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OPTIMAL ERROR ESTIMATES OF THE PARTITION OF UNITY METHOD WITH LOCAL POLYNOMIAL APPROXIMATION SPACES [*1)]

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, we provide a theoretical analysis of the partition of unity finite element method(PUFEM), which belongs to the family of meshfree methods. The usual error analysis only shows the order of error estimate to the same as the local approximations[12]. Using standard linear finite element base functions as partition of unity and polynomials as local approximation space, in 1-d case, we derive optimal order error estimates for PUFEM interpolants. Our analysis show that the error estimate is of one order higher than the local approximations. The interpolation error estimates yield optimal error estimates for PUFEM solutions of elliptic boundary value problems.

Mathematics subject classification: 65N30, 65N15, 65N50.
Key words: Meshless methods, Partition of unity finite element method(PUFEM), Error estimate.

1. Introduction

As a new family of numerical methods, in the last few years meshless methods came into the focus of interest, especially in the engineering community. This is motivated by the often encountered serious difficulties in generating meshes for problems in complex domains, or in domains evolving with the problem solution. In addition, a need to have flexibility in the selection of approximating functions (e.g., the flexibility to use non-polynomial approximating functions), played a significant role in the development of meshless methods. A recent survey of meshless and generalized finite element methods was given by [2] together with a comprehensive list of references. It states the development in this new field and provides the available mathematical theory with proofs. From its list of references, we can learn that more and more interest has been directed towards an important subclass of methods originating from the partition of Unity Method(PUM) of *Babuška* and Melenk [3]. These methods include the hp cloud method of Oden and Duarte [5], the Generalized Finite Element Method (GFEM) of Strouboulis [14]-[16] and the particle-partition of unity method (see [6]-[9] and [11]). Applying the partition of unity Huang and Xu[11] proposed a conforming finite element method for overlapping nonmatching

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grids. Why is the PUFEM so popular? The most prominent reasons are that PUFEM can include a priori knowledge about the local behavior of the solution in the finite element space and construct finite element spaces of any desired regularity. In general, we only know that the function u of interest is in some function space. As [1] points out, the local approximating spaces of PUFEM have many function spaces including polynomials and non-polynomial functions to choose for the solutions to a given differential equation. These function spaces are not unique and the choice of particular space thus depends on practical aspects (cost of constructing the functions, ease of evaluation of the functions, i.e. cost of construction of the stiffness matrix; conditioning number of the resulting stiffness matrix) and theoretical aspects (optimality of the function space). It is well known the $h-$, $p-$, and $hp-$ versions of the finite element method (FEM) use local polynomial approximating as shape functions. The success of FEM is due to the fact that a smooth function can be approximated locally by polynomials and polynomial spaces are big enough to absorb extra constraints of continuity cross interelement boundaries without loosing the approximation properties. Therefore, for sufficiently smooth function, polynomial space is preferable for the local approximating spaces in PUFEM. If the usual piecewise linear hat functions are taken as a (M, C_∞, C_G) partition of unity and the local approximation spaces are chosen to be spaces of polynomials, the PUFEM then can be referred to a generalization of the $h-$ and $p-version$ FEM. And the PUFEM have approximation properties very similar to the usual $h-$ and $p-version$ FEM [4]. Some general results on the PUFEM are provided in [3] by using technique of Taylor expansion. *Babuška* and his co-workers give main theoretical results about GFEM in [4]. Recently a posteriori estimation for GFEM similar to that for FEM can be found in [13].

The usual error analysis only shows the order of error estimate to the same as the local approximations[12]. Using standard linear finite element base functions as partition of unity and polynomials as local approximation space, in 1-d case, we derive optimal order error estimates for PUFEM interpolants. Our analysis will show that the error estimate is of one order higher than the local approximations, that is, global error estimate of order $p+1$ is achieved while the local error estimates on patches is only of order p. For this purpose, we construct a special polynomial local approximation space according to the consistence and local approximation properties of PUFEM at first, and then we derive the interpolation error estimation of PUFEM by employing the arguments in [10] and applying various techniques of Taylor expansion and theories of average polynomials interpolation. The interpolation error estimates are used to obtain optimal order error estimates for PUFEM solutions of Neumann boundary value problems. We will also show how to derive optimal order error estimates for PUFEM solutions of Dirichlet boundary value problems (BVPs) in one dimension. The error estimates we establish in this paper are in the one dimensional setting and under sufficient smoothness assumption on the functions being approximated. Error analysis for singular problems and the higher dimensional case will be addressed in the forthcoming papers.

The paper is organized as follows. In section 2, we provide a precise introduction of PUFEM, emphasizing mathematical foundation behind the development of the method. A kind of special local approximation space based on polynomials will be constructed and optimal order error estimates for PUFEM interpolants are then established in section 3. In section 4, we discuss error estimates for PUFEM solutions of boundary value problems.

2. Mathematical Foundation of the PUFEM

In this section, we present a method of constructing conforming subspaces of $H^1(\Omega)$. We construct finite element spaces which are subspaces of $H^1(\Omega)$ as an example because of their importance in applications. We would like to point out that the method leads to the construction of smoother spaces (subspaces of $H^k(\Omega), k > 1$) or subspaces of Sobolev spaces $W^{k,p}$ in a straightforward manner. we introduce the main concepts and results concerning the error estimation of PUFEM following *Babuška* and Melenk [12][3].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with boundary Γ, $\{\Omega_i\}$ be an open cover of Ω satisfying a pointwise overlap condition*

$$\exists M \in \mathbb{N}, \quad \forall x \in \Omega, \quad card\{i | x \in \Omega_i\} \leq M$$

Let $\{\varphi_i\}$ be a Lipschitz partition of unity subordinate to the cover $\{\Omega_i\}$ satisfying

$$supp\ \varphi_i \subset closure(\Omega_i) \quad \forall i, \tag{2.1}$$

$$\sum_i \varphi_i \equiv 1 \quad on\ \Omega, \tag{2.2}$$

$$\|\varphi_i\|_{L^\infty(\mathbb{R}^n)} \leq C_\infty, \tag{2.3}$$

$$\|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_G}{diam\Omega_i}, \tag{2.4}$$

where C_∞, C_G are two constants. Then$\{\varphi_i\}$ is called a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. The partition of unity $\{\varphi_i\}$ is said to be of degree $m \in \mathbb{N}$ if $\{\varphi_i\} \subset C^m(R^n)$. The covering sets $\{\Omega_i\}$ are called patches.

The usual piecewise linear hat functions on a regular (triangular) mesh in two dimensions, satisfy the above conditions of a *(M, C_∞, C_G) partition of unity* ; actually, $M = 3, C_\infty = 1$, and condition (4) is satisfied because of the regularity of the mesh, i.e. the minimum angle condition satisfied by the triangulation. Similarly, the classical bilinear finite element functions on quadrilateral meshes form a *(M, C_∞, C_G) partition of unity* ($M = 4, C_\infty = 1$).

Definition 2.2. *Let $\{\Omega_i\}$ be an open cover of $\Omega \subset \mathbb{R}^n$, and let $\{\varphi_i\}$ be a (M, C_∞, C_G) partition of unity subordinate to $\{\Omega_i\}$. Let $V_i \subset H^1(\Omega_i \cap \Omega)$ be given function spaces. Then we call the space*

$$V := \sum_i \varphi_i V_i = \{\sum_i \varphi_i v_i | v_i \in V_i\} \subset H^1(\Omega) \tag{2.5}$$

the PUFEM space. The PUFEM space is said to be of degree $m \in \mathbb{N}$ if $V \subset C^m(\Omega)$. The spaces V_i are called local approximation spaces.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be given. Let $\{\Omega_i\}$, $\{\varphi_i\}$ and V_i be as in Definitions 2.1, 2.2. Let $u \in H^1(\Omega)$ be the function to be approximated. Assume that the local approximation spaces V_i have the following approximation properties: on each patch $\Omega_i \cap \Omega$, u can be approximated by a function $v_i \in V_i$ such that*

$$\|u - v_i\|_{L^2(\Omega_i\cap\Omega)} \leq \varepsilon_1(i),$$

$$\|\nabla(u - v_i)\|_{L^2(\Omega_i\cap\Omega)} \leq \varepsilon_2(i).$$

Then the function

$$u_{ap} = \sum_i \varphi_i v_i \in V \subset H^1(\Omega)$$

satisfies

$$\|u - u_{ap}\|_{L^2(\Omega)} \leq \sqrt{M} C_\infty (\sum_i \varepsilon_1^2(i))^{\frac{1}{2}},$$

$$\|\nabla(u - u_{ap})\|_{L^2(\Omega)} \leq \sqrt{2M} (\sum_i (\frac{C_G}{diam\Omega_i})^2 \varepsilon_1^2(i) + C_\infty^2 \varepsilon_2^2(i))^{\frac{1}{2}}.$$

The detailed proof can be found in [12][3].

Theorem 1 shows that the global space V inherits the approximation properties of the local spaces V_i, i.e. the function u can be approximated on Ω by functions in V as well as the functions $u|_{\Omega_i}$ can be approximated in the local spaces V_i. Moreover, the space V inherits the smoothness of the partition of unity $\{\varphi_i\}$. In particular, the smoothness of the partition of unity enforces the conformity of the global space V. If the local approximation spaces V_i are chosen to be polynomial spaces, the PUFEM have approximation properties very similar to the usual $h-$ and $p-version$ FEM. In fact, if the polynomials in local approximation spaces have fixed degree p and the approximation in V_i is realized by decreasing the size h of the patch Ω_i, the method is referred to the $h-version$. If the patches are fixed and the local approximation is achieved by increasing the degree p of the polynomials, the method is referred to the $p-version$. In this sense, the PUFEM can be regarded as a generalization of the $h-$ and $p-version$ FEM. Then the classical FEM can be viewed as the most simply and the most special case of PUFEM, i.e., the local approximation spaces are taken as polynomials of degree 0. As [12] pointed out, in the $h-version$ of the FEM, if the approximated function u is sufficiently smooth (in H^k, say), an appropriate interpolant u_I has error estimate

$$\|u-u_I\|_{H^1} \leq C_{k,p}h^{min(k-1,p)}|u|_{H^k}$$

where $C_{k,p}$ is independent of u and h. For p=1, the order of error is 1. In the following, we shall consider whether it is possible that the order of interpolation error can be improved by choosing local special polynomial approximation spaces in $h-version$ PUFEM, that is, global error estimate of order $p+1$ is achieved while the local error estimates on patches is only of order p.

3. Interpolation Error Estimates

In this section, we will construct a PUFEM interpolant of degree p of u by a kind of special local approximation spaces consisting of polynomials of degree $p-1$ at first, then derive the error estimates of the interpolant.

3.1. Formulation of PUFEM Interpolant

In one dimension, we assume (a,b) is the problem domain. We divide the interval (a,b) into

$$a=x_1<x_2<\cdots<x_n=b.$$

set $h_i=x_i-x_{i-1},\ h=\max_i h_i,\ e_i=(x_{i-1},x_i),\ i=2,3,\cdots,n$. The usual piecewise linear hat functions in FEM are chosen to be the partition of unity functions in PUFEM. The supports of these hat functions can then be taken as the patches Ω_i in PUFEM. The local approximation spaces consisting of the polynomials of degree p are chosen to be

$$V_i = span\{1, x-x_i, (x-x_i)^2, \cdots, (x-x_i)^p\}. \tag{3.1}$$

Since the functions $\{\varphi_i\}$ form a partition of unity, we have $u=(\sum_i \varphi_i)u=\sum_i \varphi_i u$ and thus

$$(u-u_I)' = (\sum_i \varphi_i(u-v_i))' = \sum_i \varphi_i(u-v_i)' + \sum_i \varphi_i'(u-v_i).$$

A direct calculation yields

$$\|u-u_I\|_1 = O(h^p).$$

We shall provide a more precise constructive analysis to obtain the optimal interpolation error of order $p+1$. The key ingredient is to select a suitable candidate in each V_i such that $\sum \varphi_i v_i$ admits higher order accuracy.

3.2. Error Estimates of PUFEM Interpolants for $p=1$

We first consider the error estimates for $p=1$. In this case $V_i=span\{1,x-x_i\}$. To obtain the optimal error estimates we choose

$$v_i = u(x_i) + \frac{1}{2}u'(x_i)(x-x_i) \qquad \in V_i \tag{3.2}$$

and then

$$u_I(x) = \sum_{i=1}^{n} \varphi_i(x)(u(x_i) + \frac{1}{2}u'(x_i)(x-x_i)). \tag{3.3}$$

Since on each element e_i, the $\{\varphi_i\}$ form the partition of unity and have the form

$$\varphi_{i-1}(x) = \frac{x_i - x}{h_i}, \quad \varphi_i(x) = \frac{x - x_i}{h_i},$$

then by Lagrange theorem, we have

$$\begin{aligned} u - u_I|_{e_i} &= \varphi_{i-1}(x)(u(x) - u(x_{i-1}) - \frac{1}{2}u'(x_{i-1})(x - x_{i-1})) \\ &\quad + \varphi_i(x)(u(x) - u(x_i) - \frac{1}{2}u'(x_i)(x - x_i)) \\ &= \varphi_{i-1}(x)\left(\frac{1}{2}u'(x_{i-1})(x - x_{i-1}) + \frac{1}{2}u''(\xi_i)(x - x_{i-1})^2\right) \\ &\quad + \varphi_i(x)\left(\frac{1}{2}u'(x_i)(x - x_i) + \frac{1}{2}u''(\eta_i)(x - x_i)^2\right) \\ &= \frac{(x - x_{i-1})(x - x_i)}{2h_i}(u''(\eta_i) - u''(\xi_i))(x - x_i) \\ &\quad + \frac{(x - x_{i-1})(x - x_i)}{2}\left(\frac{u'(x_i) - u'(x_{i-1})}{h_i} - u''(\xi_i)\right) \\ &= \frac{1}{2}(x - x_{i-1})(x - x_i)\left((\theta_i - \xi_i)u'''(\zeta_i) + \frac{\eta_i - \xi_i}{h_i}(x - x_i)u'''(\delta_i)\right) \end{aligned} \tag{3.4}$$

where ξ_i, η_i, ζ_i, θ_i and δ_i belong to (x_{i-1}, x_i).
Therefore

$$\|u - u_I\|_{L_\infty(e_i)} \le Ch^3\|u'''\|_{L_\infty(e_i)},$$

and hence we obtain the global estimate over the whole interval (a, b)

$$\|u - u_I\|_{L_\infty(a,b)} \le Ch^3\|u'''\|_{L_\infty(a,b)}. \tag{3.5}$$

In fact, from (3.4) we see that the partition of unity finite element space can reproduce all the polynomials of order 2, i.e. on any element e_i, for any $u \in \mathcal{P}_2(e_i)$, the interpolant u_I defined by (3.3) is identical to u itself. So a direct application of the Bramble-Hilbert's lemma gives the following theorem.

Theorem 2. *Let $\{\varphi_i\}$ be piecewise linear hat function , and let u_I be the PUFEM interpolant defined in (3.3). If $u \in W^{3,q}(a, b)$, then we have the optimal order interpolation error estimates*

$$\|u - u_I\|_{l,q} \le Ch^{3-l}\|u\|_{3,q} \qquad l = 0, 1, \qquad 1 \le q \le \infty. \tag{3.6}$$

3.3. Error Estimate of PUFEM Interpolants for $p > 1$

In previous subsections, we have discussed the PUFEM interpolation error for $p = 1$. Let us now consider the case of $p > 1$. The piecewise linear hat functions in FEM are still chosen to be the partition of unity functions in PUFEM. The local approximation spaces $V_i = span\{1, x - x_i, \cdots, (x - x_i)^p\}$.

On any patch, we construct the local polynomial approximation of u as

$$\begin{aligned} v_i &= u(x_i) + \frac{p}{p+1}u'(x_i)(x - x_i) + \frac{p-1}{2(p+1)}u''(x_i)(x - x_i)^2 + \cdots + \frac{1}{(p+1)p!}u^p(x_i)(x - x_i)^p \\ &= \sum_{k=0}^{p}\frac{p+1-k}{(p+1)k!}u^{(k)}(x_i)(x - x_i)^k. \end{aligned} \tag{3.7}$$

Then the interpolant $u_I(x)$ of u in V is

$$u_I(x) = \sum_{i=1}^{n}\varphi_i(x)v_i = \sum_{i=1}^{n}\varphi_i(x)\sum_{k=0}^{p}\frac{p+1-k}{(p+1)k!}u^{(k)}(x_i)(x - x_i)^k. \tag{3.8}$$

We shall verify that this interpolation reproduces all the polynomials of order $p + 1$. Since any element e_i can be transferred into a reference element $(0, 1)$, we need only to verify the reproducing property on the reference interval.

On $(0,1)$ the shape functions of the partition of unity have the form

$$\varphi_0 = 1-x, \quad \varphi_1 = x$$

and the interpolant U_I is

$$U_I(x) = (1-x)\sum_{k=0}^{p}\frac{p+1-k}{(p+1)k!}U^{(k)}(0)x^k + x\sum_{k=0}^{p}\frac{p+1-k}{(p+1)k!}U^{(k)}(1)(x-1)^k, \tag{3.9}$$

Lemma 1. *For any polynomial $U\in\mathcal{P}_{p+1}$, the interpolation (3.9) reproduces U, i.e., $U_I = U$.*

Proof. We need only to verify the results for the polynomials $U = x^m, m = 0,1,\cdots,p+1$. It is easy to see that in the case of $U = 1$,

$$U_I = (1-x)+x = 1 = U. \tag{3.10}$$

In the case of $U = x^m$, $1\le m\le p$, it easy to see that

$$U^{(j)}(0) = j!\delta_{jm},$$

$$U(1) = 1,\ U^{(j)}(1) = \frac{m!}{(m-j)!} \quad 1\le j\le m;\ \ U^{(j)}(1) = 0 \quad m+1\le j\le p.$$

Therefore

$$\begin{aligned} U_I(x) &= (1-x)\frac{m!(p+1-m)}{(p+1)m!}x^m + x\Big(\sum_{j=0}^{m}\frac{p+1-j}{(p+1)j!}\cdot\frac{m!}{(m-j)!}(x-1)^j\Big) \\ &= (1-x)\frac{p+1-m}{p+1}x^m + x\Big(\frac{p+1-m}{p+1}x^m + \frac{m}{p+1}x^{m-1}\Big) = x^m, \end{aligned} \tag{3.11}$$

where we used the identity

$$\sum_{j=0}^{m}\frac{p+1-j}{(p+1)j!}\cdot\frac{m!}{(m-j)!}(x-1)^j = \frac{p+1-m}{p+1}x^m + \frac{m}{p+1}x^{m-1}. \tag{3.12}$$

Finally, in the case of $U = x^{p+1}$,

$$U^{(j)}(0) = 0, \qquad j = 0,1,\cdots,p,$$

$$U^{(j)}(1) = (p+1)p\cdots(p-j+2) = \frac{(p+1)!}{(p+1-j)!}\ , \quad j = 0,1,\cdots,p.$$

Therefore

$$\begin{aligned} U_I(x) &= x\sum_{j=0}^{p}\frac{p+1-j}{(p+1)j!}\frac{(p+1)!}{(p+1-j)!}(x-1)^j \\ &= x\sum_{j=0}^{p}\frac{p!}{j!(p-j)!}(x-1)^j = x(x-1+1)^p = x^{p+1}. \end{aligned} \tag{3.13}$$

That completes the proof.

From the above lemma we can derive the error estimates for the interpolation by applying the Bramble-Hilbert's lemma.

Theorem 3. *Let $\{\varphi_i\}$ be piecewise linear hat function , and let u_I be the PUFEM interpolant defined in (3.8). If $u\in W^{p+2,q}(a,b)$, then we have optimal interpolation error estimate*

$$\|u-u_I\|_{l,q,(a,b)} \le Ch^{p+2-l}\|u\|_{p+2,q,(a,b)}. \tag{3.14}$$

4. PUFEM and Error Analysis

We consider the error estimates of PUFEM in solving differential equations in this section. Let's consider a one dimensional model problem on $(0,1)$. We divide the interval $(0,1)$ into

$$0 = x_1 < x_2 < \cdots < x_n = 1,$$

and introduce the local approximation space

$$V_i = span\{1, x - x_i, \cdots, (x - x_i)^p\}. \tag{4.1}$$

Taking usual piecewise linear hat functions as the partition of unity functions in PUFEM, then the PUFEM space is

$$V = span\{\varphi_i(x)(x - x_i)^m |\ i = 1, \cdots, n\ ,\quad m = 0, \cdots, p\}. \tag{4.2}$$

Any $v \in V$ can be represented as

$$v = \sum_{i=1}^{n} \varphi_i \sum_{j=0}^{p} \alpha_j (x - x_i)^j. \tag{4.3}$$

Let the weak form of the boundary value problems be: Find $u \in W$,such that

$$B(u, v) = (f, v) \qquad \forall v \in W, \tag{4.4}$$

where $B(u, v)$ is a continuous bilinear form and positive on W. Using the Galerkin framework, the $PUFEM$ solution can be defined by

$$B(u_{pu}, v) = (f, v) \qquad \forall v \in V. \tag{4.5}$$

Céa's lemma yields the error estimates of the PUFEM solution $u_{pu} \in V$,

$$\|u - u_{pu}\|_1 \leq C \inf_{v \in V} \|u - v\|_1. \tag{4.6}$$

If the PUFEM interpolant defined in (3.8) $u_I \in V$, then theorem 3 and (4.6) give the optimal error estimate

$$\|u - u_{pu}\|_1 \leq C\|u - u_I\|_1 \leq Ch^{p+1}\|u\|_{p+2}. \tag{4.7}$$

For Neumann boundary conditions, PUFEM interpolant defined in (3.8) $u_I \in V$ since there is no constraint on the boundary value of the approximation functions. The meshless methods are known to be more difficult to deal with the Dirichlet boundary conditions. In general, most of the meshless interpolants including the PUFEM interpolants do not satisfy the Dirichlet boundary condition, since the shape functions do not have the Kronecker delta property. Thus the treatment of Dirichlet boundary value conditions is more difficult than in the FEM. But fortunately, in 1-d, the special PUFEM interpolant we define in (3.8) satisfies Dirichlet boundary condition. We check it for homogeneous Dirichlet conditions, i.e., $u(0) = u(1) = 0$. In fact, because the partition of unity $\{\varphi_i\}$ have Kronecker delta property, then we have $\varphi_j(0) = 0, j = 2, 3, \cdots, n$ and

$$v_1(0) = \sum_{k=0}^{p} \frac{p + 1 - k}{(p + 1)k!} u^{(k)}(0)x^k|_{x=0} = u(0) = 0.$$

Hence

$$u_I(0) = \sum_{j=1}^{p} \varphi_j(0)v_j(0) = 0.$$

Similarly we can prove $u_I(1) = u(1) = 0$.

Remark 1. This result does not hold in higher dimension, since a function from V does not vanish on a part of the boundary even when it is zero at all the nodes on that part of the boundary.

Since the functions in V are able to satisfy the Dirichlet condition, we can directly use the PUFEM space V as trial and test function spaces in the Galerkin procedure. Thus the same error estimate for the PUFEM solution of Dirichlet BVPs as that for Neumann BVPs can be achieved.

The Aubin-Nitche's duality argument can be used to obtain the error estimates of the PUFEM solutions as like in finite element methods.

In conclusion, we have

Theorem 4. *The PUFEM solution u_{pu} admits optimal error estimates*

$$\|u - u_{pu}\|_l \leq Ch^{p+2-l}\|u\|_{p+2}\ ,\quad l = 0, 1. \tag{4.8}$$

5. Conclusion

In this paper, we provide an error analysis for a special PUFEM which choose the usual piecewise linear hat functions to be the partition of unity and use the polynomials as the local approximating functions. We derive the optimal error estimates of the PUFEM interpolant at first, and they then are used to obtain the error estimates for the PUFEM solution of the elliptic BVPs. The results are for one dimensional case and under the assumption that the approximated function is sufficiently smooth. For us, this paper is just the starting point of working on theoretical analysis of PUFEM . In later study, we will give the error analysis for higher dimensional case and under the assumption of weaker smoothness of approximated function.

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