YB = 0 \tag{1.2}$$

with X or Y symmetric(Hermitian), nonnegative definite or positive definite and some cross combinations respectively.

In this paper, we discuss the symmetric matrix equation

$$AXA^T + BYB^T = C \tag{1.3}$$

with the unknown X and Y both symmetric, skew-symmetric, nonnegative definite , positive definite or some cross combinations respectively, which has been studied in [3] for the case X and Y are both symmetric by using the general singular value decomposition (GSVD).

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Let $R^{m\times n}$ denote the set of all real $m\times n$ matrices, $SR^{n\times n}$, $AR^{n\times n}$, $SR_0^{n\times n}$, $SR_+^{n\times n}$ and $OR^{n\times n}$ are the sets of all real symmetric , skew-symmetric, symmetric nonnegative definite, symmetric positive definite and orthogonal $n\times n$ matrices respectively. When the size is clear, we also write $A\geq 0$ or $A>0$ to denote that A is symmetric nonnegative definite or symmetric positive definite matrix, and $A\geq B(A>B)$ means $A-B\geq 0(A-B>0)$. For $A\in R^{m\times n}$, let A^T, A^+ and $R(A)$ be, respectively, the transpose, the Moore-Penrose inverse and the column space of A. $\|\cdot\|_F$ stands for the Frobenius norm of a matrix, $A*B$ represents the Hadamard product of A and B.

Let $A\in R^{m\times n}, B\in R^{m\times p}$, $C\in R^{m\times m}$, $S_1=SR^{n\times n}$, $S_2=SR_0^{n\times n}$,$S_3=SR_+^{n\times n}$, $S_4=AR^{n\times n}$, $T_1=SR^{p\times p}$, $T_2=SR_0^{p\times p}$,$T_3=SR_+^{p\times p}$, $T_4=AR^{p\times p}$, in the next sections the following problems are considered.

Problem I. Given A, B and C, and let

$$L_{ij}=\{[X,Y]: X\in S_i, Y\in T_j, AXA^T+BYB^T=C\}, \tag{1.4}$$

find the consistent conditions for $L_{ij}\neq\emptyset$, and if the conditions hold, find the expression of $[X,Y]\in L_{ij}$.

problem II. Find $[\hat{X},\hat{Y}]\in L_{ii}$, such that

$$\left\|[\hat{X},\hat{Y}]\right\|_F=\left[\|\hat{X}\|_F^2+\|\hat{Y}\|_F^2\right]^{\frac{1}{2}}=\min. \tag{1.5}$$

This paper is organized as follows. In section 2, we introduce some preliminaries and give the solutions of Problem I and Problem II on L_{11}. In section 3, we establish the solutions of Problem I on L_{12} and L_{13} . In section 4, we provide the solutions of Problem I and Problem II on L_{22}, the solutions of Problem I on L_{23} and L_{33}. Finally in section 5, we discuss the solutions of Problem I and Problem II on L_{44}.

2. Preliminaries and the Solution on L_{11}

We first introduce two lemmas about nonnegative definite and positive definite matrices, see [1], [8] and [18, p325].

Lemma 2.1. *Given matrix* $H=\begin{pmatrix} E & F\\ F^T & G\end{pmatrix}$ *with* $E\in R^{n_1\times n_1}$, $F\in R^{n_1\times n_2}$, $G\in R^{n_2\times n_2}$, *then the following statements are equivalent.*

(i) $H\geq 0$;

(ii) $E\geq 0, G-F^TE^+F\geq 0$ *and* $R(F)\subseteq R(E)$;

(iii) $G\geq 0, E-FG^+F^T\geq 0$ *and* $R(F^T)\subseteq R(G)$.

Lemma 2.2. *Given matrix* $H=\begin{pmatrix} E & F\\ F^T & G\end{pmatrix}$ *with* $E\in R^{n_1\times n_1}$, $F\in R^{n_1\times n_2}$,$G\in R^{n_2\times n_2}$, *then the following statements are equivalent.*

(i) $H>0$;

(ii) $E>0, G-F^TE^{-1}F>0$;

(iii) $G>0, E-FG^{-1}F^T>0$.

The quotient singular value decomposition (QSVD) of a matrix pair $[A,B]$ is stated as follows(cf. [4]), compared with the GSVD, it has a simple form.

Lemma 2.3. *Given two matrices* $A\in R^{m\times n}, B\in R^{m\times p}$, *the QSVD of* $[A,B]$ *is*

$$A=M{\textstyle\sum_A}U^T,\quad B=M{\textstyle\sum_B}V^T, \tag{2.1}$$

where M is a nonsingular $m \times m$ matrix, and if we let $k = rank(A, B)$, $r = k - rank(B)$, $s = rank(A) + rank(B) - k$, then in (2.1), $U \in OR^{n\times n}$, $V \in OR^{p\times p}$ and

$$\sum\nolimits_A = \begin{array}{c} \begin{pmatrix} I_A & 0 & 0 \\ 0 & S_{AB} & 0 \\ 0 & 0 & 0_A \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{c} r \\ s \\ k-r-s \\ m-k \end{array}, \\ \begin{array}{ccc} r & s & n-r-s \end{array} \end{array} \qquad \sum\nolimits_B = \begin{array}{c} \begin{pmatrix} 0_B & 0 & 0 \\ 0 & I_{AB} & 0 \\ 0 & 0 & I_B \\ 0 & 0 & 0 \end{pmatrix} \begin{array}{c} r \\ s \\ k-r-s \\ m-k \end{array}, \\ \begin{array}{ccc} p+r-k & s & k-r-s \end{array} \end{array}$$

here I_A, I_B and I_{AB} are identity matrices, 0_A and 0_B are zero matrices, and

$$S_{AB} = diag(\sigma_1, \cdots, \sigma_s), \sigma_i > 0 (i = 1, \cdots, s).$$

The solutions on L_{11} about Problem I and II can be seen in [3, Theorem 3.1], which was derived by GSVD, now we state the similar results proved by QSVD, the proof is similar to that of [3, Theorem 3.1] , so we omit the proof.

Theorem 2.1 ($X^T = X, Y^T = Y$). *Let the QSVD decomposition of the matrix pair $[A, B]$ be of the form (2.1). Partition $M^{-1}CM^{-T}$ into the following form:*

$$M^{-1}CM^{-T} = \begin{array}{c} \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{pmatrix} \begin{array}{c} r \\ s \\ k-r-s \\ m-k \end{array}, \\ \begin{array}{cccc} r & s & k-r-s & m-k \end{array} \end{array} \tag{2.2}$$

then the set L_{11} is nonempty if and only if

$$C^T = C, C_{13} = 0, C_{14} = 0, C_{24} = 0, C_{34} = 0, C_{44} = 0, \tag{2.3}$$

when the condition (2.3) is satisfied, the general expression of $[X, Y] \in L_{11}$ is

$$X = U \begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & X_{13} \\ S_{AB}^{-1}C_{12}^T & S_{AB}^{-1}(C_{22} - Y_{22})S_{AB}^{-1} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{pmatrix} U^T, \tag{2.4}$$

$$Y = V \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & C_{23} \\ Y_{13}^T & C_{23}^T & C_{33} \end{pmatrix} V^T, \tag{2.5}$$

where

$$X_{13} \in R^{r\times(n-r-s)}, X_{23} \in R^{s\times(n-r-s)}, X_{33} \in SR^{(n-r-s)\times(n-r-s)},$$
$$Y_{11} \in SR^{(p+r-k)\times(p+r-k)}, Y_{22} \in SR^{s\times s}, Y_{12} \in R^{(p+r-k)\times s}, Y_{13} \in R^{(p+r-k)\times(k-r-s)}$$

are arbitrary matrices.

In L_{11}, there exists a unique $[\hat{X}, \hat{Y}]$ that makes (1.5) hold, and $\hat{X}, \hat{Y}$ can be expressed as

$$\hat{X} = U \begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & 0 \\ S_{AB}^{-1}C_{12}^T & \Psi * (S_{AB}C_{22}S_{AB}) & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T,$$

$$\hat{Y} = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi * C_{22} & C_{23} \\ 0 & C_{23}^T & C_{33} \end{pmatrix} V^T, \tag{2.6}$$

here

$$\Psi=(\psi_{ij})\in R^{s\times s},\psi_{ij}=\frac{1}{1+\sigma_i^2\sigma_j^2},1\le i,j\le s. \tag{2.7}$$

3. The Solutions on L_{12} and L_{13}

In this section we discuss the solutions of Problem I on L_{12} and L_{13} respectively.

Theorem 3.1 **($X^T=X, Y\ge 0$).** *Let the QSVD decomposition of the matrix pair $[A,B]$ be of the form (2.1) and $M^{-1}CM^{-T}$ be of the form (2.2). Suppose the condition (2.3) hold, then the set L_{12} is nonempty if and only if*

$$C_{33}\ge 0,\quad R(C_{23}^T)\subseteq R(C_{33}). \tag{3.1}$$

When the condition (3.1) is satisfied, the general expression of $[X,Y]\in L_{12}$ is given by (2.4) and (2.5), where $X_{13},X_{23},X_{33}^T=X_{33}$ are arbitrary submatrices with appropriate sizes, and $Y_{11}^T=Y_{11},Y_{12},Y_{13},Y_{22}^T=Y_{22}$ are parameter matrices with appropriate sizes which satisfy

$$\begin{cases} R(Y_{13}^T)\subseteq R(C_{33}),\\ R(Y_1^T)\subseteq R(Y_2),\\ Y_2\ge 0,\\ Y_0-Y_1Y_2^+Y_1^T\ge 0,\end{cases} \tag{3.2}$$

where

$$\begin{cases} Y_0=Y_{11}-Y_{13}C_{33}^+Y_{13}^T,\\ Y_1=Y_{12}-Y_{23}C_{33}^+C_{23}^T,\\ Y_2=Y_{22}-C_{23}C_{33}^+C_{23}^T.\end{cases} \tag{3.3}$$

Remark. Since $L_{12}\subseteq L_{11}$, therefore condition (2.3) must be satisfied in Theorem 3.1.

Proof. The "if" part. Denote

$$X_0=U\begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & 0\\ S_{AB}^{-1}C_{12}^T & S_{AB}^{-1}(C_{22}-C_{23}C_{33}^+C_{23}^T)S_{AB}^{-1} & 0\\ 0 & 0 & 0\end{pmatrix}U^T,$$

$$Y_0=V\begin{pmatrix} 0 & 0 & 0\\ 0 & C_{23}C_{33}^+C_{23}^T & C_{23}\\ 0 & C_{23}^T & C_{33}\end{pmatrix}V^T, \tag{3.4}$$

then in view of Lemma 1, $X^T=X$, $Y^T\ge 0$ and $[X,Y]\in L_{12}$, so L_{12} is nonempty.

The "only if" part. Let $[X,Y]\in L_{12}$, then $[X,Y]\in L_{11}$, so Y has the form (2.5). Because $Y\ge 0$ implies $\begin{pmatrix} Y_{22} & C_{23}\\ C_{23}^T & C_{33}\end{pmatrix}\ge 0$, therefore (3.1) follows by Lemma 2.1.

When condition (3.1) is met, L_{12} is nonempty. X,Y can be expressed by (2.4) and (2.5). By Lemma 2.1, $Y\ge 0$ implies $R(Y_{13}^T)\subseteq R(C_{33}),C_{33}\ge 0$ and

$$\begin{pmatrix} Y_0 & Y_1\\ Y_1^T & Y_2\end{pmatrix}=\begin{pmatrix} Y_{11}-Y_{13}C_{33}^+Y_{13}^T & Y_{12}-Y_{13}C_{33}^+C_{23}^T\\ Y_{12}^T-C_{23}C_{33}^+Y_{13}^T & Y_{22}-C_{23}C_{33}^+C_{23}^T\end{pmatrix}\ge 0,$$

therefore (3.2) follows by Lemma 2.1 again. The proof is complete.

Theorem 3.2 **($X^T=X, Y>0$).** *Let the QSVD decomposition of the matrix pair $[A,B]$ be of the form (2.1) and $M^{-1}CM^{-T}$ be of the form (2.2). Suppose the conditions in (2.3) hold, then the set L_{13} is nonempty if and only if*

$$C_{33}>0. \tag{3.5}$$

When the condition (3.5) is satisfied, the general expression of $[X, Y] \in L_{13}$ *is given by (2.4) and (2.5), where* $X_{13}, X_{23}, X_{33}^T = X_{33}$ *are arbitrary submatrices with appropriate sizes, and* $Y_{11}^T = Y_{11}, Y_{12}, Y_{13}, Y_{22}^T = Y_{22}$ *are parameter matrices with appropriate sizes which satisfy*

$$\begin{cases} Y_2 > 0, \\ Y_0 - Y_1 Y_2^{-1} Y_1^T > 0, \end{cases} \tag{3.6}$$

with(3.3), here $C_{33}^+ = C_{33}^{-1}$.

4. The Solutions on L_{22}, L_{23} and L_{24}

In this section we discuss the solutions of Problem I and II on L_{22} and the solutions of Problem I on L_{23} and L_{33} respectively.

Theorem 4.1 **($X \geq 0, Y \geq 0$).** *Let the QSVD decomposition of the matrix pair* $[A, B]$ *be of the form (2.1) and* $M^{-1}CM^{-T}$ *be of the form (2.2). Suppose the condition (2.3) hold, then*

(i) The necessary and sufficient conditions for the set L_{22} *is nonempty is that*

$$C \geq 0, \quad R(C_{23}^T) \subseteq R(C_{33}), \quad R(C_{12}S_{AB}^{-1}) \subseteq R(C_{11}),$$

$$C_{22} \geq C_{12}^T C_{11}^+ C_{12} + C_{23} C_{33}^+ C_{23}^T. \tag{4.1}$$

When the condition (4.1) is satisfied, the general expression of $[X, Y] \in L_{22}$ *is given by (2.4) and (2.5), where*

$$Y_{22} = C_{23} C_{33}^+ C_{23}^T + G, \quad 0 \leq G \leq C_{22} - C_{12}^T C_{11}^+ C_{12} - C_{23} C_{33}^+ C_{23}^T. \tag{4.2}$$

$Y_{11}^T = Y_{11}, Y_{12}, Y_{13}$ *are parameter matrices with appropriate sizes which satisfy (3.2) with (3.3), and* $X_{13}, X_{23}, X_{33}^T = X_{33}$ *are parameter matrices with appropriate sizes which satisfy*

$$\begin{cases} R(X_{13}) \subseteq R(C_{11}), \\ R(Z_1^T) \subseteq R(Z_2), \\ Z_2 \geq 0, \\ Z_0 - Z_1 Z_2^+ Z_1^T \geq 0, \end{cases} \tag{4.3}$$

where

$$\begin{cases} Z_0 = S_{AB}^{-1}(C_{22} - Y_{22})S_{AB}^{-1} - S_{AB}^{-1} C_{12}^T C_{11}^+ C_{12} S_{AB}^{-1}, \\ Z_1 = X_{23} - S_{AB}^{-1} C_{12}^T C_{11}^+ X_{13}, \\ Z_2 = X_{33} - X_{13}^T C_{11}^+ X_{13}. \end{cases} \tag{4.4}$$

(ii) In L_{22}, *there exists a unique* $[\hat{X}, \hat{Y}]$ *that makes (1.5) hold, and* $\hat{X}, \hat{Y}$ *can be expressed as*

$$\hat{X} = U \begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & 0 \\ S_{AB}^{-1}C_{12}^T & S_{AB}^{-1}(C_{22} - C_{23}C_{33}^+C_{23}^T - \hat{G})S_{AB}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} U^T,$$

$$\hat{Y} = V \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{23}C_{33}^+C_{23}^T + \hat{G} & C_{23} \\ 0 & C_{23}^T & C_{33} \end{pmatrix} V^T, \tag{4.5}$$

where $\hat{G}$ *is the unique solution of following optimal problem*

$$\|S_{AB}^{-1} G S_{AB}^{-1} + S_{AB}^{-1}(C_{23}C_{33}^+C_{23}^T - C_{22})S_{AB}^{-1}\|_F^2 + \|G + C_{23}C_{33}^+C_{23}^T\|_F^2 = min, \tag{4.6}$$

for $0 \leq G \leq C_{22} - C_{12}^T C_{11}^+ C_{12} - C_{23} C_{33}^+ C_{23}^T$.

Proof. (i) Necessity. Suppose $[X,Y]\in L_{22}$, then $[X,Y]\in L_{11}$, and X has the form (2.4), Y has the form (2.5). $X\geq 0, Y\geq 0$ imply that $C\geq 0$ by (1.3) and

$$\begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} \\ S_{AB}^{-1}C_{12}^T & S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1} \end{pmatrix}\geq 0, \quad \begin{pmatrix} Y_{22} & C_{23} \\ C_{23}^T & C_{33} \end{pmatrix}\geq 0,$$

therefore by Lemma 2.1,

$$\begin{cases} R(C_{12}S_{AB}^{-1})\subseteq R(C_{11}), \\ S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1}-S_{AB}^{-1}C_{12}^TC_{11}^+C_{12}S_{AB}^{-1}\geq 0, \\ R(C_{23}^T)\subseteq R(C_{33}), \\ Y_{22}-C_{23}C_{33}^+C_{23}^T\geq 0. \end{cases} \tag{4.7}$$

Let

$$Y_{22}=C_{23}C_{33}^+C_{23}^T+G, \tag{4.8}$$

from (4.7) we know $G\geq 0$ and $C_{22}-Y_{22}-C_{12}^TC_{11}^+C_{12}\geq 0$, i.e.,

$$C_{22}-C_{12}^TC_{11}^+C_{12}-C_{23}C_{33}^+C_{23}^T\geq G\geq 0.$$

Sufficiency. Denote X_0, Y_0 by (3.4), then in view of Lemma 2.1 and condition (4.1), we know $X_0\geq 0, Y_0\geq 0$ and $[X_0,Y_0]\in L_{22}$, so L_{22} is nonempty.

When condition (4.1) is met, the expression (2.5) and the condition (3.2) about Y follows, the proof is similar to that of Theorem 3.1. While in (2.4), $X\geq 0$ implies that $R(X_{13})\subseteq R(C_{11})$ and $\begin{pmatrix} Z_0 & Z_1 \\ Z_1^T & Z_2 \end{pmatrix}\geq 0$, therefore (4.3) follows.

(ii) When $[X,Y]\in L_{22}$, from (2.4), (2.5) and (4.2), we have

$$\begin{aligned} &\|X\|_F^2+\|Y\|_F^2 \\ &=\alpha_0+\|S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1}\|_F^2+\|Y_{22}\|_F^2+2\|X_{13}\|_F^2 \\ &+2\|X_{23}\|_F^2+\|X_{33}\|_F^2+\|Y_{11}\|_F^2+2\|Y_{12}\|_F^2+2\|Y_{13}\|_F^2 \end{aligned}$$

where α_0 is a constant number, therefore $\|X\|_F^2+\|Y\|_F^2=min$ if and only if $X_{13}=0, X_{23}=0, X_{33}=0$, $Y_{11}=0, Y_{12}=0, Y_{13}=0$ and

$$\|S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1}\|_F^2+\|Y_{22}\|_F^2=min,$$

this is the optimal problem (4.6), notice that the set

$$\{G\in SR^{s\times s}|\ \ 0\leq G\leq C_{22}-C_{12}^TC_{11}^+C_{12}-C_{23}C_{33}^+C_{23}^T\}$$

is a closed convex set, therefore the optimal problem (4.6) has a unique solution.

The proof of the following results is similar to that of Theorem 4.1, so we omit the process.

Theorem 4.2 ($X\geq 0, Y>0$). *Let the QSVD decomposition of the matrix pair $[A,B]$ be of the form (2.1) and $M^{-1}CM^{-T}$ be of the form (2.2). Suppose the condition (2.3) hold and $C\geq 0$, then the necessary and sufficient conditions for the set L_{23} is nonempty is that*

$$\begin{aligned} &C_{33}>0, R(C_{12}S_{AB}^{-1})\subseteq R(C_{11}), \\ &C_{22}\geq C_{12}^TC_{11}^+C_{12}+C_{23}C_{33}^{-1}C_{23}^T. \end{aligned} \tag{4.9}$$

When the condition (4.9) is satisfied, the general expression of $[X,Y]\in L_{23}$ is given by (2.4) and (2.5), where

$$Y_{22}=C_{23}C_{33}^{-1}C_{23}^T+G, \quad 0<G\leq C_{22}-C_{12}^TC_{11}^+C_{12}-C_{23}C_{33}^{-1}C_{23}^T. \tag{4.10}$$

$Y_{11}^T = Y_{11}, Y_{12}, Y_{13}$ are parameter matrices with appropriate sizes which satisfy (3.6) with (3.3), and $X_{13}, X_{23}, X_{33}^T = X_{33}$ are parameter matrices with appropriate sizes which satisfy (4.3) with (4.4).

Theorem 4.3 ($X > 0, Y > 0$). *Let the QSVD decomposition of the matrix pair $[A, B]$ be of the form (2.1) and $M^{-1}CM^{-T}$ be of the form (2.2). Suppose the condition (2.3) hold and $C \geq 0$, then the necessary and sufficient conditions for the set L_{33} is nonempty is that*

$$C_{11} > 0, C_{33} > 0,$$

$$C_{22} > C_{12}^T C_{11}^{-1} C_{12} + C_{23} C_{33}^{-1} C_{23}^T. \tag{4.11}$$

When the condition (4.11) is satisfied, the general expression of $[X, Y] \in L_{33}$ is given by (2.4) and (2.5), where

$$Y_{22} = C_{23} C_{33}^{-1} C_{23}^T + G, \quad 0 < G < C_{22} - C_{12}^T C_{11}^{-1} C_{12} - C_{23} C_{33}^{-1} C_{23}^T. \tag{4.12}$$

$Y_{11}^T = Y_{11}, Y_{12}, Y_{13}$ are parameter matrices with appropriate sizes which satisfy (3.6) with (3.3), and $X_{13}, X_{23}, X_{33}^T = X_{33}$ are parameter matrices with appropriate sizes which satisfy

$$\begin{cases} Z_2 > 0, \\ Z_0 - Z_1 Z_2^{-1} Z_1^T > 0, \end{cases} \tag{4.13}$$

with (4.4), here $C_{11}^+ = C_{11}^{-1}$.

5. The Solutions on L_{44}

Let us first introduce a lemma.

Lemma 5.1. *Given $G, H \in R^{r\times r}$, $\Lambda = diag(\lambda_1, \cdots, \lambda_r) > 0$, $P = diag(r_1, \cdots, r_r) > 0$, there exists a unique matrix $\hat{S} \in AR^{r\times r}$, such that*

$$\|\Lambda\hat{S}\Lambda - G\|_F^2 + \|P\hat{S}P - H\|_F^2$$

$$= \min_{S\in AR^{r\times r}} \left(\|\Lambda S\Lambda - G\|_F^2 + \|PSP - H\|_F^2\right), \tag{5.1}$$

and $\hat{S}$ can be expressed as

$$\hat{S} = \frac{1}{2}\phi * \left(\Lambda(G - G^T)\Lambda + P(H - H^T)P\right), \tag{5.2}$$

where

$$\phi = (\varphi_{ij}) \in R^{r\times r}, \quad \varphi_{ij} = \frac{1}{\lambda_i^2\lambda_j^2 + r_i^2 r_j^2}, 1 \leq i, j \leq r. \tag{5.3}$$

Proof. For $S = (s_{ij}) \in AR^{r\times r}, G = (g_{ij}) \in R^{r\times r}$, and $H = (h_{ij}) \in R^{r\times r}$, we have

$$\begin{aligned} & \|\Lambda S\Lambda - G\|_F^2 + \|PSP - H\|_F^2 \\ = & \sum_{i,j} \left[(\lambda_i\lambda_j s_{ij} - g_{ij})^2 + (r_i r_j s_{ij} - h_{ij})^2\right] \\ = & \sum_{1\leq i\leq r} \left[g_{ii}^2 + h_{ii}^2\right] + \sum_{1\leq i<j\leq r} \{2(\lambda_i^2\lambda_j^2 + r_i^2 r_j^2)s_{ij}^2 + g_{ij}^2 + g_{ji}^2 \\ & + h_{ij}^2 + h_{ji}^2 + 2[\lambda_i\lambda_j(g_{ji} - g_{ij}) + r_i r_j(h_{ji} - h_{ij})]s_{ij}\}. \end{aligned} \tag{5.4}$$

From (5.4), it is easy to obtain a unique solution $\hat{S} = (\hat{s}_{ij}) \in AR^{r\times r}$ of (5.1), and

$$\hat{s}_{ij} = \frac{1}{2}\frac{1}{\lambda_i^2\lambda_j^2 + r_i^2r_j^2}[\lambda_i(g_{ij} - g_{ji})\lambda_j + r_i(h_{ij} - h_{ji})r_j], 1 \le i, j \le r, \tag{5.5}$$

Thus (5.2) is proved.

The following theorem establishes the necessary and sufficient conditions for the existence of the solutions of Problem I and Problem II on L_{44}, and under these conditions, the expressions of the solutions are obtained.

Theorem 5.1. *Let the QSVD decomposition of the matrix pair $[A, B]$ be of the form (2.1) and $M^{-1}CM^{-T}$ be of the form (2.2). Then the set L_{44} is nonempty if and only if*

$$C^T = -C, C_{13} = 0, C_{14} = 0, C_{24} = 0, C_{34} = 0, C_{44} = 0. \tag{5.6}$$

When the condition (5.6) is satisfied, the general expression of $[X, Y] \in L_{44}$ is

$$X = U\begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & X_{13} \\ -S_{AB}^{-1}C_{12}^T & S_{AB}^{-1}(C_{22} - Y_{22})S_{AB}^{-1} & X_{23} \\ -X_{13}^T & -X_{23}^T & X_{33}\end{pmatrix}U^T,$$

$$Y = V\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ -Y_{12}^T & Y_{22} & C_{23} \\ -Y_{13}^T & -C_{23}^T & C_{33}\end{pmatrix}V^T, \tag{5.7}$$

where

$$X_{13} \in R^{r\times(n-r-s)}, X_{23} \in R^{s\times(n-r-s)}, X_{33} \in AR^{(n-r-s)\times(n-r-s)},$$
$$Y_{11} \in AR^{(p+r-k)\times(p+r-k)}, Y_{12} \in R^{(p+r-k)\times s}, Y_{13} \in R^{(p+r-k)\times(k-r-s)}, Y_{22} \in AR^{s\times s}$$

are arbitrary matrices.

In L_{44}, there exists a unique $[\hat{X}, \hat{Y}]$ that makes (1.5) hold, and $\hat{X}, \hat{Y}$ can be expressed as

$$\hat{X} = U\begin{pmatrix} C_{11} & C_{12}S_{AB}^{-1} & 0 \\ -S_{AB}^{-1}C_{12}^T & \Psi * (S_{AB}C_{22}S_{AB}) & 0 \\ 0 & 0 & 0\end{pmatrix}U^T,$$

$$\hat{Y} = V\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Psi * C_{22} & C_{23} \\ 0 & -C_{23}^T & C_{33}\end{pmatrix}V^T, \tag{5.8}$$

here

$$\Psi = (\psi_{ij}) \in R^{s\times s}, \psi_{ij} = \frac{1}{1 + \sigma_i^2\sigma_j^2}, 1 \le i, j \le s, \tag{5.9}$$

Proof. If the set L_{44} is nonempty, obviously, C must be skew-symmetric. For $[X, Y] \in L_{44}$, according to Lemma 2.3, we have

$$M\Sigma_AU^TXU\Sigma_A^TM^T + M\Sigma_BV^TYV\Sigma_B^TM^T = C, \tag{5.10}$$

it is equivalent to

$$\Sigma_AU^TXU\Sigma_A^T + \Sigma_BV^TYV\Sigma_B^T = M^{-1}CM^{-T}. \tag{5.11}$$

Write

$$U^TXU = \begin{array}{cc}\begin{pmatrix} X_{11} & X_{12} & X_{13} \\ -X_{12}^T & X_{22} & X_{23} \\ -X_{13}^T & -X_{23}^T & X_{33}\end{pmatrix} & \begin{array}{c} r \\ s \\ n-r-s\end{array}\end{array}, \quad V^TYV = \begin{array}{cc}\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ -Y_{12}^T & Y_{22} & Y_{23} \\ -Y_{13}^T & -Y_{23}^T & Y_{33}\end{pmatrix} & \begin{array}{c} p+r-k \\ s \\ k-r-s\end{array}\end{array},$$

$$r \quad s \quad n-r-s \qquad\qquad p+r-k \quad s \quad k-r-s \tag{5.12}$$

by (5.11), (5.12) and (2.2), we obtain

$$\begin{pmatrix} X_{11} & X_{12}S_{AB} & 0 & 0 \\ -S_{AB}X_{12}^T & S_{AB}X_{22}S_{AB}+Y_{22} & Y_{23} & 0 \\ 0 & -Y_{23}^T & Y_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ -C_{12}^T & C_{22} & C_{23} & C_{24} \\ -C_{13}^T & -C_{23}^T & C_{33} & C_{34} \\ -C_{14}^T & -C_{24}^T & -C_{34}^T & C_{44} \end{pmatrix}, \tag{5.13}$$

Therefore

$$X_{11} = C_{11}, X_{12} = C_{12}S_{AB}^{-1}, Y_{23} = C_{23}, Y_{33} = C_{33}, \tag{5.14}$$

$$C_{13} = 0, C_{14} = 0, C_{24} = 0, C_{34} = 0, C_{44} = 0, \tag{5.15}$$

$$S_{AB}X_{22}S_{AB} + Y_{22} = C_{22}. \tag{5.16}$$

Thus when L_{44} is nonempty, the condition (5.6) hold and the expression (5.7) of L_{44} is obtained.

In addition, L_{44} is a closed convex set, so there exists a unique $[\hat{X}, \hat{Y}] \in L_{44}$ that makes (1.5) hold. When $[X, Y] \in L_{44}$,

$$\begin{aligned} &\|X\|_F^2 + \|Y\|_F^2 \\ &= \beta_0 + \|S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1}\|_F^2 + \|Y_{22}\|_F^2 + 2\|X_{13}\|_F^2 \\ &+2\|X_{23}\|_F^2 + \|X_{33}\|_F^2 + \|Y_{11}\|_F^2 + 2\|Y_{12}\|_F^2 + 2\|Y_{13}\|_F^2 \end{aligned}$$

where β_0 is a constant number, therefore $\|X\|_F^2 + \|Y\|_F^2 = min$ if and only if $X_{13} = 0$,$X_{23} = 0, X_{33} = 0$, $Y_{11} = 0, Y_{12} = 0, Y_{13} = 0$ and

$$\|S_{AB}^{-1}(C_{22}-Y_{22})S_{AB}^{-1}\|_F^2 + \|Y_{22}\|_F^2 = min$$

for $Y_{22} \in AR^{s\times s}$. Therefore, by Lemma 5.1, we obtain Y_{22}, and then $\hat{X}, \hat{Y}$. Theorem 5.1 is proved.

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