Journal of Computational Mathematics, Vol.24, No.4, 2006, 527–538.

A PROJECTION-TYPE METHOD FOR SOLVING VARIOUS WEBER PROBLEMS [*1)]

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Abstract

This paper investigates various Weber problems including unconstrained Weber problems and constrained Weber problems under l_1, l_2 and l_∞-norms. First with a transformation technique various Weber problems are turned into a class of monotone linear variational inequalities. By exploiting the favorable structure of these variational inequalities, we present a new projection-type method for them. Compared with some other projection-type methods which can solve monotone linear variational inequality, this new projection-type method is simple in numerical implementations and more efficient for solving this class of problems; Compared with some popular methods for solving unconstrained Weber problem and constrained Weber problem, a singularity would not happen in this new method and it is more reliable by using this new method to solve various Weber problems.

Mathematics subject classification: 49K05.
Key words: Linear variational inequality, Various Weber problems, Projection-type method, Slack technique.

1. Introduction

Weber problem (WP) is one of the fundamental models in location theory and has many applications in practice, see, e.g., [10]. Its objective is to site a new facility in the plane to minimize a sum of weighted distances from the new facility to a set of customers whose locations are known. Weber problem has the following formulation:

$$\text{WP:} \qquad \min_{x\in R^2} C(x) = \sum_{j=1}^{n} w_j \|x - a_j\|_p, \tag{1.1}$$

where a_i is the known location of the ith customer, $i = 1, \cdots, n$; n is the number of customers; x is the unknown location of the new facility; w_i is the weight associated with the customer a_i, $i = 1, \cdots, n$; $\| \cdot \|_p$ is the distance measuring function.

When the new facility x is restricted to be sited in a constrained area X, this model is named as constrained Weber problem (CWP).

Some efficient methods have been proposed for solving Weber problem and constrained Weber problem. Weiszfeld procedure [12] is perhaps the most popular and standard method for Weber problem with Euclidean distances; Recently, a so-called Newton-Bracketing (NB) method [8] was presented to solve Weber problem. The well-known method for constrained Weber problem whose constrained area is the union of a finite set of convex polygons was presented in [3], which consists in a search for the unconstrained solution followed by an exploration of some of the boundary parts of the polygons defining the feasible region.

* Received May 22, 2005; Final revised July 13, 2005.
1) The author was supported by the dissertation fund of Nari-Relays Corporation.

However, a singularity may happen for these popular methods: if an iterate generated by them is identical with one of customers, the next iterate is undefined. The reason for this singularity is that a "bad" initial point is chosen for these methods. Chandrasekaran and Tamir [2] showed that the set of these "bad" initial points may contain a continuum set, and thus in advance we have no way to clearly know whether one initial point is bad or not.

In this paper, we discuss various Weber problems (VWP) including Weber problems and constrained Weber problems under l_1, l_2 and l_∞-norms,

$$\text{VWP:} \qquad \min_{x\in X} C(x) = \sum_{j=1}^{n} w_j \|x - a_j\|_p, \tag{1.2}$$

where X is a constrained area which is closed and convex in R^2. Note that VWP reduce to Weber problem in the case that $X = R^2$. With a transformation technique, various Weber problems can be reformulated as min-max problems from which a class of monotone linear variational inequalities (LVIs) may be obtained,

$$\begin{cases} (x - x^*)^T(A^T z^* + q_1) \ \geq \ 0 & \forall\, x \in X, \\ (z - z^*)^T(-Ax^* + q_2) \ \geq \ 0 & \forall\, z \in Z, \end{cases} \tag{1.3}$$

where $x \in R^k,\ z \in R^{kn},\ A = (I_k, \cdots, I_k)^T \in R^{kn\times k}$, I_k is the $k \times k$ identity matrix and X and Z are closed convex sets. Thus, solving various Weber problems is equivalent to solving (1.3).

Many computational methods have been established for solving monotone linear variational inequality. The projection-type methods, e.g., projection-contraction (PC) methods, may be one class of the simplest methods for solving these problems and they are also applicable for solving (1.3). Our purpose is to exploit the favorable structure of (1.3) in practice and propose a more efficient projection-type method for it. Note that for LVI (1.3) $A^TA = nI_k$. Based on this observation, a new projection-type method is proposed. The new method is rather simple in numerical implementations. The most significance for proposing this new method is that for an arbitrarily chosen initial point the singularity would not happen for this new method, which guarantees that using this method we can acquire the optimal solution of various Weber problems. Numerical results are reported, which shows that the new projection-type method is meaningful for solving these problems.

The paper is organized as follows. Some popular methods for solving Weber problem are provided in Section 2. In Section 3 various Weber problems under l_1, l_2 and l_∞-norms are transformed into this class of variational inequalities (1.3). Some preliminaries required in coming analysis are given in Section 4. The new projection-type method for solving this class of variational inequalities is presented in Section 5. In Section 6 the convergence of the new method is provided and preliminary numerical results are reported in Section 7. Finally, some concluding remarks are drawn in the last section.

2. Some Existing Algorithms for Solving Weber Problem

In this section we discuss two popular methods for solving Weber problem: Weiszfeld procedure and Newton-Bracketing method.

2.1 Weiszfeld procedure

Since the distance measuring function is convex, as a sum of convex functions, the objective function $C(x)$ of Weber problem is convex. It is clear that the set of its optimal solutions is nonempty and convex. Whereas, the main difficulty for solving Weber problem is that $C(x)$ is non-differentiable at some locations, e.g., the locations of customers. The gradient of $C(x)$

with Euclidean distances exists for all $x \notin \{a_1, \cdots, a_n\}$, and it is given by:

$$\nabla C(x) = \sum_{j=1}^{n} w_j \frac{x - a_j}{\|x - a_j\|_2}.$$

If C is differentiable at x^*, x^* is an optimal location if and only if

$$\nabla C(x^*) = \sum_{j=1}^{n} w_j \frac{x^* - a_j}{\|x^* - a_j\|_2} = 0.$$

It follows that

$$x^* = \frac{\sum_{j=1}^{n} w_j \|x^* - a_j\|_2^{-1} a_j}{\sum_{j=1}^{n} w_j \|x^* - a_j\|_2^{-1}}.$$

The recursion of Weiszfeld procedure is

$$x^{k+1} = \frac{\sum_{j=1}^{n} w_j \|x^k - a_j\|_2^{-1} a_j}{\sum_{j=1}^{n} w_j \|x^k - a_j\|_2^{-1}}. \tag{2.1}$$

Weiszfeld procedure is perhaps the most popular and standard method for solving Weber problem with Euclidean distances. Its convergence was studied in [7, 9, 11], etc.

2.2 Newton-Bracketing method

NB method is an iterative method which works by improving bounds on the minimum value of $C(x)$, rather than by approximating a solution x^* satisfying $\nabla C(x) = 0$.

Each iteration of NB method generates a bracket $[L^k, U^k]$ and an iterate x^k satisfying $L^k \le C(x^k) \le U^k$.

Newton-Bracketing method

Given $\varepsilon > 0$, a initial point x^0 and a bracket $[L^0, U^0]$ which satisfies $L^0 \le C(x^0) \le U^0$.

For $k = 0, 1, \cdots$, if $U^k - L^k > \varepsilon$ then **do**:

$$M^k = \alpha U^k + (1-\alpha) L^k, \qquad 0 < \alpha < 1;$$

$$x^{k+1} = x^k - \frac{C(x^k) - M^k}{\|\nabla C(x^k)\|^2} \nabla C(x^k); \tag{2.2}$$

$$L^{k+1} = \begin{cases} L^k, & if \;\; C(x^{k+1}) < C(x^k), \\ M^k, & otherwise; \end{cases}$$

$$U^{k+1} = \begin{cases} C(x^{k+1}), & if \;\; C(x^{k+1}) < C(x^k), \\ U^k, & otherwise; \end{cases}$$

$$x^{k+1} = \begin{cases} x^{k+1}, & if \;\; C(x^{k+1}) < C(x^k), \\ x^k, & otherwise. \end{cases}$$

Remark 1. According to (2.1) and (2.2), it is obvious that any iterate in the sequence $\{x^k\}$ generated by Weiszfeld procedure and NB method shouldn't be identical with any of the customers. Otherwise, the methods will stop at a non-optimal solution of Weber problem.

3. LVI Reformulations of VWP

In this section various Weber problems under l_1, l_2 and l_∞-norms are reformulated as linear variational inequalities with a transformation technique.

Note that for any $d \in R^2$, we have the following properties

$$w\|d\|_1 = \max_{\xi \in B^w_\infty} d^T \xi, \tag{3.1}$$

$$w\|d\|_2 = \max_{\xi \in B^w_2} d^T \xi, \tag{3.2}$$

$$w\|d\|_\infty = \max_{\xi \in B^w_1} d^T \xi, \tag{3.3}$$

where

$$\begin{aligned} B^w_1 &= \{\xi \in R^2 \mid \|\xi\|_1 \le w\}, \\ B^w_2 &= \{\xi \in R^2 \mid \|\xi\|_2 \le w\}, \\ B^w_\infty &= \{\xi \in R^2 \mid \|\xi\|_\infty \le w\}. \end{aligned}$$

According to (3.1), (3.2) and (3.3), various Weber problems under l_p-norms,

$$\min_{x \in X} C(x) = \sum_{j=1}^{n} w_j \|x - a_j\|_p, \qquad p = 1, 2, \infty, \tag{3.4}$$

are equivalent to the following min-max problems, respectively:

$$\min_{x \in X} \max_{z_i \in B^{w_i}_t} \sum_{i=1}^{n} z_i^T (x - a_i), \qquad t = \infty, 2, 1, \tag{3.5}$$

where each z_i $(i = 1, 2, \cdots, n)$ is a vector in $B^{w_i}_t = \{\xi \in R^2 \mid \|\xi\|_t \le w_i\}$. A compact form of (3.5) is

$$\min_{x \in X} \max_{z \in \mathcal{B}_t} z^T (Ax - b), \tag{3.6}$$

where

$$\begin{array}{ll} z^T = (z_1^T, z_2^T, \cdots, z_n^T), & \mathcal{B}_t = B^{w_1}_t \times B^{w_2}_t \times \cdots \times B^{w_n}_t, \\ A = (I_2, I_2, \cdots, I_2)^T, & b^T = (a_1^T, a_2^T, \cdots, a_n^T). \end{array} \tag{3.7}$$

Let $(x^*, z^*) \in X \times \mathcal{B}_t$ be any solution of (3.6), then it follows that

$$z^T (Ax^* - b) \ \le\ z^{*T} (Ax^* - b) \ \le\ z^{*T} (Ax - b), \qquad \forall\, x \in X,\ z \in \mathcal{B}_t, \tag{3.8}$$

thus (x^*, z^*) is a solution of the following linear variational inequality:

$$x^* \in X, \ \ z^* \in \mathcal{B}_t \qquad \left\{ \begin{array}{ll} (x - x^*)^T (A^T z^*) \ \ge\ 0 & \forall\, x \in X, \\ (z - z^*)^T (-Ax^* + b) \ \ge\ 0 & \forall\, z \in \mathcal{B}_t. \end{array} \right. \tag{3.9}$$

Note that (3.9) has the same structure as (1.3) with Z taken as $\mathcal{B}_t$, thus, various Weber problems can be transformed into a class of linear variational inequalities (1.3). A compact form of the obtained LVI is given as:

$$\mathrm{LVI}(\Omega, M, q): \qquad u^* \in \Omega, \qquad (u - u^*)^T (Mu^* + q) \ge 0, \qquad \forall\, u \in \Omega, \tag{3.10}$$

where

$$u = \begin{pmatrix} x \\ z \end{pmatrix}, \quad M = \begin{pmatrix} 0 & A^T \\ -A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad A = (I_2, \cdots, I_2)^T, \quad \Omega = X \times \mathcal{B}. \tag{3.11}$$

For various Weber problems under different l_p-norms, the different forms of $\mathcal{B}$ are

$$\mathcal{B} = \begin{cases} \mathcal{B}_\infty, & p = 1; \\ \mathcal{B}_2, & p = 2; \\ \mathcal{B}_1, & p = \infty. \end{cases} \tag{3.12}$$

4. Preliminaries for Proposed Projection-type Method

In this section, we summarize some basic concepts and known properties which will be used in coming analysis.

Definition 1. *For a given vector $v \in R^n$ and a closed convex set Ω, the solution of problem*

$$\min\{\| u - v \|_2 \mid u \in \Omega\} \tag{4.1}$$

is called as the projection of v on Ω, denoted by $P_\Omega(v)$. In other words,

$$P_\Omega(v) = \text{argmin}\{\| u - v \|_2 \mid u \in \Omega\}. \tag{4.2}$$

Proposition 1.[6] *Let $\Omega \subset R^n$ be a closed convex set, then*

$$(v - P_\Omega(v))^T(u - P_\Omega(v)) \leq 0, \qquad \forall\, v \in R^n, u \in \Omega. \tag{4.3}$$

Proposition 2. *Given a closed convex set $\Omega \subset R^n$, we have*

$$\|P_\Omega(u) - P_\Omega(v)\| \leq \|u - v\|, \qquad \forall\, u,\ v\ \in R^n. \tag{4.4}$$

Theorem 1. [1, 5] *Let $\beta > 0$, then u^* is a solution of LVI(Ω, M, q) if and only if*

$$u^* - P_\Omega[\, u^* - \beta(Mu^* + q)\,] = 0. \tag{4.5}$$

Hence, solving LVI(Ω, M, q) is equivalent to finding a zero point of the residue function

$$e(u^*, \beta) := u^* - P_\Omega[\, u^* - \beta(Mu^* + q)\,]. \tag{4.6}$$

$e(u, 1)$ is commonly abbreviated to $e(u)$ and $\|e(u)\|$ is often regarded as some measure of the discrepancy between the solution and the current iterate.

5. The Proposed Projection-type Method

Note that $A^T = (I_k, \cdots, I_k)$ and $A^TA = nI_k$. Based on this observation a new projection-type method is proposed to solve the favorable class of linear variational inequalities (1.3). The general iteration of the proposed projection-type method is described as follows:
The proposed projection-type method.

Step 0. Given a tolerance $\varepsilon > 0$ and $u^0 = (x^0,\ z^0) \in \Omega$. Set $k = 0$.

Step 1. Calculate $\tilde{u}^k = (\tilde{x}^k,\ \tilde{z}^k)$:

$$\tilde{z}^k = P_Z[z^k + (Ax^k - q_2)], \tag{5.1}$$

$$\tilde{x}^k = P_X[\frac{1}{n}(nx^k - (A^T\tilde{z}^k + q_1) - A^T(\tilde{z}^k - z^k))]. \tag{5.2}$$

Step 2. Compute $\|e(\tilde{u}^k)\|$, and
if $\|e(\tilde{u}^k)\| < \varepsilon$, $\tilde{u}^k = (\tilde{x}^k,\ \tilde{z}^k)$ is the solution and stop;
else adopt slack technique to calculate $u^{k+1} = (x^{k+1}, z^{k+1})$:

$$z^{k+1} = z^k + \alpha(\tilde{z}^k - z^k), \tag{5.3}$$

$$x^{k+1} = x^k + \alpha(\tilde{x}^k - x^k), \tag{5.4}$$

and go to Step 1.

Remark 2. In fact, $\tilde{x}^k$ in (5.2) is a solution of the following linear variational inequality:

$$\tilde{x}^k \in X, \quad (x' - \tilde{x}^k)^T((A^T\tilde{z}^k + q_1) + A^T(\tilde{z}^k - z^k) + A^TA(\tilde{x}^k - x^k)) \geq 0, \quad \forall\, x' \in X. \tag{5.5}$$

This can be seen by using Theorem 1. According to Theorem 1, (5.5) is equivalent to

$$\tilde{x}^k = P_X\{\tilde{x}^k - \beta \cdot [(A^T\tilde{z}^k + q_1) + A^T(\tilde{z}^k - z^k) + A^TA(\tilde{x}^k - x^k)]\}, \qquad \forall\, \beta > 0. \tag{5.6}$$

Note that $A^TA = nI_k$. If we take $\beta = 1/n$, (5.6) will be turned into

$$\tilde{x}^k = P_X[\frac{1}{n}(nx^k - (A^T\tilde{z}^k + q_1) - A^T(\tilde{z}^k - z^k))]. \tag{5.7}$$

Thus (5.5) is equivalent to (5.2) and $\tilde{x}^k$ in (5.2) is a solution of (5.5).

Remark 3. α is the slack factor which can be chosen from the interval $(0,\ 2)$. If $\alpha = 1$, we have

$$u^{k+1} = \tilde{u}^k = u^k + \alpha(\tilde{u}^k - u^k), \qquad \alpha = 1, \tag{5.8}$$

i.e., $\tilde{u}^k$ is directly used as the next iterate u^{k+1}. This can be interpreted as follows: start with u^k and move along the direction $\tilde{u}^k - u^k$ by a step length $\alpha = 1$, then we get the next iterate u^{k+1}. Since α can be chosen from the interval $(0,\ 2)$, a natural question is whether a bigger step length could improve the computational efficiency of the proposed method. Numerical results in Section 6 reveal that $\alpha \in [1.4,\ 1.8]$ is more efficient than $\alpha = 1$.

It is obvious that given an iterate $u^k = (x^k, z^k)$ the next iterate $u^{k+1} = (x^{k+1}, z^{k+1})$ may be easily acquired by (5.1)–(5.4), and therefore the proposed method is extremely simple in numerical implementations.

6. Convergence Analysis

This section analyzes the convergence of the new projection-type method. Global convergence of it is proved under mild assumption. It is reasonable to assume that the solution set of LVI (1.3) is nonempty. In particular, we denote $u^* = (x^*, z^*)$ a solution of LVI (1.3).

Lemma 1. *The sequences $\{u^k\}$ and $\{\tilde{u}^k\}$ generated by the proposed method satisfy*

$$\{(z^k - z^*) + A(x^k - x^*)\}^T\{(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\} \geq \|(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|^2. \tag{6.1}$$

Proof. It follows from Proposition 1 that

$$(\tilde{z}^k - z)^T\{[z^k + (Ax^k - q_2)] - \tilde{z}^k\} \geq 0, \quad \forall\, z \in Z. \tag{6.2}$$

Since $u^* = (x^*, z^*)$ is a solution of LVI (1.3), $z^* \in Z$, and therefore

$$(\tilde{z}^k - z^*)^T\{[z^k + (Ax^k - q_2)] - \tilde{z}^k\} \geq 0. \tag{6.3}$$

According to the second inequality of LVI (1.3) and $\tilde{z}^k \in Z$, we have

$$(\tilde{z}^k - z^*)^T(-(Ax^* - q_2)) \geq 0. \tag{6.4}$$

Adding (6.3) and (6.4) we obtain

$$(\tilde{z}^k - z^*)^T\{(z^k - \tilde{z}^k) + A(x^k - x^*)\} \geq 0. \tag{6.5}$$

According to the first inequality of LVI (1.3) and $\tilde{x}^k \in X$, we know

$$(x^* - \tilde{x}^k)^T(-A^T z^* - q_1) \geq 0. \tag{6.6}$$

Due to (5.5) and $x^* \in X$, we achive

$$(x^* - \tilde{x}^k)^T\{(A^T\tilde{z}^k + q_1) + A^T(\tilde{z}^k - z^k) + A^TA(\tilde{x}^k - x^k)\} \geq 0. \tag{6.7}$$

Adding (6.6) and (6.7), the following inequality is obtained:

$$(x^* - \tilde{x}^k)^T\{(A^T(\tilde{z}^k - z^*) + A^T(\tilde{z}^k - z^k) + A^TA(\tilde{x}^k - x^k)\} \geq 0, \tag{6.8}$$

and consequently

$$(\tilde{z}^k - z^*)^T A(x^* - \tilde{x}^k) + (A(\tilde{x}^k - x^*))^T\{(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\} \geq 0. \tag{6.9}$$

Adding (6.5) and (6.9) we have

$$\{(\tilde{z}^k - z^*) + A(\tilde{x}^k - x^*)\}^T\{(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\} \geq 0. \tag{6.10}$$

Lemma 1 follows from (6.10) directly and the proof is complete.

Lemma 2. *Let $\{u^k\}$ and $\{\tilde{u}^k\}$ be the sequences produced by the proposed method, then the following inequality is true,*

$$\begin{aligned} &\|(z^{k+1} - z^*) + A(x^{k+1} - x^*)\|^2 \\ &\quad \leq \|(z^k - z^*) + A(x^k - x^*)\|^2 - \alpha(2-\alpha)\|(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|^2. \end{aligned} \tag{6.11}$$

Proof. By a simple manipulation, we have

$$\begin{aligned} &\|(z^{k+1} - z^*) + A(x^{k+1} - x^*)\|^2 \\ &= \|[(z^k - z^*) + A(x^k - x^*)] - \alpha[(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)]\|^2 \\ &= \|(z^k - z^*) + A(x^k - x^*)\|^2 - 2\alpha((z^k - z^*) + A(x^k - x^*))^T((z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)) \\ &\quad +\alpha^2\|(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|^2 \\ &\leq \|(z^k - z^*) + A(x^k - x^*)\|^2 - \alpha(2-\alpha)\|(z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k)\|^2. \end{aligned}$$

The last inequality follows from Lemma 1.

Now we are ready to prove the global convergence of the proposed projection-type method.

Theorem 2. *For an arbitrarily chosen initial point u^0 and a slack factor $\alpha \in (0,\ 2)$, the proposed method will generate a sequence $\{\tilde{u}^k\}$ which converges to a solution of (1.3).*

Proof. By using $v := z + Ax$, (6.11) can be written as:

$$\|v^{k+1} - v^*\|^2 \leq \|v^k - v^*\|^2 - \alpha(2-\alpha)\|v^k - \tilde{v}^k\|^2. \tag{6.12}$$

Since $\alpha \in (0,\ 2)$, (6.12) means the sequence $\{v^k\}$ is Fejér-monotone with respect to v^*. Hence $\{v^k\}$ is bounded and

$$\lim_{k\to\infty} \|v^k - \tilde{v}^k\| = 0. \tag{6.13}$$

Note that

$$e(\tilde{u}^k) \quad = \quad \begin{pmatrix} e_x(\tilde{u}^k) \\ e_z(\tilde{u}^k) \end{pmatrix} = \begin{pmatrix} \tilde{x}^k - P_X[\tilde{x}^k - (A^T\tilde{z}^k + q_1)] \\ \tilde{z}^k - P_Z[\tilde{z}^k - (-A\tilde{x}^k + q_2)] \end{pmatrix}.$$

Substituting the first $\tilde{x}^k$ in $e_x(\tilde{u}^k)$ and the first $\tilde{z}^k$ in $e_z(\tilde{u}^k)$, respectively, by

$$P_X\{\tilde{x}^k - [(A^T\tilde{z}^k + q_1) + A^T(\tilde{z}^k - z^k) + A^TA(\tilde{x}^k - x^k)]\}, \tag{6.14}$$

$$P_Z[z^k + (Ax^k - q_2)], \tag{6.15}$$

according to (5.6) and (5.1), and using the non-expansive proposition (4.4) of the projection operator, we have

$$\begin{aligned} \|e(\tilde{u}^k)\| \quad &\le \quad \left\| \begin{pmatrix} A^T(z^k - \tilde{z}^k) + A^TA(x^k - \tilde{x}^k) \\ (z^k - \tilde{z}^k) + A(x^k - \tilde{x}^k) \end{pmatrix} \right\| \\ &\le \quad \left\| \begin{pmatrix} A^T \\ I \end{pmatrix} (v^k - \tilde{v}^k) \right\|. \end{aligned}$$

It follows that

$$\lim_{k\to\infty} e(\tilde{u}^k) = 0. \tag{6.16}$$

Let $\tilde{u}^*$ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to $\tilde{u}^*$. Since $e(u)$ is a continuous function with respect to u, according to (6.16) we have

$$e(\tilde{u}^*) = \lim_{j\to\infty} e(\tilde{u}^{k_j}) = 0. \tag{6.17}$$

It follows that every cluster point of $\{\tilde{u}^k\}$ is a solution of LVI (1.3). Assume that $\{\tilde{u}^k\}$ has more than one cluster points, e.g., $(\tilde{x}_1^*, \tilde{z}_1^*)$ and $(\tilde{x}_2^*, \tilde{z}_2^*)$. Denote

$$v_1^* = \tilde{z}_1^* + A\tilde{x}_1^*, \qquad v_2^* = \tilde{z}_2^* + A\tilde{x}_2^*, \tag{6.18}$$

then v_1^* and v_2^* are two cluster points of $\{\tilde{v}^k\}$. However (6.12) and (6.13) indicate that

$$\lim_{k\to\infty} v^k = \lim_{k\to\infty} \tilde{v}^k = v_1^* = v_2^*, \tag{6.19}$$

and thus

$$(\tilde{z}_1^* - \tilde{z}_2^*) = A(\tilde{x}_2^* - \tilde{x}_1^*). \tag{6.20}$$

According to the second inequality of LVI (1.3) and $\tilde{z}_1^*, \tilde{z}_2^* \in Z$, we have

$$(\,\tilde{z}_1^* - \tilde{z}_2^*)^T(-A\tilde{x}_2^* + q_2) \ge 0, (\tilde{z}_2^* - \tilde{z}_1^*)^T(-A\tilde{x}_1^* + q_2) \ge 0. \tag{6.21}$$

Adding the two inequalities in (6.21), we obtain

$$(\tilde{z}_1^* - \tilde{z}_2^*)^T(-A(\tilde{x}_2^* - \tilde{x}_1^*)) \ge 0. \tag{6.22}$$

Substituting (6.20) into (6.22) and using $A^TA = nI_k$, we get

$$-n\|\tilde{x}_1^* - \tilde{x}_2^*\|^2 \ge 0. \tag{6.23}$$

It follows that

$$\tilde{x}_1^* = \tilde{x}_2^*, \qquad \tilde{z}_1^* = \tilde{z}_2^*, \tag{6.24}$$

thus $\{\tilde{u}^k\}$ has exactly one cluster point and

$$\lim_{k\to\infty} \tilde{u}^k = u^*. \tag{6.25}$$

The proof is complete.

Remark 4. Using Weiszfeld procedure, NB method to solve Weber problem and the algorithm in [3] to solve constrained Weber problem require comparatively little CPU time. Compared with these popular methods, the proposed projection-type method has some merits:

1. The projection-type method is applicable for solving both Weber problem and constrained Weber problem whose constrained area is closed and convex;

2. The singularity that if an iterate is identical with one of customers then the next iterate would be undefined may happen in these popular methods. Whereas, it follows from Theorem 2 that this singularity would not happen in the proposed projection-type method, thus, for an arbitrarily chosen initial point this new projection-type method can acquire the solution of Weber problem and constrained Weber problem.

7. Numerical Experience

This section has two parts:

1). Subsection 7.1 illustrates that this singularity would not happen in the proposed projection-type method.

2). Subsection 7.2 reports the performance of the projection-type method for solving a large number of various Weber problems.

All tested problems are coded with MATLAB 6.5 and numerical experiments have been carried out in PC Pentium IV with CPU 1.7G. The stopping criterion of the proposed method is chosen as

$$\|e(u)\|_\infty < 10^{-5}. \tag{7.1}$$

7.1 Singularity

When using the popular methods to solve Weber problem and constrained Weber problem, according to (2.1) and (2.2), we readily know that if one iterate x^k is identical with one of the customers then the next iterate x^{k+1} is undefined and these methods would stop at a non-optimal solution. The reason for this singularity is that we choose a "bad" initial point. To avoid this singular case an appropriate initial point should be taken. Whereas Chandrasekaran and Tamir (1989) showed that the set of "bad" initial points may contain a continuum subset and in advance we have no way to clearly know whether one initial point is "good" or "bad". However, Theorem 2 guarantees that for an arbitrarily chosen initial point this new method can acquire the optimal solution of Weber problem and constrained Weber problem.

To illustrate this, we consider the following simple Weber problem. There are nine customers a_i $(i = 1, \cdots, 9)$ in the plane whose locations are given by the columns of the following matrix. All w_i are 1. It is easy to know its optimal solution is $(0, 0)$.

$$\begin{pmatrix} -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \tag{7.2}$$

When Weiszfeld procedure or NB method is used to solve this Weber problem, the following two kinds of points can't be selected as the initial point:

1) the nine customers and,

2) some other points which are not customers and beginning with which may generate an iterate identical with one of the customers.

However, any point in the plane can be taken as the initial point of the proposed projection-type method. Table 7.1 gives the main results of the new projection-type method for different initial points. In Table 7.1 "randomly" means this problem is tested 100 times with randomly generated initial points and the numbers are the average number of iterations and the average computing time.

Table 7.1. Main results for different initial points

initial point	iterations	CPU time(s)	initial point	iterations	CPU time(s)
a_1	38	0.01643	a_6	38	0.01702
a_2	38	0.01612	a_7	38	0.01832
a_3	38	0.01562	a_8	38	0.01563
a_4	38	0.01563	a_9	38	0.01562
a_5	1	0.00090	randomly	37.45	0.01532

This simple example shows that the proposed projection-type method removes the singularity and its implementation is not effected even when some of its iterates coincides with the location of a customer. This improvement is especially meaningful for the case that the locations of customers are dense. In this sense the proposed projection-type method has the advantage of Weiszfeld procedure and NB method.

7.2 Performance of the projection-type method

In order to verify practical efficiency of the proposed projection-type method, we use this method to solve a large number of constrained Weber problems. CWPs under l_2-norm are used in our numerical experiments, however, as we have shown, the proposed method are also applicable for solving various Weber problems under l_1 and l_∞-norms. Twenty CWPs are tested for each size and the average number of iterations and computing time are computed. The customers of tested problems are randomly generated from $(-10,\ 10)^2$ and the weights are randomly generated from $(0, 5)$. The constrained area X is taken as $X = \{x| \, \|x - (-0.5,\ 0)^T\|_2 \leq 1\}$.

Table 7.2 reports numerical results obtained by using different methods to solve linear variational inequalities arising from CWPs. "/" in Table 7.2 means computing time is more than two minutes. Table 7.2 shows that

1. As the sizes of CWPs increase, the number of iterations and computing time needed in the proposed method increase too. In general, they increase rather slowly with the size.

2. The proposed method needs much fewer iterations and less computing time than the method presented in [4] requires.

We know PC method in [4] is efficient for general monotone linear variational inequality. However, since the proposed method exploits the favorable structure of LVI (1.3), it is more efficient for this class of problems.

Table 7.2. Numerical results obtained by using different methods

	PC method in [4]		New method			PC method in [4]		New method	
n	iter.	CPU(s)	iter.	CPU(s)	n	iter.	CPU(s)	iter.	CPU(s)
50	58.14	0.15812	33.88	0.03125	600	404.50	22.76633	67.28	0.66705
100	137.23	0.73237	38.93	0.06953	700	422.87	29.89183	70.75	0.81405
200	208.52	2.51645	46.22	0.15594	800	531.30	46.26600	72.20	0.95780
300	274.68	5.98351	51.82	0.25908	900	632.22	66.48472	75.40	1.12810
400	344.24	10.82863	57.26	0.38500	1000	767.65	95.64139	81.80	1.36570
500	390.91	16.78115	61.88	0.51030	2000	>1000	/	86.30	2.83280

In Step 2 of the proposed method a slack technique is applied. According to Theorem 2, slack factor α can be drawn from the interval $(0,\ 2)$. In our numerical experiences, we have found that slack factor $\alpha \in [1.4,\ 1.8]$ is more efficient than $\alpha = 1$. Table 7.3 reveals computational efficiency of the slack technique. The columns of "slack" and "noslack" give the average number of iterations and computing time for this class of problems with slack factor α taken as 1.6 and 1, respectively. It is clear that the proposed method with slack factor $\alpha = 1.6$ needs about $2/3$ times number of iterations and computing time as the method with $\alpha = 1$ requires, which indicates that slack technique can improve computational efficiency of the proposed method greatly.

Table 7.3. Main results using the proposed method with/without slack technique applied

	noslack ($\alpha = 1$)		slack ($\alpha = 1.6$)			noslack ($\alpha = 1$)		slack ($\alpha = 1.6$)	
n	iter.	CPU(s.)	iter.	CPU(s.)	n	iter.	CPU(s.)	iter.	CPU(s.)
50	47.25	0.04375	33.88	0.03125	600	105.40	1.05780	67.28	0.66705
100	61.84	0.11492	38.93	0.06953	700	103.10	1.20160	70.75	0.81405
200	75.50	0.26026	46.22	0.15594	800	113.95	1.50235	72.20	0.95780
300	85.10	0.42814	51.82	0.25908	900	118.20	1.80160	75.40	1.12810
400	102.80	0.67970	57.26	0.38500	1000	122.10	2.04840	81.80	1.36570
500	105.30	0.89314	61.88	0.51030	2000	138.90	4.55150	86.30	2.83280

8. Conclusions

This paper discusses various Weber problems which are very relevant for practical applications. A transformation technique is adopted to reformulate various Weber problems as min-max problems from which a class of monotone linear variational inequalities is obtained. Based on the favorable structure of obtained variational inequalities, a new projection-type method is suggested, which is easy to implement and promising for solving this class of problems. It also has the advantage in comparison with some popular methods for solving Weber problem and constrained Weber problem: 1) it can be used to solve both Weber problem and constrained Weber problem; 2) the singular case that when one iterate is identical with one of customers the next iterate will be undefined would not happen in this projection-type method.

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