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CONVERGENCE RATE OF A GENERALIZED ADDITIVE SCHWARZ ALGORITHM *1)

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Abstract

The convergence rate of a generalized additive Schwarz algorithm for solving boundary value problems of elliptic partial differential equations is studied. A quantitative analysis of the convergence rate is given for the model Dirichlet problem. It will be shown that a greater acceleration of the algorithm can be obtained by choosing the parameter suitably. Some numerical tests are also presented in this paper.

Mathematics subject classification: 35J67, 65N55.
Key words: Schwarz additive algorithm, Convergence rate, Dirichlet problem.

1. Introduction

A classical mathematical approach, the Schwarz alternating algorithm (see, e.g., in [12]), appears to offer promise for the parallel solution of the very large systems of linear or nonlinear elliptic problems in elasticity, fluid dynamics or other important areas. Its advantage in parallelism, wide applicability and great flexibility in implementation make Schwarz algorithm a competitive technique in parallel computations. As a result, Schwarz algorithms have attracted much attention from researchers in the field of parallel computation as well as theoreticians. The early contributions relating to Schwarz algorithms can also be seen in [9, 10, 14]. Some recent progress in this field can be seen in [8, 11, 13] and the references therein.

A generalized additive Schwarz algorithm is presented in the paper. The approach uses robin condition on the inner boundaries of the subproblems. The use of Robin boundary condition as interfacial transmission conditions in domain decomposition was introduced by P. L. Lions in [7]. Various aspects of such methods have been discussed in [1, 2, 3, 4, 5, 6, 15, 16]. Numerical experiments reported in [1, 2, 3, 6, 15, 16] show that the generalized Schwarz algorithms with appropriate parameters can accelerate the convergence dramatically. The aim of this paper is to study the convergence rate of a generalized additive Schwarz algorithm for model problems. A quantitative analysis of the convergence rate for model Dirichlet problems and some numerical results are presented in this paper.

The paper is organized as follows: In Section 2, we introduce a generalized additive Schwarz algorithm. In Sections 3 and 4, we discuss the convergence rate of the generalized additive Schwarz algorithm for one and two dimensional Dirichlet problem respectively. Finally, in Section 5, we give some preliminary numerical results.

2. A Generalized Additive Schwarz Algorithms

Before proceeding, let us introduce a generalized version of additive Schwarz algorithm. We consider the Dirichlet problem for a second order elliptic operator $\mathcal{L}$:

$$\begin{cases} \mathcal{L}u(x) = 0, & x \in \Omega, \\ u(x) = \psi(x), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

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where, Ω is a bounded domain in d-dimensional space ($d = 1, 2, 3$), $\partial\Omega$ is the boundary of Ω, ψ is a given function of $L^2(\Omega)$ and $x = (x_1, \ldots, x_d)$ is the independent variable. To simplify discussion, we consider a case for two subdomians. We also assume that the solution to this problem exists and is unique.

We decompose the solution domain Ω into two overlapping subdomains Ω_1 and Ω_2. That is $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset$. Denote by $\Gamma_1 = \partial\Omega_1 \cap \Omega$ and $\Gamma_2 = \partial\Omega_2 \cap \Omega$ the inner boundaries of Ω_1 and Ω_2, respectively.

Denote u_1 and u_2 as the restrictions of the solution u of problem (2.1) on subdomain $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively. Then, the following couplings

$$g_1(u_1)|_{\Gamma_1} = g_1(u_2)|_{\Gamma_1}$$

and

$$g_2(u_2)|_{\Gamma_2} = g_2(u_1)|_{\Gamma_2}$$

are true on the inner boundaries Γ_1 and Γ_2, where

$$g_i(v) = \alpha_i v + \beta_i \frac{\partial v}{\partial n_i}, \qquad i = 1, 2. \tag{2.2}$$

Here, for $i = 1, 2$, $\alpha_i \in [0, 1]$, $\beta_i = 1 - \alpha_i$, and n_i is the outer unit normal direction of $\partial\Omega_i$.

With these new couplings we can formulate two coupled subproblems as follows:

$$\begin{cases} \mathcal{L}u_1(x) = 0, & x \in \Omega_1, \\ u_1(x) = \psi(x), & x \in \partial\Omega_1 \cap \partial\Omega, \\ g_1(u_1(x)) = g_1(u_2(x)), & x \in \Gamma_1, \end{cases} \tag{2.3}$$

$$\begin{cases} \mathcal{L}u_2(x) = 0, & x \in \Omega_2, \\ u_2(x) = \psi(x), & x \in \partial\Omega_2 \cap \partial\Omega, \\ g_2(u_2(x)) = g_2(u_1(x)), & x \in \Gamma_2. \end{cases} \tag{2.4}$$

We have the following result (see in [15]).

Theorem 2.1. *If the boundary value problem*

$$\begin{cases} \mathcal{L}w(x) = 0, & x \in \Omega_{12}, \\ w(x) = 0, & x \in \partial\Omega_{12} \backslash (\Gamma_1 \cap \Gamma_2), \\ g_1(w(x)) = 0, & x \in \Gamma_1, \\ g_2(w(x)) = 0, & x \in \Gamma_2 \end{cases} \tag{2.5}$$

has only trivial solution and the solutions u_1, u_2 of (2.3) and (2.4) exist, then

1. $u_1(x) = u_2(x)$ *if* $x \in \Omega_{12}$.
2. $u(x) = u_1(x)$ *if* $x \in \Omega_1$ *and* $u(x) = u_2(x)$ *if* $x \in \Omega_2$,

where u is the solution of (2.1).

The original form of additive Schwarz algorithm consists of the following steps.

Let u^0 be an initial function defined on $\bar{\Omega}$ such that $u^0 - \psi$ vanishes on $\partial\Omega$. Set $u_1^0 = u^0|_{\overline{\Omega}_1}$, $u_2^0 = u^0|_{\overline{\Omega}_2}$. For $k > 0$, we define independently the following two sequences respectively:

$$\begin{cases} \mathcal{L}u_1^{k+1}(x) = 0, & x \in \Omega_1, \\ u_1^{k+1}(x) = \psi(x), & x \in \partial\Omega_1 \cap \partial\Omega, \\ u_1^{k+1}(x) = u_2^k(x), & x \in \Gamma_1, \end{cases} \tag{2.6}$$

$$\begin{cases} \mathcal{L}u_2^{k+1}(x) = 0, & x \in \Omega_2, \\ u_2^{k+1}(x) = \psi(x), & x \in \partial\Omega_2 \cap \partial\Omega, \\ u_2^{k+1}(x) = u_1^k(x), & x \in \Gamma_2. \end{cases} \tag{2.7}$$

By Theorem 2.1, in additive Schwarz algorithm, (2.6) and (2.7) could be modified by simply replacing Dirichlet inner boundary conditions by Robin ones. In such a way we obtain a generalized additive Schwarz algorithm. In the modified approach, instead of solving subproblem (2.6) and (2.7), we solve the following mixed boundary value problems

$$\begin{cases} \mathcal{L}u_1^{k+1}(x) = 0, & x \in \Omega_1, \\ u_1^{k+1}(x) = \psi(x), & x \in \partial\Omega_1 \cap \partial\Omega, \\ g_1(u_1^{k+1}(x)) = g_1(u_2^k(x)), & x \in \Gamma_1, \end{cases} \tag{2.8}$$

$$\begin{cases} \mathcal{L}u_2^{k+1}(x) = 0, & x \in \Omega_2, \\ u_2^{k+1}(x) = \psi(x), & x \in \partial\Omega_2 \cap \partial\Omega, \\ g_2(u_2^{k+1}(x)) = g_2(u_1^k(x)), & x \in \Gamma_2. \end{cases} \tag{2.9}$$

Obviously, if we let $\alpha_i = 1$, $\beta_i = 0$, $i = 1, 2$, the generalized additive Schwarz algorithm is correspond to classical additive algorithm.

In the sequel, we will analyze the convergence rate of above mentioned generalized additive Schwarz algorithm for simple model problems in one and two dimensional spaces.

3. Convergence Rate for One-dimensional Problem

In this section, we will discuss the convergence of generalized additive Schwarz algorithm when it is applied to approximate the solution of the following model problem:

$$\begin{cases} -u'' = f, & \text{in } (0,1), \\ u(0) = \phi_1, u(1) = \phi_2. \end{cases} \tag{3.1}$$

A quantitative analysis of the convergence rate is given by simple deduction. For this particular problem, we decompose domain $\Omega = (0, 1)$ into two subdomains as

$$\Omega_1 = (0, l), \qquad \Omega_2 = (1 - l', 1), \tag{3.2}$$

where the constants l and l' belong to $(0, 1)$ that satisfy $l + l' > 1$ (see Fig. 1). Obviously, $\delta = l - (1 - l') = l + l' - 1 > 0$ is the overlapping size.

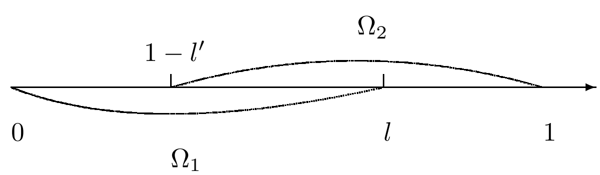

Figure 1: Domain Decomposition of $(0, 1)$.

Let u be the exact solution of problem (3.1). It is easy to verify that the solutions of (2.8) and (2.9) in this case are

$$u_1^{k+1}(x) = u(x) + [g_1(u_2^k(l)) - g_1(u(l))] \cdot \frac{x}{\alpha_1 l + \beta_1} \tag{3.3}$$

and

$$u_2^{k+1}(x) = u(x) + [g_2(u_1^k(1-l')) - g_2(u(1-l'))] \cdot \frac{1-x}{\alpha_2 l' + \beta_2}, \tag{3.4}$$

respectively. By (3.3), we have

$$u_1^{k+1}(1-l') = u(1-l') + [g_1(u_2^k(l)) - g_1(u(l))] \cdot \frac{1-l'}{\alpha_1 l + \beta_1}$$

and

$$(u_1^{k+1}(x))'|_{1-l'} = (u(x))'|_{1-l'} + [g_1(u_2^k(l)) - g_1(u(l))] \cdot \frac{1}{\alpha_1 l + \beta_1}.$$

Therefore,

$$g_2(u_1^{k+1}(1-l')) - g_2(u(1-l')) = [g_1(u_2^k(l)) - g_1(u(l))] \cdot \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1}. \tag{3.5}$$

Similarly, by (3.4), we have

$$u_2^{k+1}(l) = u(l) + [g_2(u_1^k(1-l')) - g_2(u(1-l'))] \cdot \frac{1-l}{\alpha_2 l' + \beta_2},$$

$$(u_2^{k+1}(x))'|_l = (u(x))'|_l - [g_2(u_1^k(1-l')) - g_2(u(1-l'))] \cdot \frac{1}{\alpha_2 l' + \beta_2},$$

and then

$$g_1(u_2^{k+1}(l)) - g_1(u(l)) = [g_2(u_1^k(1-l')) - g_2(u(1-l'))] \cdot \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2}. \tag{3.6}$$

Let $\epsilon_i^k = u_i^k - u|_{\overline{\Omega}_i}$, $i = 1, 2$. Then by (3.3), (3.6) and (3.5), it follows that

$$\begin{aligned}
\epsilon_1^{k+1}(x) &= [g_1(u_2^k(l)) - g_1(u(l))] \cdot \frac{x}{\alpha_1 l + \beta_1} \\
&= [g_2(u_1^{k-1}(1-l')) - g_2(u(1-l'))] \cdot \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2} \cdot \frac{x}{\alpha_1 l + \beta_1} \\
&= [g_1(u_2^{k-2}(l)) - g_1(u(l))] \cdot \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1} \cdot \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2} \cdot \frac{x}{\alpha_1 l + \beta_1} \\
&= \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1} \cdot \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2} \cdot \epsilon_1^{k-1}(x).
\end{aligned} \tag{3.7}$$

Similarly, by (3.4), (3.5) and (3.6), it follows that

$$\begin{aligned}
\epsilon_2^{k+1}(x) &= [g_2(u_1^k(1-l')) - g_2(u(1-l'))] \cdot \frac{1-x}{\alpha_2 l' + \beta_2} \\
&= [g_1(u_2^{k-1}(l)) - g_1(u(l))] \cdot \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1} \cdot \frac{1-x}{\alpha_2 l' + \beta_2} \\
&= [g_2(u_1^{k-2}(1-l')) - g_2(u(1-l'))] \\
&\quad \cdot \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2} \cdot \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1} \cdot \frac{1-x}{\alpha_2 l' + \beta_2} \\
&= \frac{\alpha_1(1-l) - \beta_1}{\alpha_2 l' + \beta_2} \cdot \frac{\alpha_2(1-l') - \beta_2}{\alpha_1 l + \beta_1} \cdot \epsilon_2^{k-1}(x).
\end{aligned} \tag{3.8}$$

Therefore, the convergence rate of the generalized additive Schwarz algorithm is determined by

$$
\begin{aligned}
& \frac{\alpha_1(1-l)-\beta_1}{\alpha_2 l'+\beta_2}\cdot\frac{\alpha_2(1-l')-\beta_2}{\alpha_1 l+\beta_1} \\
= & \frac{\alpha_1(l'-\delta)-\beta_1}{\alpha_2 l'+\beta_2}\cdot\frac{\alpha_2(l-\delta)-\beta_2}{\alpha_1 l+\beta_1} \\
= & \begin{cases}
\dfrac{l'-\delta-t_1}{l'+t_2}\cdot\dfrac{l-\delta-t_2}{l+t_1}, & \text{if } \alpha_1\alpha_2\neq 0, \\
-\dfrac{l'-\delta-t_1}{l+t_1}, & \text{if } \alpha_1\neq 0 \text{ and } \alpha_2=0, \\
-\dfrac{l-\delta-t_2}{l'+t_2}, & \text{if } \alpha_1=0 \text{ and } \alpha_2\neq 0, \\
1, & \text{if } \alpha_1=\alpha_2=0,
\end{cases}
\end{aligned} \tag{3.9}
$$

where $t_i=\beta_i/\alpha_i=(1-\alpha_i)/\alpha_i=1/\alpha_i-1$, $i=1,2$. For special case of $\alpha=\alpha_1=\alpha_2\neq 0$, we define

$$
r(t,l,l')=\sqrt{\frac{|1-l-t|}{l'+t}\cdot\frac{|1-l'-t|}{l+t}}=\sqrt{\frac{|l'-\delta-t|}{l'+t}\cdot\frac{|l-\delta-t|}{l+t}}, \tag{3.10}
$$

where $t=\beta/\alpha=1/\alpha-1$. In this case, it is easy to see that the convergence rate of the additive algorithm is $r(t,l,l')$ for $\alpha\neq 0$ and 1 for $\alpha=0$ respectively.

If $0<t\leq\min\{l'-\delta,l-\delta\}$, we have

$$
r^2(t,l,l')=(\frac{l'-t}{l'+t}-\frac{\delta}{l'+t})\cdot(\frac{l-t}{l+t}-\frac{\delta}{l+t})\leq\frac{l'-t}{l'+t}\cdot\frac{l-t}{l+t}\leq\min\{\frac{l'-t}{l'+t},\frac{l-t}{l+t}\}<1.
$$

If $t\geq\max\{l'-\delta,l-\delta\}$, we have that

$$
\begin{aligned}
r^2(t,l,l') &= \frac{1}{\frac{t+l'-\delta}{t-l'+\delta}+\frac{\delta}{t-l'+\delta}}\cdot\frac{1}{\frac{t+l-\delta}{t-l+\delta}+\frac{\delta}{t-l+\delta}} \\
&\leq \frac{t-l'+\delta}{t+l'-\delta}\cdot\frac{t-l+\delta}{t+l-\delta} \\
&\leq \min\{\frac{t-l'+\delta}{t+l'-\delta},\frac{t-l+\delta}{t+l-\delta}\} \\
&= \min\{\frac{t-1+l}{t+1-l},\frac{t-1+l'}{t+1-l'}\}<1.
\end{aligned}
$$

If $l'-\delta<l-\delta$ and $l'-\delta<t<l-\delta$,

$$
\begin{aligned}
r^2(t,l,l') &= \frac{1}{\frac{t+l'-\delta}{t-l'+\delta}+\frac{\delta}{t-l'+\delta}}\cdot(\frac{l-t}{l+t}-\frac{\delta}{l+t}) \\
&\leq \frac{t-l'+\delta}{t+l'-\delta}\cdot\frac{l-t}{l+t} \\
&\leq \frac{t-1+l}{t+1-l}\cdot\frac{l-t}{l+t}<1.
\end{aligned}
$$

And if $l-\delta<l'-\delta$ and $l-\delta<t<l'-\delta$,

$$
r^2(t,l,l')\leq\frac{l'-t}{l'+t}\cdot\frac{t-1+l'}{t+1-l'}<1.
$$

Therefore, we have that

$$
r(t,l,l')\leq\rho(t,\max\{l,l'\}),
$$

where

$$
\rho(t,l)=\max\{\sqrt{\frac{|t-l|}{t+l}},\sqrt{\frac{|t-1+l|}{t+1-l}}\}.
$$

It is easy to see that $\rho(t, \max\{l, l'\})$ is bounded away from 1 and independent of the overlapping size δ if $t \neq 0$. If $t = 0$, that is for classical additive Schwarz algorithm, the convergence rate is

$$r(0, l, l') = \frac{l' - \delta}{l'} \cdot \frac{l - \delta}{l} = (1 - \frac{\delta}{l'}) \cdot (1 - \frac{\delta}{l}), \tag{3.11}$$

which is dependent on the overlapping size δ and

$$\lim_{\delta \to 0^+} r(0, l, l') = \lim_{\delta \to 0^+} r(0, l, 1 - l + \delta) = 1. \tag{3.12}$$

(3.12) shows that the convergence rate of classical additive Schwarz algorithm (2.6)–(2.7) deteriorate as the overlapping size becomes small. However, if $\alpha = 0$, Noting that (3.7) and (3.8), $\epsilon_1^{k+1}(x) = \epsilon_1^{k-1}(x)$ and $\epsilon_2^{k+1}(x) = \epsilon_2^{k-1}(x)$ hold. That is the algorithm is divergent. Summing up above discussion, we have the following result.

Theorem 3.1. *Let generalized additive Schwarz algorithm (2.8)–(2.9) be applied to solve problem (3.1) with $\alpha = \alpha_1 = \alpha_2$ and the partition defined by (3.2). We have that*

1. *If $\alpha = 0$, that is Neunann boundary conditions are applied on the inner boundaries Γ_1 and Γ_2, the algorithm diverges.*
2. *If $\beta = 0$, that is for classical additive Schwarz algorithm, the algorithm converges to the solution with the convergence rate $r(0, l, l')$ that is defined by (3.11) and dependent on the overlapping size δ.*
3. *If $\alpha\beta \neq 0$, that is Robin boundary conditions are applied on the inner boundaries Γ_1 and Γ_2, the algorithm converges to the solution with the convergence rate $r(t, l, l')$ that is defined by (3.10) and bounded above by $\rho(t, \max\{l, l'\})$, which is less than 1 and independent of the overlapping size δ.*

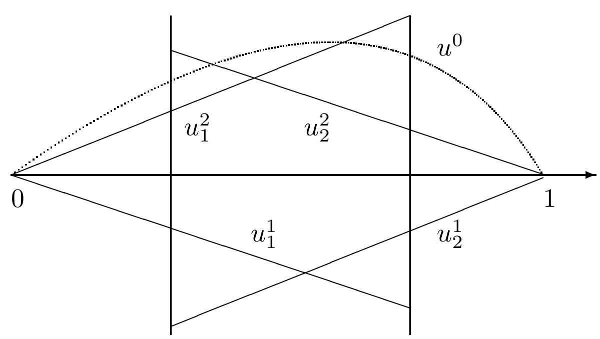

Figure 2: $u_i^{k+2} = u_i^k$, $i = 1, 2$, $k = 1, 2, \ldots$.

By Theorem 3.1, the algorithm does not converge if the interface conditions in (2.8) and (2.9) are Neumann conditions. An simple example is furnished in Fig. 2, where the generalized additive Schwarz algorithm with parameters $\alpha = \alpha_1 = \alpha_2 = 0$ is applied to approximate the (null) solution of the problem

$$\begin{cases} -u'' = 0, & \text{in } (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.13}$$

Note that for this particular problem, whatever partition is chosen, the algorithm diverges.

When $\delta \to 0^+$, the generalized additive algorithm we proposed becomes Schwarz alternating method for nonoverlapping subdomains given by P. L. Lions (see [7]). In this case, the convergence of the algorithm has been obtained in [7, 11] for $\alpha_1 = \alpha_2 \neq 0$. However, if $\alpha_1 \neq \alpha_2$, the algorithm may diverge even for $\alpha_i, \beta_i \neq 0, i = 1, 2$. In fact, let us consider the algorithm to problem (3.13) with parameters $l = \frac{1}{4}$, $l' = \frac{3}{4}$, $t_1 = \frac{1}{12}$, $t_2 = \frac{5}{4}$. It is easy to verify that for any $k \geq 1$, $\epsilon_i^{k+1} = -\epsilon_i^{k-1}$, $i = 1, 2$, which implies obvious the divergence of the algorithm.

4. Convergence Rate for Two-dimensional Problem

In this section, we will analyze quantitatively the convergence rate of generalized additive Schwarz algorithm when it is applied to approximate the solution of the following model Dirichlet problem in two-dimensional space:

$$\begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega, \\ u(x,y) = \psi(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0,1) \times (0,1)$, f and ψ are given functions of $L^2(\Omega)$, Δ is the Laplace operator.

For classical alternating Schwarz algorithm, a quantitative analysis of the convergence rate was given in [8]. Here we discuss the convergence rate for generalized additive Schwarz algorithm (2.8)–(2.9). Let $\Omega = (0,1) \times (0,1)$ and consider particular partition as

$$\Omega_1 = (0,l) \times (0,1), \qquad \Omega_2 = (1-l', 1) \times (0,1), \tag{4.2}$$

where the constants l and l' belong to $(0,1)$ that satisfy $l + l' > 1$, $\delta = l - (1-l') = l + l' - 1 > 0$ is the overlapping size.

In the algorithms, we also let $\alpha_1 = \alpha_2 = \alpha = 1 - \beta$. Moreover, we let the initial error $\epsilon^0 = u^0 - u$ have a Fourier expansion of the form as

$$\epsilon^0 = \sum_{n=1}^{\infty} a_n \sin n\pi x \sin n\pi y. \tag{4.3}$$

It is easy to see that

$$\frac{\partial \epsilon^0}{\partial x} = \sum_{n=1}^{\infty} a_n n\pi \cos n\pi x \sin n\pi y. \tag{4.4}$$

We have from the algorithm that,

$$\begin{cases} -\Delta \epsilon_1^{k+1} = 0, & \text{in } \Omega_1, \\ \epsilon_1^{k+1} = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega, \\ g_1(\epsilon_1^{k+1}) = g_1(\epsilon_2^k), & \text{on } \Gamma_1 \end{cases} \tag{4.5}$$

and

$$\begin{cases} -\Delta \epsilon_2^{k+1} = 0, & \text{in } \Omega_2, \\ \epsilon_2^{k+1} = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega, \\ g_2(\epsilon_2^{k+1}) = g_2(\epsilon_1^k), & \text{on } \Gamma_2, \end{cases} \tag{4.6}$$

where

$$\epsilon_i^k = u_i^k - u|_{\overline{\Omega}_i}, \qquad i = 1, 2, \qquad k = 0, 1, \ldots. \tag{4.7}$$

It follows from (4.3) and (4.4) that

$$\epsilon_2^0|_{\Gamma_1} = \sum_{n=1}^{\infty} a_n \sin n\pi l \cdot \sin n\pi y$$

and

$$\frac{\partial \epsilon_2^0}{\partial x}|_{\Gamma_1} = \sum_{n=1}^{\infty} a_n n\pi \cos n\pi l \cdot \sin n\pi y.$$

Therefore

$$g_1(\epsilon_2^0)|_{\Gamma_1} = \sum_{n=1}^{\infty} a_n \cdot (\alpha \sin n\pi l + \beta n\pi \cos n\pi l) \sin n\pi y.$$

Hence, we have

$$\begin{aligned} \epsilon_1^1 &= \sum_{n=1}^{\infty} a_n \cdot \frac{\alpha \sin n\pi l + \beta n\pi \cos n\pi l}{\alpha \sinh n\pi l + \beta n\pi \cosh n\pi l} \sinh n\pi x \cdot \sin n\pi y \\ &= \sum_{n=1}^{\infty} a_n^1 \sinh n\pi x \cdot \sin n\pi y, \end{aligned} \tag{4.8}$$

where

$$a_n^1 = a_n r_n^0, \qquad r_n^0 = \frac{\alpha \sin n\pi l + \beta n\pi \cos n\pi l}{\alpha \sinh n\pi l + \beta n\pi \cosh n\pi l} \tag{4.9}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Similarly

$$\epsilon_1^0|_{\Gamma_2} = \sum_{n=1}^{\infty} a_n \sin n\pi(1-l') \cdot \sin n\pi y,$$

$$\frac{\partial \epsilon_1^0}{\partial x}|_{\Gamma_2} = \sum_{n=1}^{\infty} a_n n\pi \cos n\pi(1-l') \cdot \sin n\pi y,$$

and then

$$g_2(\epsilon_1^0)|_{\Gamma_2} = \sum_{n=1}^{\infty} a_n \cdot [\alpha \sin n\pi(1-l') - \beta n\pi \cos n\pi(1-l')] \cdot \sin n\pi y.$$

Hence, we have

$$\begin{aligned} \epsilon_2^1 &= \sum_{n=1}^{\infty} a_n \cdot \frac{\alpha \sin n\pi(1-l') - \beta n\pi \cos n\pi(1-l')}{\alpha \sinh n\pi l' + \beta n\pi \cosh n\pi l'} \cdot \sinh n\pi(1-x) \cdot \sin n\pi y \\ &= \sum_{n=1}^{\infty} \tilde{a}_n^1 \sinh n\pi(1-x) \cdot \sin n\pi y, \end{aligned} \tag{4.10}$$

where

$$\tilde{a}_n^1 = a_n \tilde{r}_n^0, \qquad \tilde{r}_n^0 = \frac{\alpha \sin n\pi(1-l') - \beta n\pi \cos n\pi(1-l')}{\alpha \sinh n\pi l' + \beta n\pi \cosh n\pi l'}. \tag{4.11}$$

Similarly, for $k > 1$, we obtain that

$$\epsilon_1^k = \sum_{n=1}^{\infty} a_n^k \sinh n\pi x \cdot \sin n\pi y, \tag{4.12}$$

$$\epsilon_2^k = \sum_{n=1}^{\infty} \tilde{a}_n^k \sinh n\pi(1-x) \cdot \sin n\pi y, \tag{4.13}$$

where

$$a_n^k = \tilde{a}_n^{k-1} r_n^1, \qquad \tilde{a}_n^k = a_n^{k-1} \tilde{r}_n^1, \qquad k = 2, 3, \ldots, \tag{4.14}$$

and

$$r_n^1 = \frac{\alpha \sinh n\pi(1-l) - \beta n\pi \cosh n\pi(1-l)}{\alpha \sinh n\pi l + \beta n\pi \cosh n\pi l}, \tag{4.15}$$

$$\tilde{r}_n^1 = \frac{\alpha \sinh n\pi(1-l') - \beta n\pi \cosh n\pi(1-l')}{\alpha \sinh n\pi l' + \beta n\pi \cosh n\pi l'}. \tag{4.16}$$

Therefore

$$\begin{aligned} \epsilon_1^k|_{\Gamma_1} &= \sum_{n=1}^{\infty} a_n^k \sinh n\pi l \cdot \sin n\pi y \\ &= \sum_{n=1}^{\infty} b_n (r_n^1 \tilde{r}_n^1)^{[(k-1)/2]} \cdot \sin n\pi y, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \epsilon_2^k|_{\Gamma_2} &= \sum_{n=1}^{\infty} \tilde{a}_n^k \sinh n\pi l' \cdot \sin n\pi y \\ &= \sum_{n=1}^{\infty} \tilde{b}_n (r_n^1 \tilde{r}_n^1)^{[(k-1)/2]} \cdot \sin n\pi y, \end{aligned} \tag{4.18}$$

where

$$b_n = \begin{cases} a_n \tilde{r}_n^0 r_n^1 \sinh n\pi l, & \text{if } k \text{ is even,} \\ a_n r_n^0 \sinh n\pi l, & \text{if } k \text{ is odd,} \end{cases} \tag{4.19}$$

$$\tilde{b}_n = \begin{cases} a_n r_n^0 \tilde{r}_n^1 \sinh n\pi l', & \text{if } k \text{ is even,} \\ a_n \tilde{r}_n^0 \sinh n\pi l', & \text{if } k \text{ is odd.} \end{cases} \tag{4.20}$$

Since

$$\begin{aligned} \frac{r_{n+1}^1}{r_n^1} &= \frac{\alpha \sinh(n+1)\pi(1-l) - \beta(n+1)\pi \cosh(n+1)\pi(1-l)}{\alpha \sinh(n+1)\pi l + \beta(n+1)\pi \cosh(n+1)\pi l} \\ &\quad \div \frac{\alpha \sinh n\pi(1-l) - \beta n\pi \cosh n\pi(1-l)}{\alpha \sinh n\pi l + \beta n\pi \cosh n\pi l} \\ &= \frac{\alpha - \beta(n+1)\pi - [\alpha + \beta(n+1)\pi]e^{-2(n+1)\pi(l-\delta)}}{\alpha + \beta(n+1)\pi - [\alpha - \beta(n+1)\pi]e^{-2(n+1)\pi l}} \cdot e^{-(n+1)\pi\delta} \\ &\quad \cdot \frac{\alpha + \beta n\pi - (\alpha - \beta n\pi)e^{-2n\pi l}}{\alpha - \beta n\pi - (\alpha + \beta n\pi) \cdot e^{-2n\pi(l-\delta)}} \cdot e^{n\pi\delta} \\ &\to e^{-\pi\delta}, \qquad n \to \infty \end{aligned}$$

and

$$\frac{\tilde{r}_{n+1}^1}{\tilde{r}_n^1} \to e^{-\pi\delta}, \qquad n \to \infty,$$

respectively, we have

$$\lim_{n\to\infty} \frac{r_{n+1}^1 \tilde{r}_{n+1}^1}{r_n^1 \tilde{r}_n^1} = e^{-2\pi\delta} < 1.$$

Hence the convergence rate of the algorithm is dominant by the convergence rate of the lower frequency terms in (4.17) and (4.18). Note that the lowest frequency terms of (4.17) and (4.18) are

$$e_k = b_1 (r_1^1 \tilde{r}_1^1)^{[(k-1)/2]} \sin \pi y$$

and

$$\tilde{e}_k = \tilde{b}_1 (r_k^1 \tilde{r}_k^1)^{[(k-1)/2]} \sin \pi y,$$

respectively, it is easy to see that the convergence rate of e_k and $\tilde{e}_k$ is $\bar{r}(\alpha, l, l') = \sqrt{|r_1^1 \tilde{r}_1^1|}$. By (4.15) and (4.16), we have

$$\begin{aligned} &\bar{r}^2(\alpha, l, l') \\ =\ & \frac{|\alpha \sinh \pi(1-l) - \beta\pi \cosh \pi(1-l)|}{\alpha \sinh \pi l + \beta\pi \cosh \pi l} \cdot \frac{|\alpha \sinh \pi(1-l') - \beta\pi \cosh \pi(1-l')|}{\alpha \sinh \pi l' + \beta\pi \cosh \pi l'} \\ =\ & \frac{|\alpha \sinh \pi(l'-\delta) - \beta\pi \cosh \pi(l'-\delta)|}{\alpha \sinh \pi l + \beta\pi \cosh \pi l} \cdot \frac{|\alpha \sinh \pi(l-\delta) - \beta\pi \cosh \pi(l-\delta)|}{\alpha \sinh \pi l' + \beta\pi \cosh \pi l'} \\ \leq\ & 1. \end{aligned}$$

For small overlapping size δ,

$$\bar{r}(\alpha, l.l') \sim \frac{|\alpha-\beta\pi|}{\alpha+\beta\pi}\cdot e^{-\pi\delta}\begin{cases}\leq \dfrac{|\alpha-\beta\pi|}{\alpha+\beta\pi}\cdot e^{-\pi\delta}, & \alpha\beta\neq 0,\\ = e^{-\pi\delta}, & \alpha\beta = 0.\end{cases}$$

Theorem 4.1. *Let generalized additive Schwarz algorithm (2.8)–(2.9) be applied to solve problem (4.1) with $\alpha=\alpha_1=\alpha_2$ and the partition defined by (4.2). Let the initial error $\epsilon^0 = u^0 - u$ have an expansion of the form as (4.3). Then (4.12) and (4.13) hold. Therefore, the convergence rate of the algorithm is dominant by $\bar{r}(\alpha, l, l') = \sqrt{|r_1^1\tilde{r}_1^1|}$. Moreover, we have the following conclusion:*

1. *If $\alpha\beta = 0$,*
$$\lim_{\delta\to 0^+}\bar{r}(\alpha, l.l') = \lim_{\delta\to 0^+} e^{-\pi\delta} = 1.$$
2. *If $\alpha\beta \neq 0$,*
$$\lim_{\delta\to 0^+}\bar{r}(\alpha, l.l') \leq \frac{|\alpha-\beta\pi|}{\alpha+\beta\pi} < 1.$$

Theorem 4.1 appears that Robin conditions on the inner boundaries in the Schwarz algorithm are prefer to Dirichlet or Neumann conditions.

For three-dimensional problem, if we assume $\Omega = (0,1)\times(0,1)\times(0,1)$ and $\Omega_1 = (0,l)\times(0,1)^2$, $\Omega_2 = (l',1)\times(0,1)^2$, we can deduce similar results by analogous Fourier analysis.

5. Numerical Examples

In this section, we present some numerical results of generalized additive Schwarz algorithm (including classical additive Schwarz algorithm) when it is applied to solve the discretization of the boundary value problem (4.1) with

$$\Omega = (0,1)\times(0,1), \qquad f = \pi^2 \sin\pi x\cdot\sin\pi y.$$

We use finite element method to discrete the problem and let h be the mesh-size. In the algorithm, we choose the initial point $u^0 = 0$. The stopping criterion is that the maximum norm of the iterative error is less than $\epsilon = 10^{-5}$. We let

$$\Omega_1 = (0, 0.5+\delta/2)\times(0,1) \qquad \Omega_2 = (0.5-\delta/2, 1)\times(0,1)$$

and

$$\Omega_1 = (0, 0.5+h)\times(0,1) \qquad \Omega_2 = (0.5, 1)\times(0,1),$$

respectively

The iterative numbers of the generalized Schwarz additive algorithm are listed in the following tables.

Table 1: Iterative numbers of the algorithm with different α for the case of $h = 2^{-5}$

$l\backslash\alpha$	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
h	109	14	5	8	13	19	27	32	65	139
0.625	15	8	3	6	7	8	11	13	15	16

Table 2: Iterative numbers for the case of $\delta = 0.25$

$l\backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\alpha = 1$	15	15	15	15
$\alpha = 0.8$	5	4	3	4

Table 3: Iterative numbers for the case of $\delta = h$

$l\backslash h$	2^{-3}	2^{-4}	2^{-5}	2^{-6}
$\alpha = 1$	28	55	109	215
$\alpha = 0.8$	4	4	5	5

From the tables we may see that

(1) With an appropriate choice of α, the performances of the generalized additive Schwarz algorithm is much better than the performance of the classical additive Schwarz algorithm (see Table 1 and Fig. 3). We also see that generalized additive Schwarz algorithm with appropriate α can accelerate the convergence rate dramatically.

(2) For the case of $\delta = 0.25$, the iterative numbers seems independent of the mesh-size h for generalized and classical Schwarz algorithms (see Table 2.)

(3) For the case of $\delta = h$, the iterative numbers become very large as the mesh-size h is small for classical additive Schwarz algorithms. However, the iterative numbers are not sensitive to the mesh-size for generalized additive Schwarz algorithm with $\alpha \neq 1$(see Table 3). Since it can save much time in solving subproblems for small overlapping case, we prefer to use small overlapping in generalized additive Schwarz algorithm.

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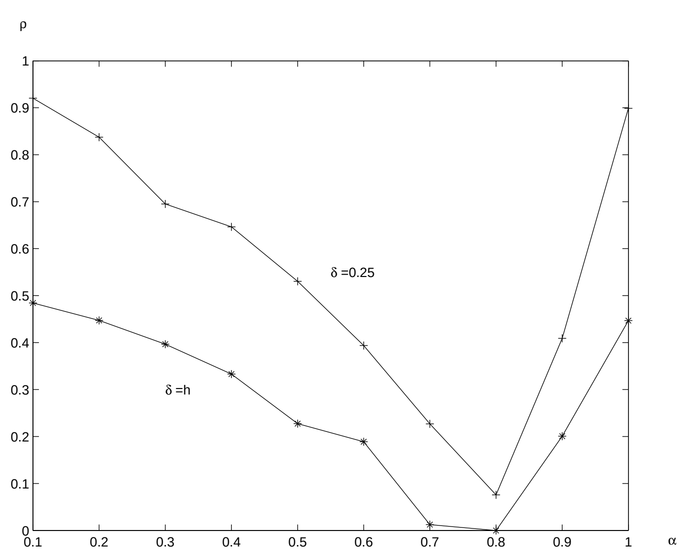

Figure 3: Approximating convergence rate $(||\epsilon^k||/||\epsilon^0||)^{1/k}$.

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