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JACOBI PSEUDOSPECTRAL METHOD FOR FOURTH ORDER PROBLEMS *1)

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Abstract

In this paper, we investigate Jacobi pseudospectral method for fourth order problems. We establish some basic results on the Jacobi-Gauss-type interpolations in non-uniformly weighted Sobolev spaces, which serve as important tools in analysis of numerical quadratures, and numerical methods of differential and integral equations. Then we propose Jacobi pseudospectral schemes for several singular problems and multiple-dimensional problems of fourth order. Numerical results demonstrate the spectral accuracy of these schemes, and coincide well with theoretical analysis.

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1. Introduction

The Jacobi polynomials play important roles in mathematical analysis and its applications. In particular, the Legendre and Chebyshev approximations have been used successfully for spectral and pseudospectral methods for non-singular differential equations, see [5, 9, 13, 14]. However, in many cases, we need to study other approximations. For instance, the usual Gauss-type interpolations are no longer available for numerical quadratures involving derivatives of functions at endpoints, and so we have to use certain specific Jacobi interpolations, see [12]. Next, in numerical analysis of finite element and boundary element methods, we took some results on the Jacobi approximations as important tools, see [1, 25, 26, 29]. The Jacobi approximations were also applied directly to numerical solutions of singular differential equations, and some problems on unbounded domains and aixymmetric domains, see [4, 15, 16, 17, 18]. Moreover, the Jacobi approximations are related to certain rational spectral methods and spectral method on triangle, see [11, 19, 20]. Recently, some authors developed the Jacobi spectral method for fourth order problems, see [23]. As for the Legendre spectral method for fourth order problems, we refer to the work of [6, 7, 27].

In actual computations, the pseudospectral method is more preferable, for which we only need to evaluate unknown functions at interpolation nodes, and thus save a lot of work. Especially, it is much easier to deal with various nonlinear problems. On the other hand, the most existing work are for second order problems. But it is also important to consider fourth order problems. For example, we may simulate incompressible fluid flows numerically, based

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on the stream function form of the Navier-Stokes equations, which fulfills the incompressibility automatically, and keeps physical boundary conditions. Since this is a nonlinear problem, we prefer to using pseudospectral method in actual computation.

The mathematical foundation of pseudospectral method for fourth order problems is the Jacobi-Gauss-type interpolations. In the early work, one considered these approximations in the standard Sobolev spaces, see [9, 14]. But, in many practical problems, the coefficients of derivatives of unknown functions involved in differential equations degenerate in different ways. Thus, the exact solutions are not in the standard Sobolev spaces. In other words, these problems are well- posed in certain non-uniformly weighted Sobolev spaces. Therefore, we have to investigate the Jacobi-Gauss-type interpolations in corresponding non-uniformly weighted Sobolev spaces. Some results on such approximations were established in [21, 22], which are very useful for pseudospectral method of second order problems. However, so far, there is no results which are appropriate for pseudospectral method of fourth order problems.

In this paper, we develop the Jacobi pseudospectral method for fourth order problems. We first establish some basic results on the Jacobi Gauss-type interpolations in certain non-uniformly weighted Sobolev spaces, which play important roles in the analysis of Jacobi pseudospectral method for fourth order problems. Then we propose the Jacobi pseudospectral schemes for several fourth order problems, such as singular differential equations and certain related problems. We also present some numerical results which demonstrate the spectral accuracy of proposed schemes, and coincide very well with theoretical analysis.

The paper is organized as follows. In the next section, we recall some recent results on the Jacobi polynomial approximation. In Section 3, we derive the basic results on the Jacobi-Gauss-type interpolations. In Section 4, we propose the Jacobi pseudospectral schemes for several problems of fourth order, and prove their convergence. In section 5, we present some numerical results. The final section is for concluding remarks.

2. Preliminaries

Let $\Lambda = \{x \mid |x| < 1\}$ and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all nonnegative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ as usual, with the inner product $(u,v)_{r,\chi}$, the semi-norm $|v|_{r,\chi}$ and the norm $\|v\|_{r,\chi}$. In particular, we denote by $(u,v)_\chi$ and $\|v\|_\chi$ the inner product and the norm of $L^2_\chi(\Lambda)$, respectively. For any $r > 0$, we define $H^r_\chi(\Lambda)$ and its norm by space interpolation as in [3]. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. When $\chi(x) \equiv 1$, we omit χ in the notations.

Denote by $J^{(\alpha,\beta)}_l(x), l = 1, 2, \ldots$ the Jacobi polynomials. Let $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta,\ \alpha, \beta > -1$. The set of Jacobi polynomials is the $L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$-orthogonal system.

For any $N \in \mathbb{N}$, $\mathcal{P}_N$ denotes the set of all algebraic polynomials of degree at most N. The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}_N$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\chi^{(\alpha,\beta)}} = 0, \qquad \forall \phi \in \mathcal{P}_N.$$

In order to describe the approximation errors, we introduce the space $H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$. For $r \in \mathbb{N}$, its semi-norm and norm are given by

$$|v|_{r,\chi^{(\alpha,\beta)},A} = \|\partial^r_x v\|_{\chi^{(\alpha+r,\beta+r)}}, \qquad \|v\|_{r,\chi^{(\alpha,\beta)},A} = (\sum_{k=0}^{r} \|\partial^k_x v\|^2_{\chi^{(\alpha+k,\beta+k)}})^{\frac{1}{2}}.$$

For any $r > 0$, we define the space and its norm by space interpolation as in [3].

Due to Theorem 2.1 of [22], we have that for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda), r \in \mathbb{N}$ and $0 \le \mu \le r$,

$$\|P_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \le cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{2.1}$$

Hereafter c denotes a generic positive constant independent of N and any function.

We now turn to the Jacobi orthogonal projections of high orders. As is well known, in many practical problems, the coefficients of derivatives of different orders involved in differential equations degenerate in different ways. Thereby the exact solutions are not in the standard Sobolev spaces, but in certain non-uniformly weighted Sobolev spaces. Consequently, we need to study the Jacobi orthogonal projections in non-uniformly weighted Sobolev spaces. To do this, let $\alpha,\beta,\gamma,\delta,\eta,\theta>-1$, and introduce the spaces $H^{\mu}_{\gamma,\delta,\eta,\theta}(\Lambda), 0\le\mu\le 1$ and $H^{\mu}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda), 0\le\mu\le 2$. For $\mu=0$, $H^0_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)=H^0_{\gamma,\delta,\eta,\theta}(\Lambda)=L^2_{\chi^{(\eta,\theta)}}(\Lambda)$. For $\mu=1$, we define the norm

$$\|v\|_{1,\gamma,\delta,\eta,\theta}=(|v|^2_{1,\chi^{(\gamma,\delta)}}+\|v\|^2_{\chi^{(\eta,\theta)}})^{\frac12}.$$

For $\mu=2$, we define the norm

$$\|v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}=(|v|^2_{2,\chi^{(\alpha,\beta)}}+|v|^2_{1,\chi^{(\gamma,\delta)}}+\|v\|^2_{\chi^{(\eta,\theta)}})^{\frac12}.$$

The spaces $H^{\mu}_{\gamma,\delta,\eta,\theta}(\Lambda)$, $0<\mu<1$ and $H^{\mu}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$, $0<\mu<2$ are defined by space interpolations as in [3], with the norms $\|v\|_{\mu,\gamma,\delta,\eta,\theta}$ and $\|v\|_{\mu,\alpha,\beta,\gamma,\delta,\eta,\theta}$, respectively.

Now, let

$$a_{\gamma,\delta,\eta,\theta}(u,v)=(\partial_x u,\partial_x v)_{\chi^{(\gamma,\delta)}}+(u,v)_{\chi^{(\eta,\theta)}},\quad \forall u,v\in H^1_{\gamma,\delta,\eta,\theta}(\Lambda),$$

$$a_{\alpha,\beta,\gamma,\delta,\eta,\theta}(u,v)=(\partial_x^2 u,\partial_x^2 v)_{\chi^{(\alpha,\beta)}}+(\partial_x u,\partial_x v)_{\chi^{(\gamma,\delta)}}+(u,v)_{\chi^{(\eta,\theta)}},\quad \forall u,v\in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda).$$

The orthogonal projection $P^1_{N,\gamma,\delta,\eta,\theta}:H^1_{\gamma,\delta,\eta,\theta}(\Lambda)\to\mathcal{P}_N$ is defined by

$$a_{\gamma,\delta,\eta,\theta}(P^1_{N,\gamma,\delta,\eta,\theta}v-v,\phi)=0,\quad\forall\phi\in\mathcal{P}_N.$$

The orthogonal projection $P^2_{N,\alpha,\beta,\gamma,\delta,\eta,\theta}:H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)\to\mathcal{P}_N$ is defined by

$$a_{\alpha,\beta,\gamma,\delta,\eta,\theta}(P^2_{N,\alpha,\beta,\gamma,\delta,\eta,\theta}v-v,\phi)=0,\ \ \forall\phi\in\mathcal{P}_N.$$

For description of approximation results, we introduce the spaces $H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r\ge 1$ and $H^r_{\chi^{(\alpha,\beta)},**}(\Lambda), r\ge 2$. For $r\in\mathbb{N}$, their semi-norms and norms are given by

$$|v|_{r,\chi^{(\alpha,\beta)},*}=\|\partial_x^r v\|_{\chi^{(\alpha+r-1,\beta+r-1)}},\qquad \|v\|_{r,\chi^{(\alpha,\beta)},*}=(\sum_{k=0}^{r-1}\|\partial_x^{k+1}v\|^2_{\chi^{(\alpha+k,\beta+k)}})^{\frac12},$$

$$|v|_{r,\chi^{(\alpha,\beta)},**}=\|\partial_x^r v\|_{\chi^{(\alpha+r-2,\beta+r-2)}},\qquad \|v\|_{r,\chi^{(\alpha,\beta)},**}=(\sum_{k=0}^{r-2}\|\partial_x^{k+2}v\|^2_{\chi^{(\alpha+k,\beta+k)}})^{\frac12}.$$

We define the spaces $H^r_{\chi^{(\alpha,\beta)},*}(\Lambda), r>1$ and $H^r_{\chi^{(\alpha,\beta)},**}(\Lambda), r>2$ by space interpolations as in [3].

We know from Theorem 3.1 of [22] that if $\gamma\le\eta+2,\delta\le\theta+2, r\in\mathbb{N}$ and $r\ge 1$, then for any $v\in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$,

$$\|P^1_{N,\gamma,\delta,\eta,\theta}v-v\|_{1,\gamma,\delta,\eta,\theta}\le cN^{1-r}|v|_{r,\chi^{(\gamma,\delta)},*}. \tag{2.2}$$

If, in addition, $\gamma\le\eta+1$ and $\delta\le\theta+1$, then for $0\le\mu\le 1$,

$$\|P^1_{N,\gamma,\delta,\eta,\theta}v-v\|_{\mu,\gamma,\delta,\eta,\theta}\le cN^{\mu-r}|v|_{r,\chi^{(\gamma,\delta)},*}. \tag{2.3}$$

Furthermore, we have from Theorem 2.1 of [23] that if $\alpha\le\min(\gamma+2,\eta+4)$, $\beta\le\min(\delta+2,\theta+4), r\in\mathbb{N}$ and $r\ge 2$, then for any $v\in H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$,

$$\|P^2_{N,\alpha,\beta,\gamma,\delta,\eta,\theta}v-v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}\le cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{2.4}$$

If, in addition, $\alpha \le \min(\gamma, \eta+2)$ and $\beta \le \min(\delta, \theta+2)$, then for $0 \le \mu \le 2$,

$$\|P^2_{N,\alpha,\beta,\gamma,\delta,\eta,\theta}v - v\|_{\mu,\alpha,\beta,\gamma,\delta,\eta,\theta} \le cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{2.5}$$

We next consider a special orthogonal projection which plays an important role in the analysis of Jacobi pseudospectral method for axisymmetric domains. To do this, let

$$\begin{aligned}
{}^0H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) &= \{v \mid v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) \quad \text{and} \quad v(1) = 0\},\\
{}^0_{00}H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) &= \{v \mid v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) \quad \text{and} \quad v(\pm 1) = \partial_x v(-1) = 0\},\\
H^2_{0,\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) &= \{v \mid v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda) \quad \text{and} \quad v(\pm 1) = \partial_x v(\pm 1) = 0\}.
\end{aligned}$$

Set $\triangle_x = \partial_x^2 - (1-x)^{-1}\partial_x$, and

$$\mathcal{B}_\beta(u,v) = (\triangle_x u, (1-x)\triangle_x((1+x)^\beta v)), \quad \forall u, v \in H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda).$$

It was shown in [23] that for any $u \in {}^0H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$, $v \in {}^0_{00}H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$ and $\beta < 1$,

$$\mathcal{B}_\beta(u,v) \le c\|u\|_{2,1,\beta,-1,\beta,1,\beta}\|v\|_{2,1,\beta,-1,\beta,1,\beta}. \tag{2.6}$$

If, in addition, $0 \le \beta \le \frac{4}{5}$, then

$$\mathcal{B}_\beta(v,v) \ge c\|v\|^2_{2,1,\beta,-1,\beta,1,\beta}. \tag{2.7}$$

Let

$$\mathcal{P}^{00}_N = \{v \mid v \in \mathcal{P}_N \text{ and } v(\pm 1) = \partial_x v(\pm 1) = 0\}.$$

We introduce the mapping $Q^2_{N,\beta} : H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda) \to \mathcal{P}_N$ such that $Q^2_{N,\beta}v - v \in H^2_{0,1,\beta,-1,\beta,1,\beta}(\Lambda)$ and

$$\mathcal{B}_\beta(Q^2_{N,\beta}v - v, \phi) = 0, \quad \forall \phi \in \mathcal{P}^{00}_N. \tag{2.8}$$

According to Theorem 3.1 of [23], we have that for any $v \in H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda) \cap H^r_{\chi^{(\bar\alpha,\beta)},**}(\Lambda)$, $r \in \mathbb{N}, r \ge 2, -1 < \bar\alpha < 1$ and $0 \le \beta \le \frac{4}{5}$,

$$\|Q^2_{N,\beta}v - v\|_{2,1,\beta,-1,\beta,1,\beta} \le cN^{2-r}|v|_{r,\chi^{(\bar\alpha,\beta)},**}. \tag{2.9}$$

We now give another orthogonal projection which will be used in the sequel.

The orthogonal projection $\tilde P^{2,0}_{N,\alpha,\beta} : H^2_{0,\chi^{(\alpha,\beta)}}(\Lambda) \to \mathcal{P}^{00}_N$ is defined by

$$(\partial_x^2(\tilde P^{2,0}_{N,\alpha,\beta}v - v), \partial_x^2\phi)_{\chi^{(\alpha,\beta)}} = 0, \quad \forall \phi \in \mathcal{P}^{00}_N.$$

We know from Theorem 2.5 of [23] that if $-1 < \alpha, \beta < 1$, $\alpha \le \gamma + 2$, $\beta \le \delta + 2$, then for any $v \in H^2_{0,\chi^{(\alpha,\beta)}}(\Lambda) \bigcap H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$ and integer $r \ge 2$,

$$\|\tilde P^{2,0}_{N,\alpha,\beta}v - v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \le cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{2.10}$$

In particular, for $0 \le \mu \le 2$,

$$\|\tilde P^{2,0}_{N,0,0}v - v\|_{\mu,0,0,\gamma,\delta,\eta,\theta} \le cN^{\mu-r}|v|_{r,\chi^{(0,0)},**}. \tag{2.11}$$

3. Jacobi-Gauss-Type Interpolations

In this section, we establish the basic results on the Jacobi-Gauss-type interpolations in certain non-uniformly weighted Sobolev spaces, which are appropriate for the analysis of Jacobi pseudospectral method for fourth order problems.

Let $\zeta^{(\alpha,\beta)}_{G,N,j}$, $\zeta^{(\alpha,\beta)}_{R,N,j}$ and $\zeta^{(\alpha,\beta)}_{L,N,j}$ be the $N+1$ zeros of polynomials $J^{(\alpha,\beta)}_{N+1}(x)$, $(1+x)J^{(\alpha,\beta+1)}_N(x)$ and $(1-x^2)J^{(\alpha+1,\beta+1)}_{N-1}(x)$, respectively. Assume that they are arranged in decreasing order.

According to [28], there exist the corresponding Christoffel numbers $\omega_{Z,N,j}^{(\alpha,\beta)}$, $0 \le j \le N$ such that

$$\int_\Lambda \phi(x)\chi^{(\alpha,\beta)}(x)dx = \sum_{j=0}^{N} \phi(\zeta_{Z,N,j}^{(\alpha,\beta)})\omega_{Z,N,j}^{(\alpha,\beta)}, \quad \forall \phi \in \mathcal{P}_{2N+\lambda_Z}, \quad Z = G, R, L, \tag{3.1}$$

where $\lambda_Z = 1$ for $Z = G$, $\lambda_Z = 0$ for $Z = R$, and $\lambda_Z = -1$ for $Z = L$, respectively. The corresponding discrete inner product and norm are defined by

$$(u, v)_{\chi^{(\alpha,\beta)},Z,N} = \sum_{j=0}^{N} u(\zeta_{Z,N,j}^{(\alpha,\beta)})v(\zeta_{Z,N,j}^{(\alpha,\beta)})\omega_{Z,N,j}^{(\alpha,\beta)}, \quad \|v\|_{\chi^{(\alpha,\beta)},Z,N} = (v, v)^{\frac{1}{2}}_{\chi^{(\alpha,\beta)},Z,N}.$$

By (3.1),

$$(\phi, \psi)_{\chi^{(\alpha,\beta)},Z,N} = (\phi, \psi)_{\chi^{(\alpha,\beta)}}, \quad \forall \phi \cdot \psi \in \mathcal{P}_{2N+\lambda_Z}, \quad Z = G, R, L. \tag{3.2}$$

Moreover, for any $\phi \in \mathcal{P}_N$ (see (2.26) of [21]),

$$\|\phi\|_{\chi^{(\alpha,\beta)}} \le \|\phi\|_{\chi^{(\alpha,\beta)},L,N} \le \sqrt{2 + \frac{\alpha+\beta+1}{N}}\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{3.3}$$

The Jacobi-Gauss-type interpolations $\mathcal{I}_{Z,N,\alpha,\beta}v \in \mathcal{P}_N$ are determined by

$$\mathcal{I}_{Z,N,\alpha,\beta}v(\zeta_{Z,N,j}^{(\alpha,\beta)}) = v(\zeta_{Z,N,j}^{(\alpha,\beta)}), \quad 0 \le j \le N, \quad Z = G, R, L. \tag{3.4}$$

They are named as the Jacobi-Gauss interpolation for $Z = G$, the Jacobi-Gauss-Radau interpolation for $Z = R$, and the Jacobi-Gauss-Lobatto interpolation for $Z = L$, respectively. These interpolations are stable. In fact, for any $v \in H^1_{\chi^{(\alpha,\beta)},A}(\Lambda)$ (see Theorems 4.1 , 4.5 and 4.9 of [22]),

$$\|\mathcal{I}_{Z,N,\alpha,\beta}v\|_{\chi^{(\alpha,\beta)}} \le c(\|v\|_{\chi^{(\alpha,\beta)}} + N^{-1}|v|_{1,\chi^{(\alpha,\beta)},A}) \tag{3.5}$$

where $Z = G$ for general $v(x)$, $z = R$ if $v(-1) = 0$, and $z = L$ if $v(-1) = v(1) = 0$.

Before deriving the error estimates of the above interpolations in the space $H^\mu_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$, we list the following inverse and imbedding inequalities:

- For any $\phi \in \mathcal{P}_N$ and $r \ge 0$,

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \le cN^{2r}\|\phi\|_{\chi^{(\alpha,\beta)}}. \tag{3.6}$$

  If, in addition, $\alpha, \beta > r - 1$, then

$$\|\phi\|_{r,\chi^{(\alpha,\beta)}} \le cN^{r}\|\phi\|_{\chi^{(\alpha-r,\beta-r)}}. \tag{3.7}$$

- If $v(x_0) = 0$ for $x_0 \in \Lambda$, then for $\gamma \le \eta + 2$ and $\delta \le \theta + 2$,

$$\|v\|_{\chi^{(\eta,\theta)}} \le c|v|_{1,\chi^{(\gamma,\delta)}}. \tag{3.8}$$

- For any measurable function $\psi(x)$, real numbers $a \le b$ and $q < 1$,

$$\int_a^b (\frac{1}{b-x}\int_x^b \psi(y)dy)^2(b-x)^q dx \le \frac{4}{1-q}\int_a^b \psi^2(x)(b-x)^q dx. \tag{3.9}$$

The results (3.6) and (3.7) come form [18]. The result (3.8) comes from [21]. The result (3.9) comes from [24].

3.1 Jacobi-Gauss Interpolation

We first estimate the approximation error of Jacobi-Gauss interpolation.

Lemma 3.1 (Theorem 4.2 of [22]). *For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, integer $r \geq 1$ and $0 \leq \mu \leq r$,*

$$\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \leq cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.10}$$

In numerical analysis of pseudospectral methods for hight order problems, it is more important to estimate the approximation errors in non-uniformly weighted Sobolev spaces, stated below.

Theorem 3.1. *If*

$$\alpha \leq \min(\gamma+2,\eta+4), \quad \beta \leq \min(\delta+2,\theta+4), \tag{3.11}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$ and integer $r \geq 4$,

$$\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \leq cN^{4-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{3.12}$$

Moreover, if

$$\alpha \leq \min(\gamma,\eta+2), \quad \beta \leq \min(\delta,\theta+2), \tag{3.13}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and integer $r \geq 2$,

$$\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{1,\gamma,\delta,\eta,\theta} \leq cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.14}$$

Furthermore, if

$$\alpha \leq \eta, \quad \beta \leq \theta, \tag{3.15}$$

then for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integer $r \geq 1$,

$$\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\eta,\theta)}} \leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.16}$$

Proof. Obviously, $\mathcal{I}_{G,N,\alpha,\beta}v(\zeta^{(\alpha,\beta)}_{G,N,j}) - v(\zeta^{(\alpha,\beta)}_{G,N,j}) = 0, \quad 0 \leq j \leq N$. Take $x_0 = \zeta^{(\alpha,\beta)}_{G,N,N}$. By Rolle theorem, there exists $x_1 \in (\zeta^{(\alpha,\beta)}_{G,N,N}, \zeta^{(\alpha,\beta)}_{G,N,N-1})$, such that $\partial_x\mathcal{I}_{G,N,\alpha,\beta}v(x_1) - \partial_x v(x_1) = 0$. For simplicity, we denote $P^1_{N,\alpha,\beta,\alpha,\beta}$ by $P^1_{N,\alpha,\beta}$ and $P^2_{N,\alpha,\beta,\alpha,\beta,\alpha,\beta}$ by $P^2_{N,\alpha,\beta}$. Then by (3.8), (3.11), (2.5), (3.6) and (3.10),

$$\begin{aligned}
\|\mathcal{I}_{G,N,\alpha,\beta}\, v - v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} &\leq c(|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{2,\chi^{(\alpha,\beta)}} + |\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\eta+2,\theta+2)}}) \\
&\leq c(|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{2,\chi^{(\alpha,\beta)}} + |\mathcal{I}_{G,N,\alpha,\beta}v - v|_{2,\chi^{(\eta+4,\theta+4)}}) \leq c|\mathcal{I}_{G,N,\alpha,\beta}v - v|_{2,\chi^{(\alpha,\beta)}} \\
&\leq c(|P^2_{N,\alpha,\beta}v - v|_{2,\chi^{(\alpha,\beta)}} + |\mathcal{I}_{G,N,\alpha,\beta}v - P^2_{N,\alpha,\beta}v|_{2,\chi^{(\alpha,\beta)}}) \\
&\leq cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},**} + cN^4(\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} + \|P^2_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}}) \\
&\leq cN^{4-r}|v|_{r,\chi^{(\alpha,\beta)},**}.
\end{aligned} \tag{3.17}$$

This leads to (3.12). If (3.13) holds, then by (3.8), (2.3), (3.6) and (3.10),

$$\begin{aligned}
\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{1,\gamma,\delta,\eta,\theta} &\leq |\mathcal{I}_{G,N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} \\
&\leq |P^1_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} + |\mathcal{I}_{G,N,\alpha,\beta}v - P^1_{N,\alpha,\beta}v|_{1,\chi^{(\alpha,\beta)}} \\
&\leq |P^1_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)}} + cN^2(\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} + \|P^1_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}}) \\
&\leq cN^{1-r}|v|_{r,\chi^{(\alpha,\beta)},*} + cN^{2-r}(|v|_{r,\chi^{(\alpha,\beta)},A} + |v|_{r,\chi^{(\alpha,\beta)},*}) \leq cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},*}.
\end{aligned} \tag{3.18}$$

This leads to (3.14). If (3.15) holds, then $\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\eta,\theta)}} \leq c\|\mathcal{I}_{G,N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}}$. This with (3.10) leads to (3.16).

3.2 Jacobi-Gauss-Radau Interpolation

For analyzing the Jacobi-Gauss-Radau Interpolation, we need the following lemma.

Lemma 3.2 (Theorem 4.6 of [22]). *For any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, integer $r \geq 1$ and $0 \leq \mu \leq r$,*

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \leq cN^{\mu-r}|v|_{\mu,\chi^{(\alpha,\beta)},A}. \tag{3.19}$$

Theorem 3.2. *If (3.11) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$ and integer $r \geq 4$,*

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \leq cN^{4-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{3.20}$$

If (3.13) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and integer $r \geq 2$,

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{1,\gamma,\delta,\eta,\theta} \leq cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},*}, \tag{3.21}$$

and if (3.15) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integer $r \geq 1$,

$$\|\mathcal{I}_{R,N,\alpha,\beta}v - v\|_{\chi^{(\eta,\theta)}} \leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.22}$$

Proof. We can prove this theorem by (3.19) and an argument similar to the proof of Theorem 3.1.

3.3 Jacobi-Gauss-Lobatto Interpolation

We now turn to the Jacobi-Gauss-Lobatto interpolation. We need the following lemma.

Lemma 3.3. *There exists a mapping $\widehat{P}^2_{N,\alpha,\beta} : H^2_{\chi^{(\alpha,\beta)},A} \to \mathcal{P}_N$ such that $\widehat{P}^2_{N,\alpha,\beta}v(-1) = v(-1), \widehat{P}^2_{N,\alpha,\beta}v(1) = v(1)$ and for any $v \in H^2_{\chi^{(\alpha,\beta)},A}(\Lambda)$,*

$$(\partial_x^2(\widehat{P}^2_{N,\alpha,\beta}v - v), \partial_x^2\phi)_{\chi^{(\alpha+2,\beta+2)}} = 0, \quad \forall \phi \in \mathcal{P}_N. \tag{3.23}$$

Moreover, for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$, integer $r \geq 2$ and $0 \leq \mu \leq r$,

$$\|\widehat{P}^2_{N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \leq cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.24}$$

Proof. Let $P_{N,\alpha,\beta}$ be the orthogonal projection as in (2.1), and set

$$\widehat{P}^2_{N,\alpha,\beta}v(x) = \int_{-1}^{x}\int_{-1}^{y} P_{N-2,\alpha+2,\beta+2}\partial_s^2 v(s)dsdy + ax + b$$

where a and b are chosen in such a way that $\widehat{P}^2_{N,\alpha,\beta}v(-1) = v(-1)$ and $\widehat{P}^2_{N,\alpha,\beta}v(1) = v(1)$. By the definition of $P_{N-2,\alpha+2,\beta+2}$, we have that

$$(\partial_x^2(\widehat{P}^2_{N,\alpha,\beta}v - v), \partial_x^2\phi)_{\chi^{(\alpha+2,\beta+2)}} = (P_{N-2,\alpha+2,\beta+2}\partial_x^2 v - \partial_x^2 v, \partial_x^2\phi)_{\chi^{(\alpha+2,\beta+2)}} = 0, \quad \forall \phi \in \mathcal{P}_N.$$

For any integer $\mu \geq 2$, we use (2.1) to verify that

$$\begin{aligned}\|\partial_x^\mu(\widehat{P}^2_{N,\alpha,\beta}v - v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}} &= \|\partial_x^{\mu-2}(P_{N-2,\alpha+2,\beta+2}\partial_x^2 v - \partial_x^2 v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}} \\ &\leq cN^{\mu-r}|\partial_x^2 v|_{r-2,\chi^{(\alpha+2,\beta+2)},A} = cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}.\end{aligned} \tag{3.25}$$

We now prove (3.24) with $\mu = 0$. Let $g \in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)$ and consider the auxiliary problem

$$(\partial_x^2 w, \partial_x^2 z)_{\chi^{(\alpha+2,\beta+2)}} = (g, z)_{\chi^{(\alpha,\beta)}}, \quad \forall z \in H^2_{\chi^{(\alpha,\beta)},A}(\Lambda). \tag{3.26}$$

Let $z(x)$ vary in $\mathcal{D}(\Lambda)$. Then in the sense of distributions,

$$\partial_x^2(\partial_x^2 w(x)\chi^{(\alpha+2,\beta+2)}(x)) = g(x)\chi^{(\alpha,\beta)}(x). \tag{3.27}$$

Accordingly,

$$\partial_x^4 w(x) = q_1(x)\chi^{(-1,-1)}(x)\partial_x^3 w(x) + q_2(x)\chi^{(-2,-2)}(x)\partial_x^2 w(x) + \chi^{(-2,-2)}(x)g(x)$$

where $q_1(x)$ and $q_2(x)$ are some polynomials of x. Hence

$$\|\partial_x^4 w\|_{\chi^{(\alpha+4,\beta+4)}} \leq c(\|\partial_x^3 w\|_{\chi^{(\alpha+2,\beta+2)}} + \|\partial_x^2 w\|_{\chi^{(\alpha,\beta)}} + \|g\|_{\chi^{(\alpha,\beta)}}). \tag{3.28}$$

So it remains to estimate $\|\partial_x^3 w\|_{\chi^{(\alpha+2,\beta+2)}}$ and $\|\partial_x^2 w\|_{\chi^{(\alpha,\beta)}}$. Since $\alpha+1, \beta+1 > 0$, we have that $\partial_x^2 w(x)\chi^{(\alpha+2,\beta+2)}(x) \to 0$ and $\partial_x(\partial_x^2 w(x)\chi^{(\alpha+2,\beta+2)}(x)) \to 0$ as $|x| \to 1$. Thus integrating (3.27) and using the Hardy inequality (3.9) twice yield that

$$\begin{aligned}\int_0^1 (\partial_x^2 w(x))^2\ \chi^{(\alpha,\beta)}(x)dx &= \int_0^1 \chi^{(-\alpha-4,-\beta-4)}(x)(\int_x^1\int_y^1 g(s)\chi^{(\alpha,\beta)}(s)dsdy)^2 dx\\ &\leq c\int_0^1 (1-x)^{-\alpha-2}(x)(\frac{1}{1-x}\int_x^1\int_y^1 g(s)\chi^{(\alpha,\beta)}(s)dsdy)^2 dx\\ &\leq c\int_0^1 (1-x)^{-\alpha-2}(x)(\int_x^1 g(y)\chi^{(\alpha,\beta)}(y)dy)^2 dx \leq c\int_0^1 g^2(x)\chi^{(\alpha,\beta)}(x)dx.\end{aligned} \tag{3.29}$$

Similarly, we use (3.27) to derive the expression of $\partial_x^3 w(x)$, and use (3.9) to deduce that

$$\begin{aligned}&\int_0^1 (\partial_x^3 w(x))^2\chi^{(\alpha+2,\beta+2)}(x)dx\\ &\leq c\int_0^1 \chi^{(-\alpha-2,-\beta-2)}(x)((\int_x^1 \partial_y^2 w(y)\chi^{(\alpha,\beta)}(y)dy)^2 + (\int_x^1 g(y)\chi^{(\alpha,\beta)}(y)dy)^2)dx\\ &\leq c\int_0^1 ((\partial_x^2 w(x))^2 + g^2(x))\chi^{(\alpha,\beta)}(x)dx.\end{aligned} \tag{3.30}$$

Also, there hold the estimates on the subinterval [-1,0], like (3.29) and (3.30). The previous statements with (3.28) implies that $\|\partial_x^4 w\|_{\chi^{(\alpha+4,\beta+4)}} \leq c\|g\|_{\chi^{(\alpha,\beta)}}$. Now, taking $z = \widehat{P}_{N,\alpha,\beta}^2 v - v$ in (3.26), we use (2.1) and (3.25) to obtain that

$$\begin{aligned}|(\widehat{P}_{N,\alpha,\beta}^2 v - v, g)_{\chi^{(\alpha,\beta)}}| &= |(\partial_x^2(\widehat{P}_{N,\alpha,\beta}^2 v - v), \partial_x^2 w)_{\chi^{(\alpha+2,\beta+2)}}|\\ &= |(\partial_x^2(\widehat{P}_{N,\alpha,\beta}^2 v - v), P_{N-2,\alpha+2,\beta+2}\partial_x^2 w - \partial_x^2 w)_{\chi^{(\alpha+2,\beta+2)}}|\\ &\leq \|\partial_x^2(\widehat{P}_{N,\alpha,\beta}^2 v - v)\|_{\chi^{(\alpha+2,\beta+2)}}\|P_{N-2,\alpha+2,\beta+2}\partial_x^2 w - \partial_x^2 w\|_{\chi^{(\alpha+2,\beta+2)}}\\ &\leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}\|\partial_x^4 w\|_{\chi^{(\alpha+4,\beta+4)}} \leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}\|g\|_{\chi^{(\alpha,\beta)}}.\end{aligned}$$

Consequently,

$$\|\widehat{P}_{N,\alpha,\beta}^2 v - v\|_{\chi^{(\alpha,\beta)}} = \sup_{\substack{g\in L^2_{\chi^{(\alpha,\beta)}}(\Lambda)\\ g\neq 0}} .\frac{|(\widehat{P}_{N,\alpha,\beta}^2 v - v, g)_{\chi^{(\alpha,\beta)}}|}{\|g\|_{\chi^{(\alpha,\beta)}}} \leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.31}$$

We next prove (3.24) with $\mu = 1$. Let $g \in L^2_{\chi^{(\alpha+1,\beta+1)}}(\Lambda)$ and consider the auxiliary problem

$$(\partial_x^2 w, \partial_x^2 z)_{\chi^{(\alpha+2,\beta+2)}} = (g, \partial_x z)_{\chi^{(\alpha+1,\beta+1)}}, \quad \forall z \in H^2_{\chi^{(\alpha,\beta)},A}(\Lambda). \tag{3.32}$$

In the sense of distributions,

$$-\partial_x(\partial_x^2 w(x)\chi^{(\alpha+2,\beta+2)}(x)) = g(x)\chi^{(\alpha+1,\beta+1)}(x). \tag{3.33}$$

Thus $\partial_x^3 w(x) = q_1(x)\chi^{(-1,-1)}(x)\partial_x^2 w(x) - \chi^{(-1,-1)}(x)g(x)$, $q_1(x)$ being a certain polynomial of x. This implies that $\|\partial_x^3 w\|_{\chi^{(\alpha+3,\beta+3)}} \leq c(\|\partial_x^2 w\|_{\chi^{(\alpha+1,\beta+1)}} + \|g\|_{\chi^{(\alpha+1,\beta+1)}})$. So it suffices to estimate $\|\partial_x^2 w\|_{\chi^{(\alpha+1,\beta+1)}}$. Since $\alpha+2, \beta+2 > 1$, we have that $\partial_x^2 w(x)\chi^{(\alpha+2,\beta+2)}(x) \to 0$ as $|x| \to 0$. Therefore integrating (3.33) and using (3.9), we derive that

$$\int_0^1 (\partial_x^2 w(x))^2 \chi^{(\alpha+1,\beta+1)}(x)dx = \int_0^1 \chi^{(-\alpha-3,-\beta-3)}(x)(\int_x^1 g(y)\chi^{(\alpha+1,\beta+1)}(y)dy)^2 dx$$
$$\leq c\int_0^1 (1-x)^{-\alpha-1}(x)(\frac{1}{1-x}\int_x^1 g(y)\chi^{(\alpha+1,\beta+1)}(y)dy)^2 dx \leq c\int_0^1 g^2(x)\chi^{(\alpha+1,\beta+1)}(x)dx. \tag{3.34}$$

A result similar to (3.34) is valid on the subinterval [-1,0]. The above facts leads to $\|\partial_x^3 w\|_{\chi^{(\alpha+3,\beta+3)}} \leq c\|g\|_{\chi^{(\alpha+1,\beta+1)}}$. Now, taking $z = \widehat{P}^2_{N,\alpha,\beta} v - v$ in (3.32), we use (2.1) and (3.25) to verify that

$$|(\partial_x(\widehat{P}^2_{N,\alpha,\beta}v - v), g)_{\chi^{(\alpha+1,\beta+1)}}| = |(\partial_x^2(\widehat{P}^2_{N,\alpha,\beta}v - v), P_{N-2,\alpha+2,\beta+2}\partial_x^2 w - \partial_x^2 w)_{\chi^{(\alpha+2,\beta+2)}}|$$
$$\leq \|\partial_x^2(\widehat{P}^2_{N,\alpha,\beta}v - v)\|_{\chi^{(\alpha+2,\beta+2)}}\|P_{N-2,\alpha+2,\beta+2}\partial_x^2 w - \partial_x^2 w\|_{\chi^{(\alpha+2,\beta+2)}}$$
$$\leq cN^{1-r}|v|_{r,\chi^{(\alpha,\beta)},A}\|\partial_x^3 w\|_{\chi^{(\alpha+3,\beta+3)}} \leq cN^{1-r}|v|_{r,\chi^{(\alpha,\beta)},A}\|g\|_{\chi^{(\alpha+1,\beta+1)}}.$$

Therefore,

$$\|\partial_x(\widehat{P}^2_{N,\alpha,\beta}v - v)\|_{\chi^{(\alpha+1,\beta+1)}} = \sup_{\substack{g\in L^2_{\chi^{(\alpha+1,\beta+1)}}(\Lambda)\\ g\neq 0}} .\frac{|(\partial_x(\widehat{P}^2_{N,\alpha,\beta}v - v), g)_{\chi^{(\alpha+1,\beta+1)}}|}{\|g\|_{\chi^{(\alpha+1,\beta+1)}}} \leq cN^{1-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.35}$$

Finally, (3.24) follows from (3.25), (3.31), (3.35) and space interpolation.

Lemma 3.4. *For any $v \in H^\mu_{\chi^{(\alpha,\beta)},A}(\Lambda)$, integer $r \geq 2$ and $0 \leq \mu \leq r$,*

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{\mu,\chi^{(\alpha,\beta)},A} \leq cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.36}$$

Proof. Due to (3.5), (3.7) and (3.24), for integer $\mu \geq 0$,

$$\|\partial_x^\mu(\mathcal{I}_{L,N,\alpha,\beta}v - \widehat{P}^2_{N,\alpha,\beta}v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}} \leq cN^\mu\|\mathcal{I}_{L,N,\alpha,\beta}(\widehat{P}^2_{N,\alpha,\beta}v - v)\|_{\chi^{(\alpha,\beta)}}$$
$$\leq cN^\mu(\|\widehat{P}^2_{N,\alpha,\beta}v - v\|_{\chi^{(\alpha,\beta)}} + N^{-1}|\widehat{P}^2_{N,\alpha,\beta}v - v|_{1,\chi^{(\alpha,\beta)},A}) \leq cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}.$$

Moreover, using (3.24) again gives that

$$\|\partial_x^\mu(\mathcal{I}_{L,N,\alpha,\beta}v - v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}} \leq \|\partial_x^\mu(\widehat{P}^2_{N,\alpha,\beta}v - v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}}$$
$$+\|\partial_x^\mu(\mathcal{I}_{L,N,\alpha,\beta}v - \widehat{P}^2_{N,\alpha,\beta}v)\|_{\chi^{(\alpha+\mu,\beta+\mu)}} \leq cN^{\mu-r}|v|_{r,\chi^{(\alpha,\beta)},A}.$$

We complete the proof with space interpolation.

We are now in position to estimate the approximation error of $\mathcal{I}_{L,N,\alpha,\beta}$ in the space $H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}$ (Λ).

Theorem 3.3. *If (3.11) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$ and integer $r \geq 4$,*

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \leq cN^{4-r}|v|_{r,\chi^{(\alpha,\beta)},**}. \tag{3.37}$$

If (3.13) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},*}(\Lambda)$ and integer $r \geq 2$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{1,\gamma,\delta,\eta,\theta} \leq cN^{2-r}|v|_{r,\chi^{(\alpha,\beta)},*}. \tag{3.38}$$

Moreover, if (3.15) holds, then for any $v \in H^r_{\chi^{(\alpha,\beta)},A}(\Lambda)$ and integer $r \geq 1$,

$$\|\mathcal{I}_{L,N,\alpha,\beta}v - v\|_{\chi^{(\eta,\theta)}} \leq cN^{-r}|v|_{r,\chi^{(\alpha,\beta)},A}. \tag{3.39}$$

Proof. We prove this theorem by using Lemma 3.4 and the same argument as in the proof of Theorem 3.1.

4. Jacobi Pseudospectral Method for Fourth Order Problems

In this section, we propose the Jacobi pseudospectral schemes for several model problems of fourth order, prove their convergence and present some numerical results.

Model Problem 1. We first consider the singular problem

$$\partial_x^2(a(x)\partial_x^2 U(x)) - \partial_x(b(x)\partial_x U(x)) + c(x)U(x) = f(x), \qquad x \in \Lambda. \tag{4.1}$$

For simplicity, we assume that

$$a(x) = a_1(x)\chi^{(\alpha,\beta)}(x), \quad b(x) = b_1(x)\chi^{(\gamma,\delta)}(x), \quad c(x) = c_1(x)\chi^{(\eta,\theta)}(x), \quad \alpha,\beta,\gamma,\delta > 0, \tag{4.2}$$

$$a_1(x) \in H^s(\Lambda), \quad b_1(x) \in H^{s'}(\Lambda), \quad c_1(x) \in H^{s''}(\Lambda), \qquad s, s', s'' > \frac{3}{4}, \tag{4.3}$$

$$a_1(x) \geq a_{\min} > 0, \quad b_1(x) \geq b_{\min} > 0, \quad c_1(x) \geq c_{\min} > 0, \quad \forall x \in \bar{\Lambda}. \tag{4.4}$$

Clearly, for any $v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$, we have that $(1-x)^{\alpha+1}(1+x)^{\beta+1}(\partial_x^2 v(x))^2 \to 0$, $(1-x)^{\gamma+1}(1+x)^{\delta+1}(\partial_x v(x))^2 \to 0$ as $|x| \to 1$. Consequently, for $m \in \mathbb{N}$,

$$(1-x)^{\frac{\alpha}{2}-\frac{3}{2}+m}(1+x)^{\frac{\beta}{2}+\frac{3}{2}+m}\partial_x^m v(x) \to 0, \quad (1-x)^{\frac{\gamma}{2}-\frac{1}{2}+m}(1+x)^{\frac{\delta}{2}-\frac{1}{2}+m}\partial_x^m v(x) \to 0, \quad \text{as } |x| \to 1.$$

Therefore for any $u, v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$,

$$\partial_x(a(x)\partial_x^2 u(x))v(x), \quad a(x)\partial_x^2 u(x)\partial_x v(x), \quad b(x)\partial_x u(x)v(x) \to 0, \quad \text{as } |x| \to 1. \tag{4.5}$$

Now, let

$$\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(u,v) = (a_1\partial_x^2 u, \partial_x^2 v)_{\chi^{(\alpha,\beta)}} + (b_1\partial_x u, \partial_x v)_{\chi^{(\gamma,\delta)}} + (c_1 u, v)_{\chi^{(\eta,\theta)}}.$$

We multiply (4.1) by $v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$, integrate the resulting equation by parts, and then use (4.5) to derive a weak form of (4.1). It is to find $U \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda)$ such that

$$\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(U,v) = (f,v), \qquad \forall v \in H^2_{\alpha,\beta,\gamma,\delta,\eta,\theta}(\Lambda). \tag{4.6}$$

Next, let N be any positive even number. We introduce the bilinear form

$$\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(u,v) = (\widetilde{a}_1\partial_x^2 u, \partial_x^2 v)_{\chi^{(\alpha,\beta)},G,N} + (\widetilde{b}_1\partial_x u, \partial_x v)_{\chi^{(\gamma,\delta)},G,N} + (\widetilde{c}_1 u, v)_{\chi^{(\eta,\theta)},G,N}$$

where $\widetilde{a}_1 = P_{N/2,0,0}a_1$, $\widetilde{b}_1 = P_{N/2,0,0}b_1$ and $\widetilde{c}_1 = P_{N/2,0,0}c_1$. Furthermore, let $\widetilde{f}(x) = \chi^{(-\eta,-\theta)}(x)\, f(x)$. The pseudospectral scheme for (4.6) is to find $u_N \in \mathcal{P}_N$ such that

$$\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(u_N,\phi) = (\widetilde{f},\phi)_{\chi^{(\eta,\theta)},G,N}, \qquad \forall \phi \in \mathcal{P}_N. \tag{4.7}$$

We now consider the existence of solution of (4.7). In fact, by virtue of (3.2), for any $\phi, \psi \in \mathcal{P}_N$,

$$|\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(\phi,\psi)| \leq (\|\widetilde{a}_1\|_{L^\infty(\Lambda)} + \|\widetilde{b}_1\|_{L^\infty(\Lambda)} + \|\widetilde{c}_1\|_{L^\infty(\Lambda)})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}\|\psi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}.$$

Moreover, for any $v \in H^r(\Lambda)$ and $r \geq 3/4$ (see [9]),

$$\|P_{N,0,0}v - v\|_{L^\infty(\Lambda)} \leq cN^{3/4-r}\|v\|_r. \tag{4.8}$$

The above with imbedding theorem implies that $\|\widetilde{a}_1\|_{L^\infty(\Lambda)} \leq c\|a_1\|_{3/4}$, etc.. Therefore

$$|\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(\phi,\psi)| \leq c(\|a_1\|_{3/4} + \|b_1\|_{3/4} + \|c_1\|_{3/4})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}\|\psi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}. \tag{4.9}$$

On the other hand, we have from (3.2), (4.3), (4.4) and (4.8) that for large N,

$$\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(\phi,\phi) \geq c\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}, \qquad \forall \phi \in \mathcal{P}_N. \tag{4.10}$$

By (4.9), (4.10) and the Lax-Milgram lemma, (4.7) has a unique solution such that

$$\|u_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \leq c\|\mathcal{I}_{G,N,\eta,\theta}\widetilde{f}\|_{\chi^{(\eta,\theta)}}.$$

We have the following result on the convergence of (4.7).

Theorem 4.1. *Let* $\alpha \leq \min(\gamma+2,\delta+4)$, $\beta \leq \min(\delta+2,\theta+4)$ *and* $r,\sigma \in \mathbb{N}$. *If* $U \in H^r_{\chi^{(\alpha,\beta)},**}(\Lambda)$ *with* $r \geq 2$, *and* $\widetilde{f} \in H^\sigma_{\chi^{(\eta,\theta)},A}(\Lambda)$ *with* $\sigma \geq 1$, *then*

$$\begin{aligned}\|U - u_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} &\leq c^* N^{2-r}\|U\|_{r,\chi^{(\alpha,\beta)},**} + c(N^{3/4-s}\|a_1\|_s|U|_{2,\chi^{(\alpha,\beta)}} \\ &+N^{3/4-s'}\|b_1\|_{s'}|U|_{1,\chi^{(\gamma,\delta)}} + N^{3/4-s''}\|c_1\|_{s''}\|U\|_{\chi^{(\eta,\theta)}} + N^{-\sigma}\|\widetilde{f}\|_{\sigma,\chi^{(\eta,\theta)},A}),\end{aligned} \tag{4.11}$$

c^* *being a positive constant depending only on the norms* $\|a_1\|_{3/4}$, $\|b_1\|_{3/4}$ *and* $\|c_1\|_{3/4}$.

Proof. In order to obtain better error estimate, we set $U_N = P^2_{N,\alpha,\beta,\gamma,\delta,\eta,\theta}U$. Then by (4.6), (4.7) and the ellipticity (4.10),

$$\begin{aligned} c\|U_N - u_N\|^2_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} &\leq \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N - u_N, U_N - u_N) \\ &= \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N, U_N - u_N) - (\widetilde{f}, U_N - u_N)_{\chi^{(\eta,\theta)},G,N} \\ &= \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N, U_N - u_N) - \mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(U, U_N - u_N) \\ &+(\widetilde{f}, U_N - u_N)_{\chi^{(\eta,\theta)}} - (\widetilde{f}, U_N - u_N)_{\chi^{(\eta,\theta)},G,N}. \end{aligned} \tag{4.12}$$

For simplicity, let

$$\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(u,v) = (\widetilde{a}_1\partial_x^2 u, \partial_x^2 v)_{\chi^{(\alpha,\beta)}} + (\widetilde{b}_1\partial_x u, \partial_x v)_{\chi^{(\gamma,\delta)}} + (\widetilde{c}_1 u, v)_{\chi^{(\eta,\theta)}},$$

$$G_1(U,\phi) = \mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(U,\phi) - \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(U,\phi),$$

$$G_2(U,U_N,\phi) = \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(U,\phi) - \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N,\phi),$$

$$G_3(\widetilde{f},\phi) = (\widetilde{f},\phi)_{\chi^{(\eta,\theta)}} - (\widetilde{f},\phi)_{\chi^{(\eta,\theta)},G,N}.$$

Then (4.12) implies that

$$\|U_N - u_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \leq c \sup_{\phi\in\mathcal{P}_N,\phi\neq 0} \frac{|G_1(U,\phi)| + |G_2(U,U_N,\phi)| + |G_3(\widetilde{f},\phi)|}{\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}}. \tag{4.13}$$

We now estimate the right side of (4.13). By (4.8),

$$\begin{aligned}|G_1(U,\phi)| &\leq (\|\widetilde{a}_1 - a_1\|_{L^\infty(\Lambda)}|U|_{2,\chi^{(\alpha,\beta)}} + \|\widetilde{b}_1 - b_1\|_{L^\infty(\Lambda)}|U|_{1,\chi^{(\gamma,\delta)}} \\ &+\|\widetilde{c}_1 - c_1\|_{L^\infty(\Lambda)}\|U\|_{\chi^{(\eta,\theta)}})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \\ &\leq c((\tfrac{N}{2})^{3/4-s}\|a_1\|_s|U|_{2,\chi^{(\alpha,\beta)}} + (\tfrac{N}{2})^{3/4-s'}\|b_1\|_{s'}|U|_{1,\chi^{(\gamma,\delta)}} \\ &+(\tfrac{N}{2})^{3/4-s''}\|c_1\|_{s''}\|U\|_{\chi^{(\eta,\theta)}})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}.\end{aligned} \tag{4.14}$$

On the other hand, (3.2) implies that

$$\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U,\phi) = \widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U,\phi), \quad \forall\phi\in\mathcal{P}_N$$

whence

$$|G_2(U,U_N,\phi)| \leq |\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U - U,\phi)| + |\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U - U_N,\phi)|.$$

According to (4.9),

$$\begin{aligned}&|\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U - U_N,\phi)| \\ &\quad\leq c^*(\|P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U - U\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} + \|U - U_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}.\end{aligned}$$

The above with (2.4) and a similar argument leads to that

$$\begin{aligned}|G_2(U,U_N,\phi)| &\le c^*(\|P^2_{N/2,\alpha,\beta,\gamma,\delta,\eta,\theta}U-U\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}+\|U-U_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta})\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}\\ &\le c^*N^{2-r}\|U\|_{r,\chi^{(\alpha,\beta)},**}\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}.\end{aligned} \tag{4.15}$$

Furthermore, we use (3.10) to obtain that

$$|G_3(\widetilde{f},\phi)| \le \|\mathcal{I}_{G,N,\eta,\theta}\widetilde{f}-\widetilde{f}\|_{\chi^{(\eta,\theta)}}\|\phi\|_{\chi^{(\eta,\theta)}} \le cN^{-\sigma}\|\widetilde{f}\|_{\sigma,\chi^{(\eta,\theta)},A}\|\phi\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta}. \tag{4.16}$$

Finally, a combination of (4.13)-(4.16) and (2.5) leads to (4.11).

Remark 4.1. If $a_1 \in \mathcal{P}_{k+5}$, $b_1 \in \mathcal{P}_{k+3}$, $c_1 \in \mathcal{P}_{k+1}$, $k \ge 0$ and (4.4) holds, then we can approximate (4.6) in another way. In this case, let

$$\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(u,v) = (a_1\partial_x^2 u,\partial_x^2 v)_{\chi^{(\alpha,\beta)},G,N}+(b_1\partial_x u,\partial_x v)_{\chi^{(\gamma,\delta)},G,N}+(c_1 u,v)_{\chi^{(\eta,\theta)},G,N}.$$

The corresponding pseudospectral scheme is to find $u_N \in \mathcal{P}_N$ such that

$$\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(u_N,\phi) = (\widetilde{f},\phi)_{\chi^{(\eta,\theta)},G,N}, \quad \forall \phi \in \mathcal{P}_N.$$

The ellipticity is also valid for $\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(\phi,\psi)$. So we can derive the estimates like (4.12) and (4.13). But in their derivations, $\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U,\phi)$ and $\widetilde{\mathcal{A}}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N,\phi)$ are now replaced by $\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U,\phi)$ and $\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(U_N,\phi)$, respectively. Thanks to (3.2),

$$\mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta,N}(P^2_{N-k,\alpha,\beta,\gamma,\delta,\eta,\theta}U,\phi) = \mathcal{A}_{\alpha,\beta,\gamma,\delta,\eta,\theta}(P^2_{N-k,\alpha,\beta,\gamma,\delta,\eta,\theta}U,\phi), \quad \forall \phi \in \mathcal{P}_N.$$

Therefore, if conditions of Theorem 4.1 are fulfilled, then we use (2.5) and the same argument as in the proof of Theorem 4.1 to conclude that

$$\|U-u_N\|_{2,\alpha,\beta,\gamma,\delta,\eta,\theta} \le c(N^{2-r}\|U\|_{r,\chi^{(\alpha,\beta)},**}+N^{-\sigma}\|\widetilde{f}\|_{\sigma,\chi^{(\eta,\theta)},A}).$$

Model Problem 2. We now turn to another model problem. Let $\Omega = \{(y_1,y_2) \mid y_1^2+y_2^2 < 4\}$. We consider the following problem on an axisymmetric domain,

$$\begin{cases} \triangle^2 W = G, & \text{in } \Omega,\\ W = g_0, \quad \partial_n W = -g_1, & \text{on } \partial\Omega \end{cases} \tag{4.17}$$

where g_0, g_1 and G are given functions.

Let $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. Accordingly, $V(r,\theta) = W(r\cos\theta, r\sin\theta)$ and $F(r,\theta) = G(r\cos\theta, r\sin\theta)$. Furthermore, let $\triangle_{r,\theta} = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$. Then the problem (4.17) is changed to

$$\begin{cases} \triangle^2_{r,\theta} V = F, & (r,\theta) \in (0,2)\times[0,2\pi),\\ V(2,\theta) = g_0, \quad \partial_r V(2,\theta) = -g_1, & \theta \in [0,2\pi). \end{cases} \tag{4.18}$$

Obviously, $\partial_\theta^m V(r,0) = \partial_\theta^m V(r,2\pi)$. In addition, the solution $V(r,\theta)$ satisfies the pole condition: $\partial_\theta V(0,\theta) = 0, \quad \theta \in [0,2\pi)$.

We can use the mixed Jacobi-Fourier pseudospectral method to solve (4.18) numerically. For clarity, we now focus on the case in which F is invariant under the rotation in the sense of [4]. Then $V(r,\theta)$ and $F(r,\theta)$ become $V(r)$ and $F(r)$, and g_0 and g_1 become two constants, respectively. Consequently, (4.18) is reduced to

$$\begin{cases} (\partial_r^2 + r^{-1}\partial_r)^2 V = f, \quad r \in (0,2),\\ V(2) = g_0, \quad \partial_r V(2) = -g_1. \end{cases} \tag{4.19}$$

In order to solve (4.19) numerically, we make the variable transformation: $x = 1 - r$. Accordingly, $U(x) = V(1-x)$ and $f(x) = F(1-x)$. Let $\triangle_x$ and $\mathcal{B}_\beta(u,v)$ be the same as in Section 2 (see (2.6)-(2.8)). Then the problem (4.19) is reformed to

$$\begin{cases} \triangle_x^2 U(x) = f(x), \quad x \in \Lambda, \\ U(-1) = g_0, \quad \partial_x U(-1) = g_1. \end{cases} \tag{4.20}$$

For simplicity, we assume that $U(1) = 0$ and $\partial_x U(1) = 0$. For $U(1) \neq 0$ or $\partial_x U(1) \neq 0$, we may use a variable transformation, as discussed in [2].

We now derive a weak formulation of (4.20). Let $0 \leq \beta \leq \frac{4}{5}$ and $v \in H^2_{0,1,\beta,-1,\beta,1,\beta}(\Lambda)$. By multiplying the first formula of (4.20) by $v(x)\chi^{(1,\beta)}(x)$ and integrating the result over Λ, we obtain that

$$(\triangle_x^2 U, v)_{\chi^{(1,\beta)}} = (f, v)_{\chi^{(1,\beta)}}.$$

Furthermore, integration twice gives that

$$\begin{aligned} (\triangle_x^2 U, v)_{\chi^{(1,\beta)}} &= \int_\Lambda \partial_x^2 \triangle_x U(x) v(x) \chi^{(1,\beta)}(x) dx - \int_\Lambda \partial_x \triangle_x U(x) v(x) (1+x)^\beta dx \\ &= \int_\Lambda \triangle_x U(x) \partial_x^2 (v(x)\chi^{(1,\beta)}(x)) dx + \int_\Lambda \triangle_x U(x) \partial_x (v(x)(1+x)^\beta) dx \\ &= \int_\Lambda \triangle_x U(x)((1-x)\partial_x^2(v(x)(1+x)^\beta) - \partial_x(v(x)(1+x)^\beta) dx = \mathcal{B}_\beta(U,v). \end{aligned}$$

Let

$${}^{00}H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda) = \{v \mid v \in H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda) \text{ and } v(1) = \partial_x v(1) = 0\}.$$

A variational formulation of (4.20) is to find $U \in {}^{00}H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda)$, such that $U(-1) = g_0$, $\partial_x U(-1) = g_1$, and

$$\mathcal{B}_\beta(U,v) = (f,v)_{\chi^{(1,\beta)}}, \quad \forall v \in H^2_{0,1,\beta,-1,\beta,1,\beta}(\Lambda). \tag{4.21}$$

If $f \in (H^2_{0,1,\beta,-1,\beta,1,\beta}(\Lambda))'$, then by (2.6), (2.7) and the Lax-Milgram Lemma, (4.21) has a unique solution.

Let ${}^{00}\mathcal{P}_N = \{v \mid v \in \mathcal{P}_N \text{ and } v(1) = \partial_x v(1) = 0\}$, and

$$\mathcal{B}_{\beta,N}(u,v) = (\triangle_x u, (1+x)^{-\beta} \triangle_x ((1+x)^\beta v))_{\chi^{(1,\beta)},R,N}.$$

A direct calculation shows that $\triangle_x \phi$, $(1+x)^{-\beta} \triangle_x ((1+x)^\beta \phi)) \in \mathcal{P}_N$ for any $\phi \in \mathcal{P}_N^{00}$. Thus by (3.2),

$$\mathcal{B}_\beta(\phi,\psi) = (\triangle_x \phi, (1+x)^{-\beta} \triangle_x ((1+x)^\beta \psi)))_{\chi^{(1,\beta)}} = \mathcal{B}_{\beta,N}(\phi,\psi), \quad \forall \phi,\psi \in \mathcal{P}_N^{00}. \tag{4.22}$$

The pseudospectral scheme for (4.21) is to find $u_N \in {}^{00}\mathcal{P}_N$ such that $u_N(-1) = g_0$, $\partial_x u_N(-1) = g_1$, and

$$\mathcal{B}_{\beta,N}(u_N,\phi) = (f,\phi)_{\chi^{(1,\beta)},R,N}, \quad \forall \phi \in \mathcal{P}_N^{00}, \tag{4.23}$$

We now prove the convergence of (4.23). Let $Q^2_{N,\beta}$ be the same as in (2.8). Set $U_N = Q^2_{N,\beta} U$. By virtue of (2.8) and (4.21),

$$\mathcal{B}_\beta(U_N,\phi) = \mathcal{B}_\beta(U,\phi) = (f,\phi)_{\chi^{(1,\beta)}}, \quad \forall \phi \in \mathcal{P}_N^{00}. \tag{4.24}$$

Furthermore, we use (2.7), (3.2) and (4.22)-(4.24) to deduce that

$$\begin{aligned} &c\|U_N - u_N\|^2_{2,1,\beta,-1,\beta,1,\beta} \\ &\leq \mathcal{B}_\beta(U_N - u_N, U_N - u_N) = (f, U_N - u_N)_{\chi^{(1,\beta)}} - (f, U_N - u_N)_{\chi^{(1,\beta)},R,N} \\ &= (f - \mathcal{I}_{R,N,1,\beta} f, U_N - u_N)_{\chi^{(1,\beta)}} \leq \|f - \mathcal{I}_{R,N,1,\beta} f\|_{\chi^{(1,\beta)}} \|u_N - U_N\|_{2,1,\beta,-1,\beta,1,\beta}. \end{aligned}$$

This fact implies that

$$\|U_N - u_N\|_{2,1,\beta,-1,\beta,1,\beta} \le c\|\mathcal{I}_{R,N,1,\beta}f - f\|_{\chi^{(1,\beta)}}.$$

Finally we use (2.9) and (3.10) to reach the following result.

Theorem 4.2. *Let* $-1 < \bar{\alpha} < 1, 0 \le \beta \le \frac{4}{5}, r \in \mathbb{N}$ *and* $r \ge 2$. *If* $U \in H^2_{1,\beta,-1,\beta,1,\beta}(\Lambda) \cap H^r_{\chi^{(\bar{\alpha},\beta)},**}(\Lambda)$ *and* $f \in H^{r-2}_{\chi^{(1,\beta)},A}(\Lambda)$, *then*

$$\|U - u_N\|_{2,1,\beta,-1,\beta,1,\beta} \le cN^{2-r}(|U|_{r,\chi^{(\bar{\alpha},\beta)},**} + |f|_{r-2,\chi^{(1,\beta)},A}).$$

Model Problem 3. The third model problem is as follows,

$$\begin{cases} \partial_t^2 U(x,t) + U^3(x,t) + \partial_x^4 U(x,t) = f(x,t), & x \in \Lambda,\ 0 < t \le T, \\ \partial_x U(\pm 1,t) = U(\pm 1,t) = 0, & 0 < t \le T, \\ \partial_t U(x,0) = U_1(x), & x \in \bar{\Lambda}, \\ U(x,0) = U_0(x), & x \in \bar{\Lambda}, \end{cases} \tag{4.25}$$

where $U_1(\pm 1) = U_0(\pm 1) = \partial_x U_0(\pm 1) = 0$. Its weak formulation is to find $U(t) \in H^1(0,T;L^2(\Lambda)) \cap L^2(0,T;H_0^2(\Lambda) \cap L^4(\Lambda))$ such that $\partial_t U(0) = U_1$, $U(0) = U_0$, and

$$(\partial_t^2 U(t) + U^3(t), v) + (\partial_x^2 U(t), \partial_x^2 v) = (f(t), v), \quad \forall v \in H_0^2(\Lambda), \quad 0 < t \le T. \tag{4.26}$$

The corresponding pseudospectral scheme is to find $u_N(t) \in \mathcal{P}_N^{00}$ for all $0 \le t \le T$ such that

$$\begin{cases} (\partial_t^2 u_N(t) + u_N^3(t), \phi)_{\chi^{(0,0)},L,N} + (\partial_x^2 u_N(t), \partial_x^2 \phi)_{\chi^{(0,0)},L,N} \\ \quad = (f(t), \phi)_{\chi^{(0,0)},L,N}, \qquad \forall \phi \in \mathcal{P}_N^{00}, \quad t \in (0,T], \\ \partial_t u_N(0) = u_{N,1}, \\ u_N(0) = u_{N,0}, \end{cases} \tag{4.27}$$

where $u_{N,j}$ are some approximations to $U_j,\ j = 0, 1$. For instance, we may take $u_{N,j} = \widetilde{P}^{2,0}_{N,0,0} U_j$ or $u_{N,j} = k_N^2 U_j$, k_N^2 being the interpolation operator given in (13.17) of [5].

We first derive a prior estimate. Putting $\phi = 2\partial_t u_N(t)$ in (4.27) and using (3.2), we obtain that

$$\begin{aligned} &\partial_t \|\partial_t u_N(t)\|^2_{\chi^{(0,0)},L,N} + \tfrac{1}{2}\partial_t \|u_N(t)\|^4_{l^4,L,N} + \partial_t |u_N(t)|_2^2 \\ &\quad = 2(f(t), \partial_t u_N(t))_{\chi^{(0,0)},L,N} \le \|f(t)\|^2_{\chi^{(0,0)},L,N} + \|\partial_t u_N(t)\|^2_{\chi^{(0,0)},L,N}, \end{aligned} \tag{4.28}$$

where $\|v\|_{l^4,L,N} = \|v^2\|^{\frac{1}{2}}_{\chi^{(0,0)},L,N}$.

For simplicity of statements, let

$$E(v,t) = \|\partial_t v(t)\|^2_{\chi^{(0,0)},L,N} + \frac{1}{2}\|v(t)\|^4_{l^4,L,N} + |v(t)|_2^2.$$

As in [30] , we multiply (4.28) by e^{-t}, and then obtain that

$$\partial_t(e^{-t}E(u_N,t)) \le e^{-t}\|f(t)\|^2_{\chi^{(0,0)},L,N}.$$

Integrating the above with respect to t, we obtain from (3.3) that

$$E(u_N,t) \le e^t(E(u_N,0) + \textstyle\int_0^t e^{-s}\|f(s)\|^2_{\chi^{(0,0)},L,N}ds) \le cB(u_{N,0},u_{N,1},f,t) \tag{4.29}$$

where

$$B(u,v,w,t) = e^t(\frac{1}{2}\|u\|^4_{l^4,L,N} + |u|_2^2 + \|v\|^2 + \int_0^t e^{-s}\|w(s)\|^2_{\chi^{(0,0)},L,N}ds).$$

Furthermore, by (3.36) and (3.37), we obtain that for $0 \le t \le T$,

$$B(u_{N,0}, u_{N,1}, f, t) \le cB_1(U_0, U_1, f, T) \tag{4.30}$$

where

$$\begin{aligned} B_1(U_0, U_1, f, T) = \|U_0\|^4_{L^\infty(\Lambda)} + |U_0|^2_2 + |U_0|^2_{4,\chi^{(0,0)},**} + \|U_1\|^2 + N^{-2}|U_1|^2_{1,\chi^{(0,0)},A} \\ + \|f\|^2_{L^2(0,T;L^2(\Lambda))} + N^{-2}\|f\|^2_{L^2(0,T;H^1_{\chi^{(0,0)},A}(\Lambda))}. \end{aligned}$$

We now analyze the stability of scheme (4.27). Clearly, it is a nonlinear problem, and so does not possess the stability in the sense of Courant et al. [10]. But it might be stable in the sense of Guo [14]. Suppose that $u_{N,0}$, $u_{N,1}$ and f have the errors $\tilde{u}_{N,0}$, $\tilde{u}_{N,1}$ and $\tilde{f}$, respectively, which induce the error of u_N, denoted by $\tilde{u}_N$. By (4.27), we get that for any $\phi \in \mathcal{P}^{00}_N$ and $t \in (0, T]$,

$$\begin{cases} (\partial_t^2 \tilde{u}_N(t) + \tilde{u}_N^3(t) + F_0(t), \phi)_{\chi^{(0,0)},L,N} + (\partial_x^2 \tilde{u}_N(t), \partial_x^2 \phi)_{\chi^{(0,0)},L,N} \\ \quad = (\tilde{f}(t), \phi)_{\chi^{(0,0)},L,N}, \\ \partial_t \tilde{u}_N(0) = \tilde{u}_{N,1}, \\ \tilde{u}_N(0) = \tilde{u}_{N,0}, \end{cases} \tag{4.31}$$

where $F_0(t) = 3\tilde{u}_N^2(t)u_N(t) + 3\tilde{u}_N(t)u_N^2(t)$. Taking $\phi = 2\partial_t \tilde{u}_N(t)$ in (4.31), we use (3.2) to obtain that

$$\begin{aligned} \partial_t \|\partial_t \tilde{u}_N(t)\|^2_{\chi^{(0,0)},L,N} + \tfrac{1}{2}\partial_t \|\tilde{u}_N(t)\|^4_{l^4,L,N} + 2(F_0(t), \partial_t \tilde{u}_N(t))_{\chi^{(0,0)},L,N} + \partial_t |\tilde{u}_N(t)|^2_2 \\ \le \|\partial_t \tilde{u}_N(t)\|^2_{\chi^{(0,0)},L,N} + \|\tilde{f}(t)\|^2_{\chi^{(0,0)},L,N}. \end{aligned} \tag{4.32}$$

We next estimate $|2(F_0(t), \partial_t \tilde{u}_N(t))_{\chi^{(0,0)},L,N}|$. By a prior estimate (4.29), (4.30), the embedding inequality and the Poincaré inequality, we have

$$\|u_N(t)\|_\infty \le c\|u_N\|_{C(0,T;H^1(\Lambda))} \le c\|u_N\|_{C(0,T;H^2(\Lambda))} \le cB_1(U_0, U_1, f, T). \tag{4.33}$$

Therefore, by (3.3) and the Poincaré inequality,

$$\begin{aligned} &|2(F_0(t), \partial_t \tilde{u}_N(t))_{\chi^{(0,0)},L,N}| \\ &\le \|\tilde{u}_N(t)\|^4_{l^4,L,N} + \|\tilde{u}_N(t)\|^2_{\chi^{(0,0)},L,N} + c(\|u_N(t)\|^2_\infty + \|u_N(t)\|^4_\infty)\|\partial_t \tilde{u}_N(t)\|^2_{\chi^{(0,0)},L,N} \\ &\le \|\tilde{u}_N(t)\|^4_{l^4,L,N} + c|\tilde{u}_N(t)|^2_2 + c_1\|\partial_t \tilde{u}_N(t)\|^2_{\chi^{(0,0)},L,N}, \end{aligned} \tag{4.34}$$

where c_1 depends only on $B_1(U_0, U_1, f, T)$. Thus, substituting (4.34) into (4.32) and integrating the resulting inequality, we have that

$$E(\tilde{u}_N, t) \le \rho(\tilde{u}_{N,0}, \tilde{u}_{N,1}, \tilde{f}, t) + c_1 \int_0^t E(\tilde{u}_N, s)ds, \tag{4.35}$$

where

$$\rho(\tilde{u}_{N,0}, \tilde{u}_{N,1}, \tilde{f}, t) = E(\tilde{u}_N, 0) + c\int_0^t \|\tilde{f}(s)\|^2_{\chi^{(0,0)},L,N} ds.$$

Finally, we use the Gronwall inequality to reach the following result.

Theorem 4.3. *Let u_N be the solution of (4.27), and $\tilde{u}_N$ be its error induced by $\tilde{u}_{N,0}$, $\tilde{u}_{N,1}$ and $\tilde{f}$. Then for all $0 \le t \le T$,*

$$E(\tilde{u}_N, t) \le \rho(\tilde{u}_{N,0}, \tilde{u}_{N,1}, \tilde{f}, t)e^{c_1 t}. \tag{4.36}$$

We next analyze the convergence of scheme (4.27). For simplicity of statements, we assume $U_0(x) = U_1(x) \equiv 0$. Otherwise, we may reformulate (4.25) by the variable transformation

$V(x,t)=U(x,t)-U_0(x)-tU_1(x)$, so that $V(x,0)=\partial_t V(x,0)\equiv 0$. Now, putting $U_N=\tilde{P}^{2,0}_{N,0,0}U$, we obtain from (4.26) that

$$\begin{aligned}(\partial_t^2 U_N(t) &+U_N^3(t),\phi)_{\chi^{(0,0)},L,N}+(\partial_x^2 U_N(t),\partial_x^2\phi)_{\chi^{(0,0)},L,N}\\ &=\sum_{j=1}^{3}G_j(t,\phi)+(f(t),\phi)_{\chi^{(0,0)},L,N},\quad \phi\in\mathcal{P}_N^{00},\quad t\in(0,T]\end{aligned}\tag{4.37}$$

where

$$\begin{aligned}G_1(t,\phi)&=(\partial_t^2 U_N(t),\phi)_{\chi^{(0,0)},L,N}-(\partial_t^2 U(t),\phi),\\ G_2(t,\phi)&=(U_N^3(t),\phi)_{\chi^{(0,0)},L,N}-(U^3(t),\phi),\\ G_3(t,\phi)&=(f(t),\phi)-(f(t),\phi)_{\chi^{(0,0)},L,N}.\end{aligned}$$

Let $\widetilde{U}_N=u_N-U_N$. By subtracting (4.37) from (4.27), we obtain that for any $\phi\in\mathcal{P}_N^{00}$ and $t\in(0,T]$,

$$\begin{cases}(\partial_t^2\widetilde{U}_N(t)+\widetilde{U}_N^3(t)+G_0(t),\phi)_{\chi^{(0,0)},L,N}+(\partial_x^2\widetilde{U}_N(t),\partial_x^2\phi)_{\chi^{(0,0)},L,N}=-\displaystyle\sum_{j=1}^{3}G_j(t,\phi),\\ \partial_t\widetilde{U}_N(0)=\widetilde{U}_N(0)=0,\end{cases}\tag{4.38}$$

where $G_0(t)=3\widetilde{U}_N^2(t)U_N(t)+3\widetilde{U}_N(t)U_N^2(t)$. Taking $\phi=2\partial_t\tilde{U}_N(t)$ in (4.38) and comparing (4.38) with (4.31), we can derive an estimate like (4.35). But u_N, $\tilde{u}_N$ and $\|u_N\|_{C(0,T;H^1(\Lambda))}$ are now replaced by U_N, $\widetilde{U}_N$ and $\|U_N\|_{C(0,T;H^1(\Lambda))}$, respectively. Thus, it remains to estimate $|G_j(t,\partial_t\widetilde{U}_N(t))|$. Firstly, by (2.11), the imbedding inequality and the Poincaré inequality,

$$\|U_N(t)\|_\infty\le\|U_N\|_{C(0,T;H^1(\Lambda))}\le\|U_N\|_{C(0,T;H^2(\Lambda))}\le c\|U\|_{C(0,T;H^2(\Lambda))}.$$

Next, as in the derivation of (4.34), we deduce that

$$\begin{aligned}|2(\widetilde{G}_0(t),\partial_t\widetilde{U}_N(t))_{\chi^{(0,0)},L,N}|&\le\|\widetilde{U}_N(t)\|^4_{l^4,L,N}+c|\widetilde{U}_N(t)|_2^2\\ &+c(\|U(t)\|^2_{C(0,T;H^2(\Lambda))}+\|U(t)\|^4_{C(0,T;H^2(\Lambda))})\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N}.\end{aligned}$$

Furthermore, by (3.3), (2.11) and (3.36), we obtain that

$$\begin{aligned}\|\partial_t^2U_N(t)-\partial_t^2U(t)\|_{\chi^{(0,0)},L,N}&\le c\|\mathcal{I}_{L,N,0,0}(\partial_t^2U_N(t)-\partial_t^2U(t))\|\\ &\le c\|\partial_t^2U_N(t)-\partial_t^2U(t)\|+c\|\partial_t^2U(t)-\mathcal{I}_{L,N,0,0}\partial_t^2U(t)\|\\ &\le cN^{-r}\|\partial_t^2U(t)|_{r,\chi^{(0,0)},**}.\end{aligned}$$

Similarly, by (3.3), (2.11) and (3.36),

$$\begin{aligned}\|\partial_t^2U(t)&-\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t)\|_{\chi^{(0,0)},L,N}=\|\mathcal{I}_{L,N,0,0}\partial_t^2U(t)-\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t)\|_{\chi^{(0,0)},L,N}\\ &\le c\|\mathcal{I}_{L,N,0,0}\partial_t^2U(t)-\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t)\|\\ &\le c(\|\mathcal{I}_{L,N,0,0}\partial_t^2U(t)-\partial_t^2U(t)\|+\|\partial_t^2U(t)-\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t)\|)\\ &\le cN^{-r}\|\partial_t^2U(t)|_{r,\chi^{(0,0)},**}.\end{aligned}$$

The above two estimates with (3.2) and (2.11) lead to that for integer $r\ge 2$,

$$\begin{aligned}|G_1(t,\partial_t\widetilde{U}_N)|&\le|(\partial_t^2U_N(t)-\partial_t^2U(t),\partial_t\widetilde{U}_N(t))_{\chi^{(0,0)},L,N}|\\ &+|(\partial_t^2U(t)-\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t),\partial_t\widetilde{U}_N(t))_{\chi^{(0,0)},L,N}|\\ &+|(\tilde{P}^{2,0}_{N-1,0,0}\partial_t^2U(t)-\partial_t^2U(t),\partial_t\widetilde{U}_N(t))|\\ &\le cN^{-2r}|\partial_t^2U(t)|^2_{r,\chi^{(0,0)},**}+c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N}.\end{aligned}$$

Now,let N be suitably large and $M = [\frac{N-1}{3}]$. Then by (3.2), (3.3), (2.11) and the previous estimate,

$$\begin{aligned}
&|G_2(t,\partial_t\widetilde{U}_N(t)| \\
\le\ & |(U^3(t) - (\tilde{P}^{2,0}_{M,0,0}U(t))^3, \partial_t\widetilde{U}_N(t))| + |((\tilde{P}^{2,0}_{M,0,0}U(t))^3 - U_N^3(t), \partial_t\widetilde{U}_N(t))_{\chi^{(0,0)},L,N}| \\
\le\ & c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N} + c(\|U(t)\|^4_\infty + \|\tilde{P}^{2,0}_{M,0,0}U(t)\|^4_\infty)\|U(t) - \tilde{P}^{2,0}_{M,0,0}U(t)\|^2 \\
&+c(\|U_N(t)\|^4_\infty + \|\tilde{P}^{2,0}_{M,0,0}U(t)\|^4_\infty)\|\tilde{P}^{2,0}_{M,0,0}U(t) - U_N(t)\|^2_{\chi^{(0,0)},L,N} \\
\le\ & c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N} + cN^{-2r}(\|U\|^4_{C(0,T;L^\infty(\Lambda))} + \|U\|^4_{C(0,T;H^2(\Lambda))})|U(t)|^2_{r,\chi^{(0,0)},**}.
\end{aligned}$$

Thanks to (3.2), (3.3) and (3.36), we get that for integer $s \ge 1$,

$$\begin{aligned}
&|G_3(t,\partial_t\widetilde{U}_N)| \\
\le\ & |(f(t) - \mathcal{I}_{L,N-1,0,0}f(t), \partial_t\widetilde{U}_N(t))| + |(\mathcal{I}_{L,N-1,0,0}f(t) - \mathcal{I}_{L,N,0,0}f(t), \partial_t\widetilde{U}_N(t))_{\chi^{(0,0)},L,N}| \\
\le\ & \|f(t) - \mathcal{I}_{L,N-1,0,0}f(t)\|^2 + \|\mathcal{I}_{L,N-1,0,0}f(t) - \mathcal{I}_{L,N,0,0}f(t)\|^2 + c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N} \\
\le\ & 2\|f(t) - \mathcal{I}_{L,N-1,0,0}f(t)\|^2 + \|f(t) - \mathcal{I}_{L,N,0,0}f(t)\|^2 + c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N} \\
\le\ & cN^{-2s}|f(t)|^2_{s,\chi^{(0,0)},A} + c\|\partial_t\widetilde{U}_N(t)\|^2_{\chi^{(0,0)},L,N}.
\end{aligned}$$

Finally, a combination of the previous estimates and (2.11) leads to that for all $0 \le t \le T$,

$$E(U - U_N, t) \le c_2(N^{-2r} + N^{-2s}) \tag{4.39}$$

where c_2 is a positive constant depending only on $\|U\|_{H^2(0,T;H^r_{\chi^{(0,0)},**}(\Lambda))\cap C(0,T;H^2(\Lambda))}$ and $\|f\|_{L^2(0,T;H^s_{\chi^{(0,0)},A}(\Lambda))}$, and integers $r \ge 2$, $s \ge 1$.

5. Numerical Results

In this section, we present some numerical results.

Example 1. We first consider problem (4.1) with $a(x) = (1-x^2)^2$, $b(x) = 1-x^2$ and $c(x) = 1$. For description of numerical errors, let

$$E(v) = \left(\sum_{j=0}^{N}(U(\zeta^{(0,0)}_{G,N,j}) - v(\zeta^{(0,0)}_{G,N,j}))^2\omega^{(0,0)}_{G,N,j}\right)^{1/2}.$$

We take the test function

$$U(x) = \ln(x + \varepsilon^k + 1), \quad \text{with } \varepsilon > 0 \text{ and } 0 < k < 1.$$

For small ε, the solution $U(x)$ varies very rapidly at $x \sim -1$. Moreover, for $j \ge 1$, $|\partial_x^j U(x)| \to \infty$, as $\varepsilon \to 0$ and $x \to -1$. Let $\varepsilon = 10^{-4}$ and $u_N(x)$ be the numerical solution given by (4.7). In Figure 1, we plot $\log_{10} E(u_N)$ vs. $\sqrt{N}$ with $k = \frac{1}{3}$ and $k = \frac{1}{4}$, respectively. It shows that scheme (4.7) provides very accurate numerical results even for small N and very small ε. It also indicates the rapid convergence as N increases, which coincides well with the theoretical analysis.

Next, we take the test function

$$U(x) = (1-x^2)^{k-p}, \text{ with } k = 2, 3 \text{ and } 0 < p < 1.$$

Clearly, $|\partial_x^k U(x)| \to \infty$, as $|x| \to 1$. We use (4.7) to solve (4.1) with the same $a(x), b(x)$ and $c(x)$ as in the previous case. In Figure 2, we plot $\log_{10} E(u_N)$ vs. $\sqrt{N}$ with $k = 2, 3$ and $p = 0.01, 0.001$, respectively. It demonstrates again the high accuracy of numerical solutions of (4.7), and its rapid convergence. We can also see from Figure 2 that the convergence rate

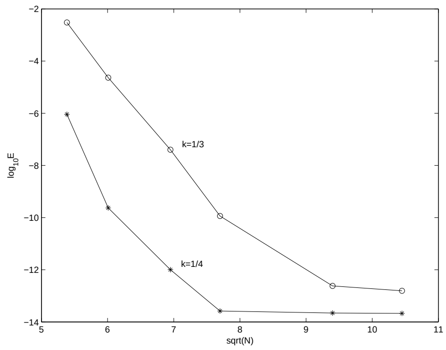

Figure 1. Convergence rates: case 1

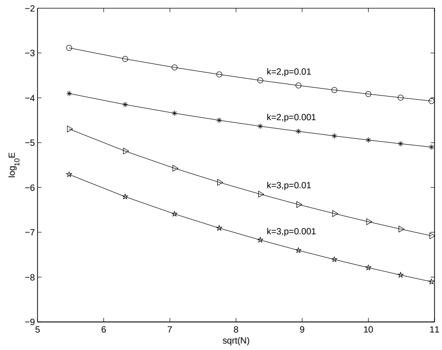

Figure 2. Convergence rates: case 2

depends on the regularity of the exact solution. This coincides again with the theoretical analysis.

Example 2. We next consider problem (4.20). According to the boundary conditions, we can take in (4.23), $\phi(x) = (1-x^2)^2 q_{N-4}(x),\ q_{N-4} \in \mathcal{P}_{N-4}$. Moreover, the numerical solution

$$u_N(x) = \frac{1}{4}(1-x)^2(4(1+x)^2 p_{N-4}(x) + (1+x)(g_0+g_1) + g_0), \quad p_{N-4} \in \mathcal{P}_{N-4}.$$

Substituting the above expressions into (4.23), we can find $p_{N-4}(x)$. In actual computation, we take the monic Jacobi polynomials $J_l^{(1,\beta)}(x)$ as the base functions of $\mathcal{P}_{N-4}$. We take the test function

$$U(x) = (1-x)^3(1+x)^{5-\beta}.$$

In Figure 3, we plot $\log_{10} \|U - u_N\|_{\chi^{(1,\beta)},R,N}$ vs. $\log_{10} N$, with $\beta = 0.2$ (the upper line), $\beta = 0.3$ (the middle line) and $\beta = 0.4$ (the lower line). Clearly, the numerical solution converges fast as N increases, as predicted in the theoretical analysis.

Example 3. Finally, we consider problem (4.25), and take the test function

$$U(x,t) = (1-x^2)^k \sin(x + \frac{\pi}{2}t)e^{2t}.$$

We use scheme (4.27) with $u_{N,j} = \widetilde{P}^{2,0}_{N,0,0} U_j, j = 0,1$, to solve (4.25) with $k = 2,3$. Let $u_N(x,t)$ be the numerical solution. In actual computation, we introduce the auxiliary function $v_N(t) = \partial_t u_N(t)$. Then we use the Runge-Kutta method of fourth order in time discretization, with mesh size τ. For description of numerical errors, let

$$E(u_N,t) = \left(\sum_{j=0}^{N}(U(\zeta^{(0,0)}_{L,N,j},t) - u_N(\zeta^{(0,0)}_{L,N,j},t))^2 \omega^{(0,0)}_{L,N,j}\right)^{1/2}.$$

The numerical errors at $t = 1$, with $\tau = 0.001$ and different N, are illustrated in Figure 4, i.e., $\log_{10}(E(u_N,1))$ vs. N. It can be seen that the errors decay very quickly as N increases . This also coincides very well with the theoretical analysis.

6. Concluding Remarks

In this work, we developed the Jacobi pseudospectral method for fourth order problems. The pseudospectral schemes were proposed for three model problems. The first one is a singular problem. The second is a problem on an axisymmetric domain, which is related to a singular

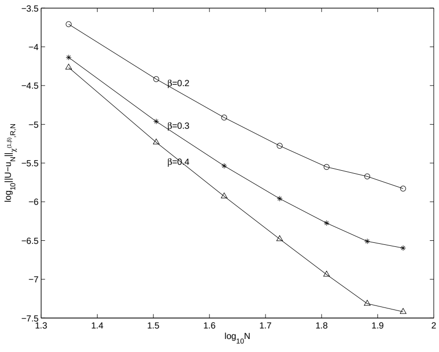

Figure 3. Convergence rate of (4.23)

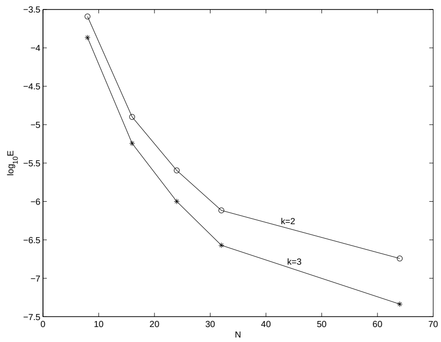

Figure 4. Convergence rate of (4.27)

problem. The third is a nonlinear parabolic equation of high order. Their convergences were proved. The numerical results demonstrated their spectral accuracy, and coincided very well with theoretical analysis. Although we only considered three model problems in this paper, the proposed method is also applicable to many singular or non-singular, steady or unsteady, and linear or nonlinear problems of fourth order. Clearly, it is not difficult to generalize this approach to multiple-dimensional problems. Furthermore, we may use suitable variable transformations to change some fourth order problems on unbounded domains to singular problems on bounded domains, and then use the proposed method in this paper to solve them numerically, such as the stream function form of the Navier-Stokes equations in an infinite strap, the oscillation of a very long beam and so on.

In this work, we established some basic results on the Jacobi-Gauss-type interpolations in certain non-uniformly Jacobi-weighted Sobolev spaces. They play an important role in numerical analysis of pseudospectral method for fourth order problems, especially, for various singular problems and nonlinear problems.

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