

# ARTIFICIAL BOUNDARY METHOD FOR BURGERS' EQUATION USING NONLINEAR BOUNDARY CONDITIONS <sup>\*1)</sup>

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Dedicated to the 70th birthday of Professor Lin Qun

## Abstract

This paper discusses the numerical solution of Burgers' equation on unbounded domains. Two artificial boundaries are introduced and boundary conditions are obtained on the artificial boundaries, which are in nonlinear forms. Then the original problem is reduced to an equivalent problem on a bounded domain. Finite difference method is applied to the reduced problem, and some numerical examples are given to show the effectiveness of the new approach.

*Mathematics subject classification:* 65M06, 65M12, 65M15.

*Key words:* Burgers' equation, Unbounded domain, Artificial boundary condition.

## 1. Introduction

For numerical solutions of partial differential equations on unbounded domains, the artificial boundary method is the most efficient method and has been applied to many application problems [10, 16, 11, 5, 19, 15, 7, 8]. The artificial boundary method generally means to introduce artificial boundaries, find boundary conditions on the artificial boundaries, and reduce the original problem to an equivalent or approximate problem defined on a bounded domain. In general, the boundary conditions on the artificial boundaries are obtained by considering the exterior problems outside the artificial boundaries. In most cases, the basic assumption of the artificial boundary method is that the equation is linear. Then analytic forms of the boundary conditions on the artificial boundaries can be obtained. Usually, the artificial boundary method can not be applied directly to nonlinear problems. However, for some problems, if the equation can be linearized outside the artificial boundaries, then it is possible to find the boundary conditions on the artificial boundaries [9, 14, 6].

In this paper, we consider the numerical solution of the Burgers' equation on unbounded domains

$$u_t + uu_x - \nu u_{xx} = f(x, t), \quad \forall (x, t) \in R \times (0, T], \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad (1.2)$$

$$u(x, t) \rightarrow 0, \quad |x| \rightarrow +\infty, \quad (1.3)$$

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where  $\nu > 0$  is viscous coefficient, the source term  $f(x, t)$  and initial data  $u_0(x)$  have compact supports satisfying

$$\text{supp}\{f(x, t)\} \subset [x_l, x_r] \times [0, T], \quad \text{supp}\{u_0(x)\} \subset [x_l, x_r].$$

Burgers' equation is a simple, but important model in fluid dynamics. Burgers' equation itself models various kind of physical phenomena such as turbulence [3]. Besides, due to its similarity to the Navier-Stokes equation, the solution of Burgers' equation is a natural first step towards the designing of new numerical methods for flow problems. When the domain is bounded, various numerical approaches have been discussed by many researches (see recent papers [2, 4] and references therein). In this paper, we mainly deal with the difficulty of the unboundedness of the solution domain. The idea is to introduce artificial boundaries to make the computational domain finite, and find boundary conditions on the artificial boundaries. Unlike linear problems, these boundary conditions are nonlinear. Then we solve the problem on the finite domain, the reduced problem is equivalent to the original problem in the sense that the solution is the same as the restriction of the original problem. In section 2, we describe the artificial boundary method using nonlinear boundary conditions. In section 3, we consider the numerical approximation of the reduced problem. In section 4, we give some numerical examples to show the effectiveness of the new approach.

## 2. The Artificial Boundary Method

Consider the problem (1.1)-(1.3). We introduce two artificial boundaries

$$\Gamma_l = \{(x, t) \mid x = x_l, 0 \leq t \leq T\},$$

$$\Gamma_r = \{(x, t) \mid x = x_r, 0 \leq t \leq T\}.$$

Then the unbounded domain  $\Omega = R \times [0, T]$  is divided into three parts (see Fig. 1),

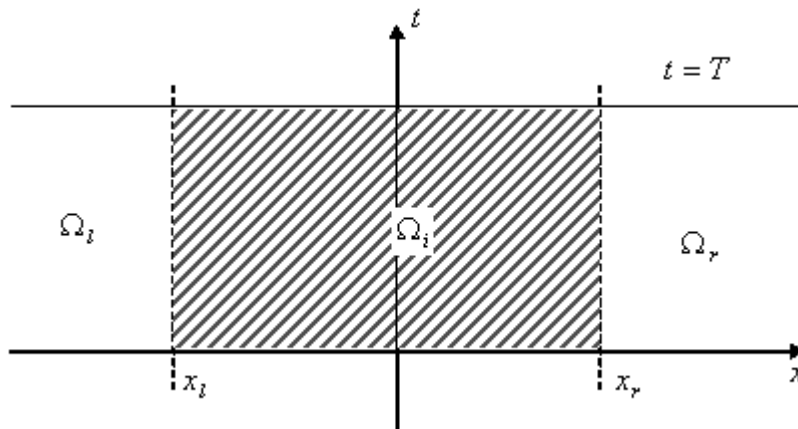


Fig. 1. The artificial boundaries

$$\Omega_l = \{(x, t) \mid x \leq x_l, 0 \leq t \leq T\},$$

$$\Omega_r = \{(x, t) \mid x \geq x_r, 0 \leq t \leq T\},$$

$$\Omega_i = \{(x, t) \mid x_l < x < x_r, 0 \leq t \leq T\},$$

where  $\Omega_i$  is the computational domain and its two boundary conditions at  $x = x_l$  and  $x = x_r$  are to be determined.

In order to obtain boundary conditions on  $\Gamma_l$  and  $\Gamma_r$ , we consider firstly the restriction of  $u$  on the right semi-infinite domain  $\Omega_r$ , where the solution satisfies

$$u_t + uu_x - \nu u_{xx} = 0, \quad \text{in } \Omega_r, \quad (2.1)$$

$$u|_{t=0} = 0, \quad (2.2)$$

$$u \rightarrow 0, \quad x \rightarrow +\infty, \quad (2.3)$$

$$u|_{x=x_r} = u(x_r, t), \quad 0 < t \leq T. \quad (2.4)$$

This problem can not be solved independently due to the unknown function  $u(x_r, t)$ . However, if we assume that the boundary value  $u(x_r, t)$  is given, then this is a well-posed problem. To solve this problem analytically, we use the Cole-Hopf transformation [17] to reduce the nonlinear Burgers equation to a linear heat equation. Let

$$\omega(x, t) = - \int_x^\infty u(y, t) dy, \quad x_r \leq x < +\infty,$$

then

$$\omega_t = - \int_x^\infty u_t(y, t) dy = \nu u_x - \frac{1}{2} u^2, \quad (2.5)$$

$$\omega_x = u, \quad \omega_{xx} = u_x. \quad (2.6)$$

Substituting into (2.1)-(2.4), we obtain,

$$\omega_t + \frac{1}{2} \omega_x^2 - \nu \omega_{xx} = 0,$$

$$\omega|_{t=0} = 0,$$

$$\omega|_{x=x_r} = \omega(x_r, t),$$

$$\omega \rightarrow 0, \quad \text{when } x \rightarrow +\infty.$$

Let  $v = \psi(\omega) - 1$  with  $\psi(\omega) = e^{-\frac{\omega}{2\nu}}$ , then  $v$  satisfies

$$v_t = \nu v_{xx}, \quad \forall (x, t) \in \Omega_r, \quad (2.7)$$

$$v|_{t=0} = 0, \quad \forall x \in [x_r, +\infty), \quad (2.8)$$

$$v \rightarrow 0 \quad \text{when } x \rightarrow +\infty \quad (2.9)$$

$$v|_{x=x_r} = v(x_r, t). \quad (2.10)$$

This is the standard heat equation. By solving (2.7)-(2.10) we obtain the boundary condition on the artificial boundary  $x = x_r$ ,

$$\frac{\partial v(x_r, t)}{\partial x} = - \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{1}{\sqrt{t-\tau}} \frac{\partial v(x_r, \tau)}{\partial \tau} d\tau. \quad (2.11)$$

For the numerical solution of heat equation on an unbounded domain, Han and Huang [12, 13] proposed a class of artificial boundary conditions for one and two dimensional cases. The convergence of difference scheme using artificial boundaries was given by Wu and Sun [18]. To transform the boundary condition (2.11) back into the original variable  $u$ , considering

$$v_x = \phi'(\omega)u, \quad v_t = \phi'(\omega)(\nu u_x - \frac{1}{2}u^2),$$

we have

$$G(t)u(x_r, t) = - \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{G(\tau)}{\sqrt{t-\tau}} \{ \nu u_x(x_r, \tau) - \frac{1}{2} u(x_r, \tau)^2 \} d\tau$$

where

$$G(t) = \psi'(\omega)|_{x=x_r} = - \frac{1}{2\nu} \exp(-\frac{\omega}{2\nu})|_{x=x_r}.$$

Hence we obtain

$$u(x_r, t) = - \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{G(\tau)}{G(t)\sqrt{t-\tau}} \{ \nu u_x(x_r, \tau) - \frac{1}{2} u(x_r, \tau)^2 \} d\tau.$$

This is the boundary condition in the original variables, but it is rather difficult to be used directly for computation, since the function  $G(t)$  contains an infinite integral. To simply this boundary condition, let

$$g_r(t) = -\frac{1}{2\nu}\omega|_{x=x_r} = \frac{1}{2\nu} \int_{x_r}^{+\infty} u(y, t)dy,$$

then from equation (2.5),  $g_r(t)$  is the solution of the following initial value problem,

$$\frac{dg_r(t)}{dt} = -\frac{1}{2}u_x(x_r, t) + \frac{1}{4\nu}u(x_r, t)^2, \quad g_r(0) = 0.$$

So we have

$$g_r(t) = \int_0^t \left( -\frac{1}{2}u_x(x_r, \tau) + \frac{1}{4\nu}u(x_r, \tau)^2 \right) d\tau. \tag{2.12}$$

Thus, we have the first artificial boundary condition

$$u(x_r, t) = -\frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{e^{g_r(\tau)-g_r(t)}}{\sqrt{t-\tau}} \{ \nu u_x(x_r, \tau) - \frac{1}{2}u(x_r, \tau)^2 \} d\tau. \tag{2.13}$$

By using the Abel transform [1], we have the second artificial boundary condition from (2.13),

$$u_x(x_r, t) = \frac{1}{2\nu}u(x_r, t)^2 - \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{\partial}{\partial \tau} (e^{g_r(\tau)-g_r(t)}u) \frac{1}{\sqrt{t-\tau}} d\tau, \tag{2.14}$$

which is simpler than the first artificial boundary condition (2.13). Therefore we obtain the nonlinear artificial boundary conditions by coupling (2.13) (or (2.14)) and (2.12) with the auxiliary function  $g_r(t)$ . For abbreviation, we write two artificial boundary conditions (2.13) and (2.14) as

$$u(x_r, t) = R_1(u(x_r, t), g_r(t)), \quad \text{and} \quad u_x(x_r, t) = R_2(u(x_r, t), g_r(t)).$$

with  $g_r(t)$  given in (2.12).

For the left semi-infinite domain  $\Omega_l$ , we can also derive the similar artificial boundary conditions,

$$u(x_l, t) = \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{e^{g_l(\tau)-g_l(t)}}{\sqrt{t-\tau}} \{ \nu u_x(x_l, \tau) - \frac{1}{2}u(x_l, \tau)^2 \} d\tau. \tag{2.15}$$

and the second version

$$u_x(x_l, t) = \frac{1}{2\nu}u(x_l, t)^2 + \frac{1}{\sqrt{\nu\pi}} \int_0^t \frac{\partial}{\partial \tau} (e^{g_l(\tau)-g_l(t)}u) \frac{1}{\sqrt{t-\tau}} d\tau, \tag{2.16}$$

where

$$g_l(t) = \int_0^t \left( -\frac{1}{2}u_x(x_l, \tau) + \frac{1}{4\nu}u(x_l, \tau)^2 \right) d\tau. \tag{2.17}$$

We denote the two conditions (2.15) and (2.16) by

$$u(x_l, t) = L_1(u(x_l, t), g_l(t)), \quad \text{and} \quad u_x(x_l, t) = L_2(u(x_l, t), g_l(t)).$$

Using these nonlinear artificial boundary conditions, the original problem (1.1)-(1.3) is reduced to the following problem defined on the finite domain  $\Omega_i$ :

$$\text{Find } u(x, t) \in \Omega_i \text{ such that} \tag{2.18}$$

$$u_t + uu_x - \nu u_{xx} = f(x), \quad \text{in } \Omega_i; \tag{2.19}$$

$$u(x, 0) = u_0(x), \quad x_l \leq x \leq x_r; \tag{2.20}$$

$$u(x_r, t) = R_1(u(x_r, t), g_r(t)), \quad \text{or} \quad u_x(x_r, t) = R_2(u(x_r, t), g_r(t)); \tag{2.21}$$

$$u(x_l, t) = L_1(u(x_l, t), g_l(t)), \quad \text{or} \quad u_x(x_l, t) = L_2(u(x_l, t), g_l(t)), \tag{2.22}$$

where  $g_r(t)$  and  $g_l(t)$  are given in (2.12) and (2.17), respectively.

### 3. Numerical Approximation

In this section we consider the numerical solution of the reduced problem (2.18)-(2.22). In the computational region  $[x_l, x_r]$ , let  $h = \Delta x = (x_r - x_l)/M$  be the spatial mesh size, and let  $k = \Delta t = T/N$  be the time step, where  $M$  and  $N$  are positive integers. Let the grid points and time steps be

$$x_j = x_l + jh, \quad t_n = nk, \quad j = 0, 1, \dots, M, \quad n = 0, 1, 2, \dots, N,$$

and denote the approximation of  $u(x_j, t^n)$  by  $u_j^n$ . For the approximation of the Burgers' equation, we use the second order implicit Crank-Nicolson scheme:

$$\frac{u_j^{n+1} - u_j^n}{k} + u_j^{n+\frac{1}{2}} \frac{u_{j+1}^{n+\frac{1}{2}} - u_{j-1}^{n+\frac{1}{2}}}{2h} - \nu \frac{u_{j+1}^{n+\frac{1}{2}} - 2u_j^{n+\frac{1}{2}} + u_{j-1}^{n+\frac{1}{2}}}{h^2} = f(x_j, (n + \frac{1}{2})k),$$

$$u_j^0 = u_0(x_j)$$

for  $j = 1, \dots, M - 1$ , where

$$u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^n + u_j^{n+1}).$$

The scheme is unconditionally stable with the truncation error  $O(k^2 + h^2)$ . The resulting system of equations from this approximation has  $M - 1$  equations and  $M + 1$  unknowns. Thus, two extra conditions are needed to complete the system. These two conditions are provided by the two artificial boundary conditions. Here we only provide discrete formulae of the second conditions of (2.14) and (2.16). For the right boundary, we use,

$$\frac{u_M^{n+1} - u_{M-2}^{n+1}}{2h} = \frac{1}{2\nu}(u_{M-1}^{n+1})^2 - \frac{2}{\sqrt{\nu\pi}e^{g_r^{n+1}}} \cdot \sum_{p=0}^n \frac{e^{g_r^{p+1}} u_{M-1}^{p+1} - e^{g_r^p} u_{M-1}^p}{k} (\sqrt{t_{n+1} - t_p} - \sqrt{t_{n+1} - t_{p+1}}). \quad (3.1)$$

where

$$g_r^{n+1} = g_r^n + \frac{1}{2}(Q_r^n + Q_r^{n+1})k,$$

and

$$Q_r^p = -\frac{u_M^p - u_{M-2}^p}{4h} + \frac{(u_{M-1}^p)^2}{4\nu}.$$

And on the left boundary, we also have

$$\frac{u_2^{n+1} - u_0^{n+1}}{2h} = \frac{1}{2\nu}(u_1^{n+1})^2 + \frac{2}{\sqrt{\nu\pi}e^{g_l^{n+1}}} \cdot \sum_{p=0}^n \frac{e^{g_l^{p+1}} u_1^{p+1} - e^{g_l^p} u_1^p}{k} (\sqrt{t_{n+1} - t_p} - \sqrt{t_{n+1} - t_{p+1}}). \quad (3.2)$$

where

$$g_l^{n+1} = g_l^n + \frac{1}{2}(Q_l^n + Q_l^{n+1})k,$$

and

$$Q_l^p = -\frac{u_2^p - u_0^p}{4h} + \frac{(u_1^p)^2}{4\nu}.$$

The approximations (3.1), (3.2) to artificial boundary conditions have the accuracy  $O(k^{\frac{3}{2}} + h^2)$  [18]. The scheme is implicit, we must use iteration methods to numerically solve the system. Here the simple iteration method is used in our numerical experiments.

#### 4. Numerical Examples

To show the effectiveness of the new approach using artificial boundaries, we present three numerical examples in this section. In the first two examples, we consider the Burgers' equation without source term. In this case, the exact solutions are given, and the numerical solutions are compared with the exact solution. In the third example, we consider the Burgers' equation with a source term. The exact solution to this problem is unknown. We take a numerical solution computed on a very fine mesh as the exact solution, and then compare the numerical solutions with it.

**Example 1.** We first consider the Burgers equation without source term

$$u_t + uu_x - \nu u_{xx} = 0, \quad (4.1)$$

which has the exact solution

$$u(x, t) = -\frac{\frac{x}{t+1}}{1 + \sqrt{\frac{t+1}{t_0} \exp\left(\frac{x^2}{4\nu(t+1)}\right)}},$$

where  $t_0 = \exp(1/8\nu)$ . The solution represents two waves propagating to the left and right respectively with amplitudes gradually decreasing. Using the proposed numerical method in section 2 and section 3 with the initial condition obtained from the exact solution, we simulate the result for two cases:

- i)  $\nu = 1, T = 16, [x_l, x_r] = [-8, 8]$ ; and
- ii)  $\nu = 0.1, T = 12, [x_l, x_r] = [-3, 3]$ .

Table 1 and 2 show the numerical errors and orders of accuracy for two cases with time step  $k = h$ , where the  $L_\infty$  and  $L_1$  errors are defined by

$$E_\infty = \max |u(x_j, t^n) - u_j^n|, \quad j = 0, 1, \dots, M, \quad n = 0, 1, \dots, N,$$

$$E_1 = \frac{1}{(N+1)(M+1)} \sum_{n=0}^N \sum_{j=0}^M |u(x_j, t^n) - u_j^n|.$$

We can see that the method gives second order of accuracy in both cases. The  $x - t$  solution contours are plotted for  $N = 256$  in Fig. 2. From the figures, no reflective waves can be seen near the artificial boundaries, so artificial boundary condition is very effective.

Table 1.  $\nu = 1, L_\infty$  and  $L_1$  errors and orders of accuracy

N	$L_\infty$ order	$L_\infty$ order	$L_1$ order	$L_1$ order
16	1.544e-2	–	1.076e-3	–
32	5.202e-3	1.570	2.774e-4	1.956
64	1.248e-3	2.059	6.953e-5	1.996
128	2.951e-4	2.080	1.722e-5	2.014
256	7.363e-5	2.003	4.211e-6	2.032

Table 2.  $\nu = 0.1, L_\infty$  and  $L_1$  errors and orders of accuracy

N	$L_\infty$ order	$L_\infty$ order	$L_1$ order	$L_1$ order
16	7.544e-3	–	1.111e-3	–
32	1.950e-3	1.952	2.766e-4	2.006
64	4.854e-4	2.006	6.968e-5	1.989
128	1.222e-4	1.990	1.749e-5	1.994
256	3.053e-5	2.001	4.371e-6	2.000

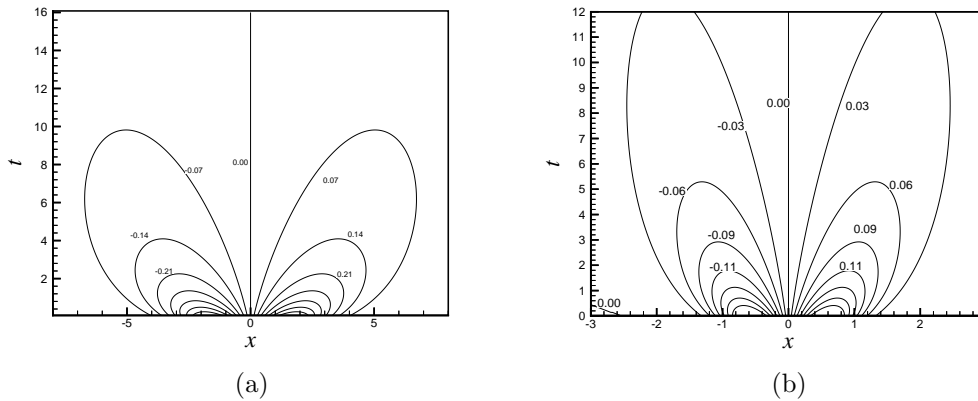


Fig. 2. (a)  $\nu = 1$ , 15 contours from -0.5 to 0.5; (b)  $a = 0.1$ , 15 contours from -0.2 to 0.2

**Example 2.** Let  $\alpha, \mu$  and  $\gamma$  be constants, and let  $\eta = \frac{\alpha(x-\mu t-\gamma)}{\nu}$ . We consider the Burgers' equation (4.1) with

$$u(x, t) = -\frac{\alpha + \mu + (\mu - \alpha)e^\eta}{1 + e^\eta},$$

as the exact solution. This solution represents a travelling wave moving to the right with speed  $\mu$ . We take  $\alpha, \mu, \gamma = 1$  and time step  $k = h$ . Similarly, we compute the solution for two cases

- i)  $\nu = 1, T = 24, [x_l, x_r] = [-8, 16]$ ;
- ii)  $\nu = 0.1, T = 4, [x_l, x_r] = [-1, 3]$ .

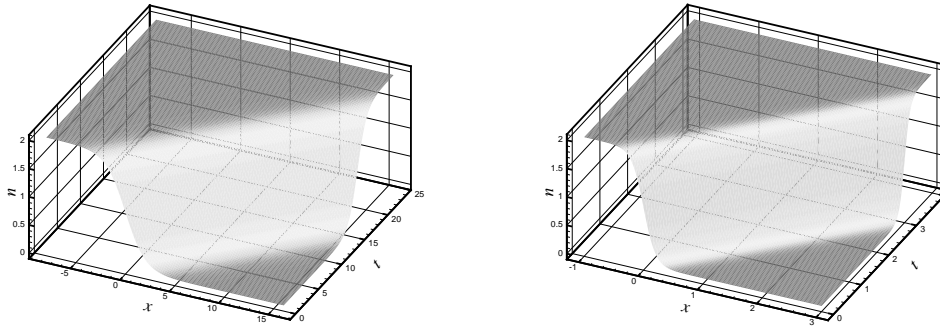
We use the exact solution as the initial condition and left boundary condition. On the right boundary, the artificial boundary condition is imposed. The  $L_\infty$  and  $L_1$  errors and orders of accuracy in  $x - t$  plane are listed in Tables 3 and 4. Here we can see that the numerical orders approach to 1.5 when the mesh is refined. Fig. 3 is the solution plots in  $x - t$  plane for  $\nu = 1$  and  $\nu = 0.1$  with the same grid points  $N = 1024$ . Fig. 4 shows the error evolutions in time  $t$  at the artificial boundaries for different grids. Clearly, the errors decrease rapidly as  $N$  increases.

Table 3.  $\nu = 1, L_\infty$  and  $L_1$  errors and orders of accuracy

N	$L_\infty$ order	$L_\infty$ order	$L_1$ order	$L_1$ order
32	5.809e-1	–	1.264e-2	–
64	1.582e-1	1.877	2.880e-3	2.134
128	4.528e-2	1.805	7.183e-4	2.003
256	1.429e-2	1.664	1.949e-4	1.882
512	4.799e-3	1.574	5.640e-5	1.789
1024	1.664e-3	1.528	1.716e-5	1.717

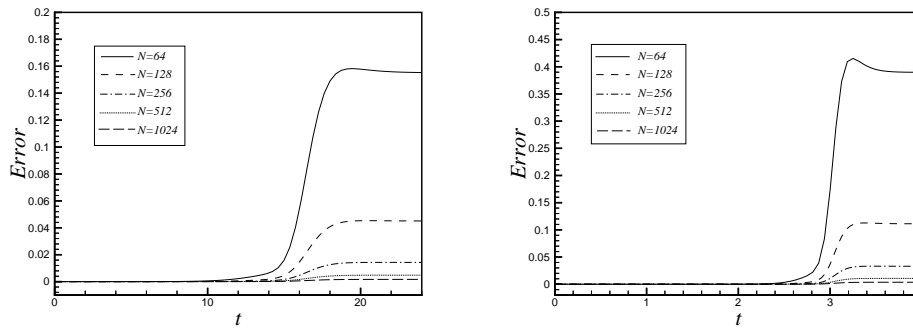
Table 4.  $\nu = 0.1, L_\infty$  and  $L_1$  errors and orders of accuracy

N	$L_\infty$ order	$L_\infty$ order	$L_1$ order	$L_1$ order
32	1.411	–	2.216e-2	–
64	4.153e-1	1.764	5.046e-3	2.135
128	1.127e-1	1.882	1.182e-3	2.094
256	3.316e-2	1.765	2.994e-4	1.981
512	1.068e-2	1.635	8.807e-5	1.888
1024	3.624e-3	1.559	2.296e-5	1.816



(a) (b)

Fig. 3. Solution in  $x - t$  plane.  $N = 1024$ . (a)  $\nu = 1$  ; (b)  $\nu = 0.1$



(a) (b)

Fig. 4. Errors evolutions in time  $t$  at artificial boundary with different grids.

(a)  $\nu = 1, x = 16$  ; (b)  $\nu = 0.1, x = 3$

**Example 3.** In this example, we consider the Burgers equation with a source term:

$$u_t + uu_x - \frac{1}{5}u_{xx} = \left(\frac{1}{2} + \sin\pi t\right)e^{-x^2},$$

$$u(x, 0) = 0.$$

We compute the solution using the artificial boundary conditions in the domain  $-4 \leq x \leq 4$ ,  $0 \leq t \leq 8$ . The exact solution to this problem is unknown. We take the numerical solution computed on a very fine mesh (2048 grid points) and in a larger domain  $x \in [-8, 8]$  as the "exact" solution for the purpose of comparison. Fig. 5(a) shows the surface plot of the numerical solution for  $N = 256$ . The "exact" solution is shown in Fig. 5(b). We can see that the numerical solution well agrees with the "exact" solution.

In Table 5, we show the numerical errors and orders in  $x - t$  plane. Fig. 6 are the solutions for  $x \in [2, 4]$  at  $t = 6$  and  $t = 8$ . When  $t = 6$ , the errors are very small. When  $t = 8$ , the errors appear, but converge to the "exact" solution rapidly when the mesh is refined.



Table 5.  $L_\infty$  and  $L_1$  errors and orders of accuracy

N	$L_\infty$ order	$L_\infty$ order	$L_1$ order	$L_1$ order
16	8.009e-1	–	3.192e-2	–
32	2.441e-1	1.714	7.271e-3	2.145
64	6.216e-2	1.973	1.697e-3	2.099
128	1.685e-2	1.886	4.168e-4	2.026
256	5.078e-3	1.730	1.031e-4	2.015

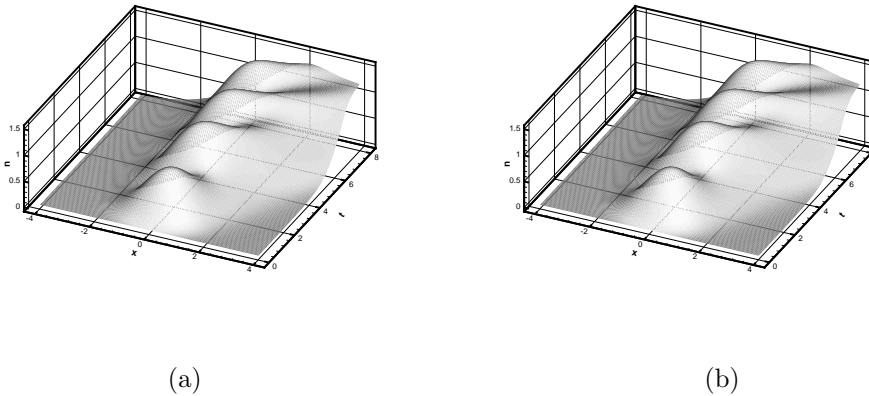


Fig. 5.  $x - t$  solutions. (a)  $N = 256$ ; (b) "exact" solution

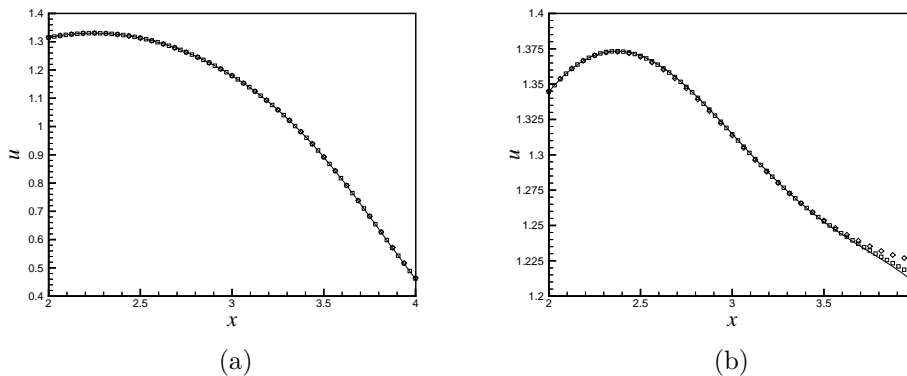


Fig. 6. Diamond:  $N = 128$ , square:  $N = 256$ , solid: "exact" solution. (a)  $t = 6$ , (b)  $t = 8$

### 5. Conclusion

The artificial boundary method has been applied to the Burgers' equation on unbounded domains. Using the Cole-Hopf transformation we obtained the boundary conditions on the artificial boundaries. These boundary conditions are in nonlinear forms. With the artificial boundaries, we can solve the original unbounded problem in a much smaller domain, the computational work can be greatly reduced. The numerical examples showed that the new approach is very effective, the numerical solutions converge fast to the exact solutions.

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